Resonant and Invariant Advection in Single-Screw Extrusion

1 Introduction

In this paper, we discussed how unmixed and well-mixed zones are determined by the practical design parameters of the Chaos Screw (CS) system for the single-screw extruder, e.g., the helix angle, the magnitude of adverse pressure gradient, and the fraction of no-barrier zone. The key concept is the frequency ratio distribution of the unperturbed system. Through the extensive three-dimensional numerical simulations with finite element methods and particle tracing techniques, successive creation and breakup of resonant orbits and KAM tori in terms of design parameters are presented. The results show that one can predict the spatial configuration of well-mixed and unmixed zones from the frequency ratio distribution and the amount of the perturbation strength. Finally, the exit time distributions caused by dynamical structures are discussed.

It has been well known that the frequency ratios in the unperturbed system determine the future of the perturbed system. For the unsteady two-dimensional area-preserving flows, the subharmonic Melnikov method precisely describes the resonance mechanism and the associated resonant conditions; the Moser Twist theorem states an explicit condition (the Diophantine condition) for the sufficiently irrational orbits to survive and to form invariant KAM surfaces under perturbation [1]. Also in the three-dimensional flow systems, the same phenomena have been investigated even at the earliest stage of the history of the chaotic mixing: Dombre et al. [2] found the coexistence of chaotic regions with resonance bands and KAM tori in the ABC (Arnold-Beltrami-Childress) flow. However, the corresponding theory for the three-dimensional system was rather recent; Mezić and Wiggins [3] introduced a complete Hamiltonian canonical description, the so-called action-angle-angle transformation that is well suited for the family of periodic orbits. They presented the associated KAM-like theory and the Melnikov method to describe homoclinic manifolds. However, general three-dimensional flows cannot be written in the Hamiltonian form by the action-angle-angle transformation; the motion of a fluid particle in the unperturbed system should be of the following form [3]:

\[
\begin{align*}
\dot{x} &= u(x, y) = \frac{\partial \Psi(x, y)}{\partial y}, \\
\dot{y} &= v(x, y) = -\frac{\partial \Psi(x, y)}{\partial x}, \\
\dot{z} &= w(x, y),
\end{align*}
\]

where \((x, y, z)\) and \((u, v, w)\) are the position of the fluid particle and the fluid velocity at the particle location, respectively; \(\Psi(x, y)\) is the stream function satisfying the two-dimensional continuity, \(\partial u/\partial x + \partial v/\partial y = 0\). The flow in Eq. 1 is called the regular duct flow [4]. Regarding configuration of resonant and invariant orbits, rigorous use of the theory has not been made until Fountain et al. [5] presented their study on mixing in a stirred vessel flow. They showed creation and breakup of resonance bands and KAM tori according to the frequency ratio distribution by the numerical simulations and the experiments. Hwang and Kwon [6] also presented similar numerical results for a non-Newtonian viscous flow in the CS single-screw extrusion system that belongs to the class exactly. Hwang et al. [7] also presented flow-visualization results through the 3-D channel flow experiment in the CS system and showed the existence of resonance bands, KAM tori and chaotic zones experimentally.

In this paper, we present extensive numerical results on the resonant and invariant motions of particles in the CS system. In section 2 are a brief introduction of CS for the sake of com-
pleteness and a useful formula for obtaining the frequency ratio distribution in the metering section of the single-screw extruder. In section 3, the extensive numerical results will be presented through the Poincaré section analyses under the various design parameters in the CS system. The aim is to show how one can predict the configuration of the perturbed dynamical structures by the frequency-ratio distributions and the perturbation values. Equivalently, it means how the well-mixed and unmixed zones are determined by the practical design variables of the CS system. The exit time distribution caused by these dynamical structures will be discussed also at the end of section 3.

