Truss/Cable and Tensegrity structures
1 Truss structures

A truss is a conceptual construction element, which can only transfer a load in one direction, the axis of the truss. It is assumed that the axis is a straight line, with length \( l \). The cross-sectional area \( A \) is assumed to be uniform, so the truss is cylindrical. The axial force \( f \) can be tensile, which leads to elongation of the truss, or compressive, which will result in a reduction of its length. A truss with a compressive load is sometimes referred to as a strut.

The axial force results in an axial stress \( \sigma \), which, for linear elastic material behavior, is related to the linear axial strain \( \varepsilon \) through Young’s modulus \( E \). Poisson’s ratio relates the axial strain to the cross-sectional strain \( \varepsilon_d \). The material behavior is taken to be uniform and isotropic.

\[
\begin{align*}
 f &= \sigma A = E \varepsilon A = E A \frac{\Delta l}{l} = \frac{EA}{l} \Delta l = k \Delta l
\end{align*}
\]

A truss structure is an assemblage of trusses. In mechanical and civil engineering, truss structures are used extensively, because they allow lightweight structures which can transfer high loads.
1.1 Möbius’ rule

In a truss structure the truss (or strut and cable) elements are connected in nodes. In three
dimensional space \( (d = 3) \) nodes can move in the three global coordinate directions which
are referred to as degrees of freedom (dof’s). The structure is also connected to the fixations,
which means that in those nodes a number of dof’s is zero, so they are called kinematic
constraints, the number of which is indicated to be \( nk \).

Designing a truss structure invariably implies the calculation of reaction forces in the
fixation nodes. To this purpose the three force equilibrium equations have to be solved.
Möbius rule indicates whether a unique solution can be determined. It is based on counting
the number of unknowns and results in the number \( D_e \), as is shown in the examples below
for some two-dimensional truss structures.

\[
D_e = (d - 1) \times 3 - nk
\]

\( D_e = 0 \)  \( \rightarrow \)  externally statically determinate
\( D_e < 0 \)  \( \rightarrow \)  externally statically indeterminate
\( D_e > 0 \)  \( \rightarrow \)  rigid body movement

\[
\begin{align*}
\text{Fig. 1.3 : Two-dimensional truss structures with } D_e\text{-count} \\
d = 2 ; nk = 3 & \quad d = 2 ; nk = 3 & \quad d = 2 ; nk = 3 \\
D_e = 3 - 3 = 0 & \quad D_e = 3 - 3 = 0 & \quad D_e = 3 - 3 = 0
\end{align*}
\]

1.2 Maxwell’s rule

Although the reaction forces can be calculated for all three systems, it is obvious that the
third one is a mechanism and cannot carry an external load. Möbius rule does not indicate
this. But there is more.

Besides of the fixation forces, the axial forces in the individual trusses also have to be
calculated and this is where Maxwell’s rule can be used to calculated, again by counting,
the number \( D_i \) from the dimension \( d \), the number of kinematic constraints \( nk \), the number
of nodes \( nn \) and the number of elements \( ne \). Again this number is calculated for the three
examples.

\[
D_i = d \times nn - nk - ne
\]

\( D_i = 0 \)  \( \rightarrow \)  internally statically determinate
\[ D_i < 0 \quad \rightarrow \quad \text{internally statically indeterminate} \]

\[ D_i > 0 \quad \rightarrow \quad \text{kinematically indeterminate} \]

Maxwell’s rule indicates that the third system is a mechanism. It also indicates that the second one is overdetermined. However, there are still some issues to consider, which is clear from the examples below, which all have \( D_i = 0 \). Obviously, however, the second system is a mechanism. The third system is also a mechanism, because it cannot carry a transverse force in the middle node. So \( D_i = 0 \) is not always indicating the unique existence of a solution.

\[ D_i = 12 - 3 - 9 = 0 \]

\[ D_i = 12 - 3 - 9 = 0 \]

\[ D_i = 6 - 4 - 2 = 0 \]

1.3 Calladine’s rule

This is where Calladine’s rule comes into play which says that \( D_i \) is the difference of the number of internal mechanisms \((nm)\) and internal stress states \((ni)\). These numbers cannot be counted, but come from the analysis of the system of equilibrium equations.

\[ D_i = d \times nn - nk - ne = nm - ni \]
1.4 Prestressing

The right part of the second system in the figure above, is a finite mechanism and cannot carry load. The third system is an infinitesimal mechanism. This structure can carry a transverse force in the middle node after applying a prestress in the trusses. The figure below shows the deformation. The trusses are axially loaded with a prestress $\sigma_o = 1$ kPa and subsequently with a vertical force $F_V = -100$ N in node 2.

![Figure 1.7: Prestressed trusses loaded with transversal force](image)

1.5 Buckling

Compressive loads in a truss (or strut) may provoke buckling, which mostly leads to damage of the truss and failure on a structural scale. Buckling occurs when the compressive force exceeds a limit value, the buckling force $P_b$. This can be calculated by taking into account the bending stiffness of the truss, which is a bit out of its conceptual character. The buckling force appears to be a function of the cross-sectional moment of inertia $I$, which is a function of the shape of the cross-section.

$$P_b = \frac{\pi^2 EI}{\ell^2}$$

- rectangular: $I = \frac{1}{12}wh^3$ ; $w =$ width ; $h =$ height
- circular: $I = \frac{1}{64}\pi d^4$ ; $d =$ diameter

2 Truss/cable structures

A cable is a special truss, which cannot carry a compressive load. As a consequence the axial stiffness of a cable is zero under compression. In the model figures of the following sections, cables are drawn in blue and trusses in red.