2 Modeling

2.1 System

The unwound geometry of CS is shown in Fig. 1. (For details of the CS system, see [6, 7].) The coordinate system is also indicated in the figure; x is chosen along the width direction, y along the screw depth, and z along the channel direction from hopper to die. The xy plane is called the cross-sectional plane and the z direction is called the longitudinal direction throughout the study. We denote length of barrier zone by a, the length of no-barrier zone by b, the total length of period by L (= a + b), the height of barrier by h, the height of flight by H, the width of barrier c, and the width of channel by W. As indicated in [6, 7], the periodic removal of barrier is regarded as the geometric perturbation.

\[ \beta = b/L. \] (2)

The geometry of the periodic unit is fixed as the dimensionless width of channel \( W/H = 4 \), the dimensionless height of barrier \( h/H = 0.66 \), the dimensionless length of period \( L/H = 10 \), and the dimensionless width of barrier \( c/H = 0.1 \). The drag velocity on the top plate is denoted by \( V_d \) in the direction of the helix angle \( \alpha \). The rest of geometry and flow conditions are considered as the parameters: the helix angle \( \alpha \), the fraction of the no-barrier zone \( \beta \), and the fraction of the pressure-driven flow \( \kappa \) due to the (adverse) pressure gradient \( \partial p/\partial z \) (Fig. 1). The value of \( \kappa \) is defined as the ratio of the pressure-driven flowrate \( Q_p \) to the drag-driven flowrate \( Q_d \).

To focus our study on the kinematics only, the flow in the CS system is assumed to be the Stokes flow. With the height of flight \( H \) for the characteristic length and with the z directional component of the drag velocity \( V_{dz} \) for the characteristic velocity, the dimensionless variables can be expressed in the following way:

\[ x' = \frac{x}{H}, \quad v' = \frac{v}{V_{dz}}, \quad t' = \frac{V_{dz}t}{H}, \quad p' = \frac{1}{\mu} \frac{H}{V_{dz}}p, \] (3)

where \( t \) is time and \( \mu \) is the viscosity. Hereafter, we exclusively use the dimensionless variables, omitting the superscript ‘*’.

The continuity equation and equations of motions are as follows.

\[ \nabla \cdot v = 0, \] (4)

\[ \nabla p = \nabla^2 v. \] (5)

Boundary conditions are also indicated in Fig. 1. The drag velocity is specified on the top plate with the helix angle; no-slip condition is prescribed on the screw root, the barrier surface, and the flights. On the inlet and outlet boundaries, it is assumed that the velocity distributions on those surfaces are the same as the velocity distribution of the no-barrier case that depends only on the helix angle \( \alpha \) and the pressure gradient \( \kappa \).

According to the definition in Eq. 1, the unperturbed system (\( \beta = 0 \)) is the full-barrier system (see [6, 7] for dynamics of the unperturbed system). It would be useful to obtain the solution form of the unperturbed velocity fields in that they can be used as boundary conditions on the inlet and outlet surfaces and in that they can be used in the derivation of frequency ratios. Since the governing equations (Eq. 5) are linear and the boundary conditions can be separated into homogeneous ones, the solution form of the unperturbed velocity \( \mathbf{v}_0(x, y) \) can be easily obtained by the superposition of simple flow solutions. Let us consider the decomposition of boundary conditions into three homogeneous boundary conditions in the unperturbed system; (i) the unit drag on the top surface in z, \( V_{dz} = 1 \), (ii) the drag in \( x \), \( V_{dx} = \tan \alpha \), and (iii) the pressure gradient due to die effect between the inlet and outlet (Fig. 2). Let \( \mathbf{v}_d = (0, w_d, 0) \) be the flow field in the z direction by the unit drag, \( \mathbf{v}_c = (u_c, v_c, 0) \) be the cross-sectional flow solution in the xy plane by the unit drag \( V_{dx} = 1 \), and \( \mathbf{v}_p = (0, 0, w_p) \) be the pressure-driven flow solution with the pressure \( P \) at the outlet that leads to \( Q_d = \int -w_d da - \int w_c da = 0 \) \((Q_d \) is the flow rate solely due to \( u_c \) and \( Q_p \) is the flow rate solely due to the preferable pressure gradient, \( \partial p/\partial z < 0 \)). We call them the basic flow solutions in the unperturbed system (Fig. 2). Then, the unperturbed velocity field \( \mathbf{v}_0(x, y) \) can be expressed in terms of the basic flows \( \mathbf{v}_d, \mathbf{v}_c \), and \( \mathbf{v}_p \) with the helix angle \( \alpha \) and the fraction of backflow \( \chi (= Q_p/Q_d) \) as parameters:

\[ \mathbf{v}_0(x, y; \alpha, \chi) = \mathbf{v}_d(x, y) + \tan \alpha \mathbf{v}_c(x, y) - \chi \mathbf{v}_p(x, y). \] (6)

(The minus sign in front of \( \chi \) indicates that the \( \mathbf{v}_p \) usually flows in the reverse direction.)

2.2 Frequency Ratios

From the geometric periodicity of the CS, the motions of particle on the inlet surface are the same as those on the outlet. As indicated in [6], one can safely connect two ends together to...
form a set of nested tori with the new phase variable $\varphi$ rather than $z$ such that

$$\varphi = z/\lambda, \mod(1), \quad \varphi \in S,$$

(7)

where $S$ is a circle of period 1. As mentioned in [6], there are two competing frequencies $f_1$ and $f_2$ for a helical orbit on each torus (Fig. 3). $f_1$ is the frequency associated with the motion around the large circumference in the $\varphi$ direction; and $f_2$ is associated with the cross-sectional motion in the $xy$ plane. The ratio between the two frequencies is called the frequency ratio and is denoted by $\Omega$. Since the values of $f_1$ and $f_2$ depend on the toroidal stream surface, the frequency ratio is a function of the toroidal stream surface. The variable $\zeta$ is chosen as an index of each torus, which is the coordinate from the center of elliptic rotation in the negative $y$ direction [6]. The frequency ratio as a function of $\zeta$ can be expressed as follows:

$$\Omega(\zeta) = \frac{\lambda}{d(\zeta)} = \frac{\lambda}{\int w_0(u_0^2 + v_0^2)^{-1/2} dl},$$

(8)

where $d(\zeta)$ is the longitudinal distance traveled by a particle during one cross-sectional rotation and $l$ is a coordinate along the (projected) cross-sectional rotation (Fig. 4) [6]. To obtain the explicit dependence of the helix angle $\alpha$ and the fraction of backflow $\kappa$, the unperturbed velocity components in Eq. 8 are replaced by Eq. 6, and one gets

$$\Omega(\zeta; \alpha, \kappa) = \frac{\lambda \tan \alpha}{\int (w_d - \kappa w_p)(u_x^2 + v_x^2)^{-1/2} dl} = \frac{\lambda \tan \alpha}{d_d(\zeta) - \kappa d_p(\zeta)}.$$

(9)

The $d_d(\zeta)$ and $d_p(\zeta)$ are the distances traveled along the $z$ direction by a particle during one cross-sectional rotation under the basic drag flow, and under the basic pressure flow, respectively:

$$d_d(\zeta) = \int w_d(u_x^2 + v_x^2)^{-1/2} dl, \quad d_p(\zeta) = \int w_p(u_x^2 + v_x^2)^{-1/2} dl.$$

(10)