A truss/cable structure is generally a mechanism if the cables are not prestressed. This prestress has to be large enough to ensure that it will remain after applying an external load.
When the prestress is too low, *slacking* of a cable may occur, which leads to instability of the structure.

In mechanical and civil engineering, truss/cable structures are used extensively, because they allow lightweight structures which can transfer high loads.

![Truss/cable structures](image)

**Fig. 2.8 : Truss/cable structures**

### 2.1 Elementary truss/cable systems (simplexes)

When we want to design a prestressed truss/cable structure which in our case is also rather elongated on a larger scale (mast!) it is recommended to try building it from some sort of elementary truss/cable units or systems. These units, also called "simplexes" have been reported in literature and are shown below. Their names are respectively: prism, anti-prism, reciprocal prism, di-pyramide, and crystal-cell pyramide.

![Simplexes](image)

**Fig. 2.9 : Prism, anti-prism, reciprocal prism, di-pyramide, crystal-cell pyramide**
The FEM program trus3d is used to model these elementary systems, load the cables with a prestress and subsequently load the system with a vertical load. Structures, deformed by prestressing and subsequently by loading, are shown below. Deformation is small but is magnified in the figures.

Fig. 2.10 : Prestressed and loaded truss/cable simplexes; height = 200 mm; prestress = 1000 MPa; load = -1000 $\vec{e}_3$ N; magnification = [38 2.5 123 86]

3 Tensegrity structures

In the elementary truss/cable systems, trusses are mutually connected. There is a class of truss/cable structures in which these connections are prevented. They are called "tensegrities" and defined formally as :

"An assemblage of tension and compression components arranged in a discontinuous compression system" (DCCT systems)

The above definition comes from a paper of Richard Buckminster Fuller, one of the pioneers on the use of these structures. He also devised the name: TENsional intEGRITY.

Another pioneer is the artist Kenneth Snelson, who has designed some fabulous tensegrities. The pictures below show only the most known examples.
The artist Marcelo Pars has also built some large and beautiful tensegrities.

3.1 Rule of Emmerich

Some counting can be done on the basis of Emmerich’s rule, which gives a ratio of the number of struts ($n_s$) and the number of cables ($n_c$) in a tensegrity structure.
ns = \frac{1}{2} \times nn \quad ; \quad nc = \frac{3}{2} \times nn \quad \rightarrow \quad nn = \text{even} \quad \land \quad \frac{ns}{nc} = \frac{1}{3}

ne = ns + nc = 2 \times nn \quad : \quad \text{minimal tensegrity}

ne > 2 \times nn \quad : \quad \text{non-minimal tensegrity}

3.2 Elementary tensegrities (simplexes)

Just as we have seen for the truss/cable systems, some classes of elementary tensegrities are used as a starting point to build larger systems. It can be shown that stable tensegrities can be constructed from some Platonic polyhedrons and all Archimedean polyhedrons. Much used are 'prisms' and some other elementary tensegrities.

3.2.1 Platonic polyhedrons

There are 5 Platonic polyhedrons, which are shown below and which have all kind of geometric properties, that can be studied elsewhere. Here it suffices to mention that only the icosahedron and the dodecahedron can be used to build stable tensegrities.

Fig. 3.13 : Platonic polyhedrons

The icosahedron is, although not minimal according to Emmerich's rule, the most simple one and thus used in structures. It has 6 struts and 24 cables, which have to be prestressed. Below it is shown as it is modelled in Matlab, prestresses and loaded with an external load.
3.2.2 Prisms

The prism shown below is T3-prism, having a triangular shaped bottom and top plane. The prism is in fact an anti-prism, because the top plane is rotated with respect to the bottom plane. This rotation angle is between $90^\circ$ and $180^\circ$. The prism is modelled in Matlab, prestressed and subsequently loaded with an 'axial' load.

![Prestressed and loaded triangular anti-prism](image)

3.2.3 Other tensegrity elementaries

Three two-dimensional tensegrities are shown in the initial configuration in the first row of figures below. The number of internal mechanisms and internal stress states is given.

The second row of figures shows the systems after prestressing the 'diagonal' and 'side' cables. The prestress $\sigma_0$ is given. Horizontal cables are not prestressed. Deformation is scaled up with a magnification factor $\text{magf}$. 

![Initial, prestressed and loaded geometry of icosahedron](image)
The third row of figures shows the deformation after applying a force \((f_6)\) in the upper-right corner node (node 6 in the model). Again the magnification factor is given.

\[
\begin{bmatrix}
85 \\
60 \\
50
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.8 \\
9.8 \\
64
\end{bmatrix}
\]

Fig. 3.16: Two-dimensional tensegrities

Three-dimensional versions of the above two-dimensional elementaries can also be made, but it is no longer possible to find a combination of cable prestresses such that the prestressed state is in equilibrium. Such an equilibrium prestress state has to be determined with an optimization procedure.

Fig. 3.17: Three-dimensional tensegrities