Once one evaluates the $d_d(\zeta)$ and $d_p(\zeta)$ from the basic flow solutions, it is easy to obtain the frequency-ratio distribution for any combinations of $(\alpha, \kappa)$ from Eq. 9. In addition, even though we fixed the value of $\lambda$ as 10 throughout the study, Eq. 9 is not restricted to those case. One can see that the effect of the value $\lambda$ is the same as that of $\tan \alpha$. Bearing in mind the limited choice in the value of the helix angle for productivity reason, it seems fortunate that one can control the frequency-ratio distribution, which affects directly the perturbed dynamics, by changing the length of period $\lambda$, while the helix angle left unchanged.
In the unperturbed system, if the frequency ratio is rational, the corresponding orbit is periodic and the Poincaré section of the orbit possesses only $n$ periodic points, when $n$ is the denominator of the ratio $m/n$. On the other hand, if the ratio is irrational, the orbit completely fills the torus (stream surface) where the orbit is located and the Poincaré section of the orbit possesses the infinite number of points that constitute the ‘drift ring.’ More importantly, one can predict the future of orbits under perturbation from the distribution of the ratios. An orbit on a torus whose frequency ratio is rational, say $m/n$, is perturbed to give the subharmonic orbit that consists of perturbed heteroclinic orbit with $2n$ fixed points, $n$ elliptic and $n$ hyperbolic, in turns. Nearby orbits can also be perturbed to give the same frequency ratio so that these orbits form an annulus in the three-dimensional space and form so called the resonance band in the Poincaré section. Transports of fluid particles through the resonant annulus are limited due to the successive trapping and release by the associated lobe dynamics of the heteroclinic tangles. Smaller denominator makes larger and stronger resonance, in general. The rational frequency ratio whose denominator is relatively small in the frequency-ratio distribution, e.g. 1, 2, 3, or 4, is called the dominant rational frequency ratio of the system. Thus, the resonance corresponding to the dominant rational frequency ratio can be easily visualized under sufficiently small perturbation [1]. For the irrational frequency ratio orbit, if the orbit is located sufficiently far from the orbit of strong rational frequency ratio, then it may remain invariant. In other words, if the ratio is sufficiently irrational, or, sufficiently difficult to be approximated by the rational number, the corresponding orbit remains invariant and forms the KAM torus. The KAM tori prevent transports completely between two neighboring regions and, in this regard, they are considered as complete barrier for material transports. However, if the perturbation becomes sufficiently large so that neighboring rational frequency-ratio orbits (with rather large denominator) may be perturbed, the KAM torus could vanish [1].

3 Numerical Analyses

3.1 Numerical Methods

The full three-dimensional multi-variant $Q_1^*P_0$ finite element scheme was used in solving the velocity fields as the author’s previous studies [6 to 8]. As described in Eq. 6 and in Fig. 2, the unperturbed velocity fields can be determined by the superposition of the basic flow solutions; the unit drag flow $v_d(x, y)$, the unit cross flow $v_c(x, y)$, and the pressure-driven flow solution $v_p(x, y)$ satisfying $\int w_d(x, y) dA = \int w_p(x, y) dA$. The unperturbed velocity field with the parameters $\alpha$ and $\kappa$ is determined by Eq. 6 using the basic flow solutions.

The perturbed velocity field with the parameters $(\alpha, \kappa, \beta)$ can be obtained by specifying the corresponding unperturbed velocity field with the parameters $(\alpha, \kappa)$ on the inlet and outlet surfaces of the periodic unit (section 2.1). This assumption is valid as long as the perturbation strength is sufficiently small ($\beta \ll 1$). (The value of $\beta$ in this study ranges from 0.01 to 0.15.) A representative example finite element mesh for $\beta = 0.1$ is shown in Fig. 5. Plotted in Fig. 6 are the cross-sectional velocity and pressure distributions for $\alpha = \pi/6$, $\kappa = 0.25$ and $\beta = 0.05$ at the center of the no-barrier zone ($\varphi = 1/2$). Finally, numerical integration has been carried out by using the fourth order Runge-Kutta method, for the purpose of visualization of dynamical structures via Poincaré sections [6, 7].

3.2 Frequency Ratio Distribution

Resonant and invariant advections in the perturbed nonlinear oscillation strongly depend on the distribution of the frequency ratios $\Omega(\zeta; \alpha, \kappa)$ of the unperturbed system and, to obtain the ratio, one has to determine $d_d(\zeta)$ and $d_p(\zeta)$ in Eq. 9. The
cross-sectional rotation center \((x_c, y_c)\) with \(\zeta = 0\) has been found from the cross flow solution and is located at \((1.031750, 0.670551)\). Numerical results for \(\Delta_d(\xi)\) and \(\Delta_p(\xi)\) are indicated in Fig. 7. Both approach infinity near the homoclinic orbit \((\xi_h = 0.4106)\) where the cross-sectional period is infinity.

With \(\Delta_d(\xi)\) and \(\Delta_p(\xi)\), the frequency-ratio distributions of any \((\alpha, \kappa)\) set are easily determined from Eq. 9. Four different \((\alpha, \kappa)\) combinations were selected as example cases and they are listed in Table 1. The frequency ratio distributions for the four cases are shown in Fig. 8. As the helix angle \(\alpha\) and the fraction of backflow \(\kappa\) increase, the frequency ratio also increases, since \(\Omega\) is proportional to \(\tan \alpha\) and larger adverse pressure gradient leads to smaller \(\Delta(\xi)\). In Table 1, the dominant rational frequency ratios of each case are listed as well. They were obtained by the \(\text{mod}(1)\) function, because only the decimal parts are important. In this regard, the rational frequency ratios associated with the case IV are listed as \(1/3, 1/4, 1/5, 1/6\) (not \(4/3, 5/4, 6/5, \text{and } 7/6\)).

### Overall Structures

Let us consider overall dynamical behaviors of the system, using the case I \((\alpha = \pi/12 \text{ and } \kappa = 0)\) with \(\beta = 0.02\). Fig. 9A to D indicate the Poincaré sections at four different phases; (A) \(\varphi = 0\) (center of the barrier zone), (B) \(\varphi = 0.2\), (C) \(\varphi = 0.5\) (center of the no-barrier zone), and (D) \(\varphi = 0.8\). The Poincaré sections have been obtained by the forward and backward integrations of 66 equally distributed initial points at \(\varphi = 0\) for 150 periods. The phase where the Poincaré section is evaluated has been varied to reconstruct the three-dimensional dynamical structure in \(\mathbb{R}^3 \times S\). In addition, the corresponding longitudinal section at \(x = x_c\) plane is also indicated in Fig. 10. Since the longitudinal section is composed of two transversal planes (above and below \(y = y_c\)), the section can represent the longitudinal dynamical structure.

In Fig. 9 one can see four nested regular rotations (KAM tori) and a ring of two unmixed islands (resonance bands) floating in the chaotic region. (Unmixed islands are also KAM tori and they are helical cylinders in \(\mathbb{R}^3\). But we consistently keep using the term unmixed islands to refer to the KAM tori in the resonance band, in order to avoid confusion.) The unmixed islands move half a period during one longitudinal period so that they will return to their original positions after two longitudinal periods. Thus the corresponding resonance band is called period 2, which can be predicted from the most dominant rational frequency ratio \(1/2\) in this case (Table 1). In Fig. 10, the longitudinal section, the whole \(y_p\) plane has been separated by eight horizontal lines. They correspond to the four KAM tori. Again, one can see the two periodic unmixed islands; one in the upper

<table>
<thead>
<tr>
<th>Case No.</th>
<th>(\alpha)</th>
<th>(\kappa)</th>
<th>DRFR’s, mod(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>(\pi/12)</td>
<td>0</td>
<td>1/2, 2/5, ...</td>
</tr>
<tr>
<td>Case II</td>
<td>(\pi/12)</td>
<td>0.25</td>
<td>2/3, 4/7, ...</td>
</tr>
<tr>
<td>Case III</td>
<td>(\pi/6)</td>
<td>0</td>
<td>1, ...</td>
</tr>
<tr>
<td>Case IV</td>
<td>(\pi/6)</td>
<td>0.25</td>
<td>1/3, 1/4, 1/5, 1/6 ...</td>
</tr>
</tbody>
</table>

Table 1. Four different \((\alpha, \kappa)\) combinations and their corresponding dominant rational frequency ratios (DRFRs) (see Fig. 8). The bold faced ratio indicates the most dominant rational frequency ratio of each case.

Fig. 8. Frequency-ratio distributions for four parametric compositions (see Table 1)

Fig. 9. Poincaré sections for case I with \(\beta = 0.02\) at four different phases
the period-2 resonance band. In Fig. 11B (strongly under small value of more dominant, the associated orbit is supposed to be resonant Since the rational frequency ratio of the period-2 resonance is breakup.) With around it. (See [10] for detailed description of the KAM torus becomes unstable, with high period resonance bands band has contracted greatly and the outermost (largest) KAM tori survive. When \( b = 0.05 \) (Fig. 11C), the period-2 resonance band has vanished. Also the largest KAM torus cannot be seen, but the remnant of KAM torus still prevents the material transports. Further, the innermost (the smallest) KAM torus has disappeared due to the existence of the period-5 resonance band. When \( \beta = 0.10 \) (Fig. 11E), the remaining two KAM tori have disappeared also. When \( \beta = 0.12 \) (Fig. 11F), the period-5 resonance band becomes clear and it must be associated with \( \Omega = 2/5 \). The chaotic region covers the whole region except inside the resonance band.

### Case II.

A period-3 resonance band is expected to occur under the sufficiently small \( \beta \) value; and a period-7 to occur with rather larger \( \beta \) value at the inner region, since the dominant rational frequency ratio are given in the order of 2/3 and 4/7. For the smallest \( \beta \) value, \( \beta = 0.02 \) (Fig. 12A), one can see a period-3 resonance band and four nested KAM tori surrounded by the resonance band. When \( \beta = 0.05 \) (Fig. 12B), the period-3 resonance band has vanished and the outermost KAM torus becomes unstable. For \( \beta = 0.06 \) (Fig. 12C), the outer three KAM tori have been broken; but they still divide the phase space, somehow preventing material transports across them. When \( \beta = 0.07 \) (Fig. 12D), dynamical structures have been changed a lot: the period-7 resonance band in the intermediate region and the period-2 resonance band in the innermost region. The period-7 resonance band must be associated with \( \Omega = 4/7 \) according to the frequency ratio distribution, but the period-2 resonance band cannot be predicted

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**Fig. 10.** Longitudinal section evaluated at \( x = x_c \) in Fig. 9

**Fig. 11.** Poincaré sections at \( \varphi = 0 \) for case I under various perturbation values
A) \( \beta = 0.02 \); B) 0.03; C) 0.05; D) 0.07; E) 0.1; F) 0.12
from the frequency-ratio distribution. This could be understood by the small difference between the frequency ratio of the unperturbed orbit and the actual winding number of the perturbed orbit. As shown in Fig. 8, the value of the frequency ratio near the rotation center is close to 1/2, and, therefore, the winding number of the perturbed system must be changed to give rise to the 1/2 winding number near the rotation center. In case of $b = 0.10$ (Fig. 12E), the period-2 resonance band preserves its dynamics, but the period-7 resonance band has disappeared. For the large value of perturbation $b = 0.15$ (Fig. 12F), the overall structure does not change, but the period-2 resonance band becomes weak. This means that more materials can penetrate the resonance band than in the previous case.

Case III. The dominant rational frequency ratio is 1/1, thus one can expect the resonance band of period 1. There can be found a period-1 resonance band in Figs. 13A to C. It would be the largest resonance band of all the cases, because $\Omega = 1/1$ is the most dominant frequency ratio possible. For this reason, one can even find a subharmonic orbit by which we mean a nested resonance band formed around the elliptic fixed point in the resonance band of period 1. This kind of chaotic structure is called the fractal, i.e., the self-similarity. There can be seen a number of satellites that rotate around the period-1 resonance band in Figs. 13A and 13B. The number seems to be five in case of Fig. 13A and two in case of Fig. 13B. (If the subsubharmonic resonance band consists of $n'$ unmixed islands and the parent resonance band has $n$ unmixed islands, then the period of the subsubharmonic resonance band becomes $n'n$.) For $b = 0.05$, the subsubharmonic orbit no longer exists (Fig. 13C). When $b = 0.07$, the resonance band of period 1 has vanished as well and only the innermost KAM torus survives. For $b = 0.10$ and 0.12 (Figs. 13E and 13F), the whole region becomes chaotic. This is the only case in our study that shows completely chaotic behaviors on the whole phase space.

Case IV. Since this case has comparable rational frequency ratios 1/3, 1/4, 1/5, and 1/6, the Poincaré sections in the perturbed system might show the cascade of all the resonance bands simultaneously. In Fig. 14C ($b = 0.05$), one can see period-3, 4, 5, and 6 resonance bands at the same time. As usual, the period-3 resonance band can be investigated under small perturbation in case of $b = 0.01$ (Fig. 14A) and, for $b = 0.02$ (Fig. 14B), period-3 and period-5 resonance bands become apparent. For $b = 0.07$ (Fig. 14D), all the resonance bands have disappeared and there can be seen only two nested KAM tori near the rotation center. When $b = 0.10$ (Fig. 14E), the outer KAM torus has been broken and the inner KAM torus becomes unstable. When $b = 0.12$ (Fig. 14F), there can be seen a period-6 resonance band with six unmixed islands near the center of rotation.
3.5 Longitudinal Mixing and Exit Time Distribution

Chaotic mixing also affects the longitudinal mixing characteristics. In the chaotic region, randomness in the cross-sectional motion leads to random experience of the longitudinal velocity component. As a result, random longitudinal exit time distribution can be achieved in the chaotic region [7]. Plotted in Fig. 15 are the two exit time distributions for the case I with $b = 0.02$ and $b = 0.05$. To obtain the distribution, we started with the equally spaced 200 initial points along the $y$ axis at $x = x_c$ and $z = 0$; then recorded the exit time that can be identified by the initial location for a certain longitudinal length. Thus, the $y$ coordinate value of the initial point is used as the index of the point.

Let us consider Fig. 15A for the case of $b = 0.02$ in which the period-2 resonance band and four KAM tori were observed (Figs. 9 and 10). After rather short longitudinal length, $z = 10a$, the exit time in Fig. 15A seems to be regular except the boundary. But, the exit time distributions gradually resemble the dynamic structures in the Poincaré section. At $z = 50a$, there can be found apparent distinction between the resonant tori, KAM tori, and chaotic regions. Mezic and Wiggin [11] verified this analogy in the distributions between the exit time and the Poincaré section by saying that “a set of constant (rescaled) residence time is composed of the orbits of the Poincaré map.” The exit time in the resonance band is nearly constant, which indicates the annulus containing the resonance band rotates almost linearly [7]. The same arguments can be applied to the second case when $b = 0.05$ in Fig. 15B (see also Fig. 11C). As the traveling longitudinal length increases, the corresponding exit time distribution gradually reflects the dynamical structures: outer chaotic zone, nested KAM tori, and small period-2 resonance band.

The exit time distribution for the fully chaotic case (the case III with $b = 0.12$) is presented in Fig. 16, which is caused by the cross-sectional chaotic motion of particles. Random exit time distribution has been achieved even in the early periods.

4 Conclusions

In this study, we discussed how configuration of unmixed and well-mixed zones is determined from the practical design parameters – the helix angle $\alpha$, the fraction of backflow $\kappa$, and the fraction of no-barrier zone $b$ – in the CS system through extensive three-dimensional numerical simulations. The equation of the frequency ratio distribution has been first derived in terms of $(\alpha, \kappa)$. By the numerical simulation, the frequency ratio distributions of four representative cases were evaluated and, by varying the perturbation value $b$, the corresponding Poincaré sections for $(\alpha, \kappa, b)$ have been scrutinized. Numerical results show that, as the perturbation increases, the chaotic region is
extended to cover larger and larger region in the phase space, with successive creation and destruction of the resonance bands and KAM tori being experienced. The period and the size of unmixed islands in the resonance band have been found to be highly dependent on the frequency ratios and the magnitude of the perturbation value. In fact, the periods of resonance bands have been found to be exactly the same as the predicted values from the dominant rational frequency ratios in most cases. In the strongest resonance of period 1, the higher order (subsubharmonic) resonance has been also observed, indicating the characteristics of the fractal structure. In this case with relatively large $\beta$ value, completely chaotic behavior in the whole region has been found. In addition, the exit time distributions were presented and their dependency on the dynamical structures are discussed. In the fully chaotic case, the distribution is found to be random even in the early periods.

References


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