1 Linear plate bending

A plate is a body of which the material is located in a small region around a surface in the three-dimensional space. A special surface is the mid-plane. Measured from a point in the mid-plane, the distance to both surfaces of the plate is equal. The geometry of the plate and position of its points is described in an orthonormal coordinate system, either Cartesian (coordinates \(\{x, y, z\}\)) or cylindrical (coordinates \(\{r, \theta, z\}\)).

1.1 Geometry

The following assumptions about the geometry are supposed to hold here:

- the mid-plane is planar in the undeformed state,
- the mid-plane coincides with the global coordinate plane \(z = 0\),
- the thickness \(h\) is uniform.

Along the edge of the plate a coordinate \(s\) is used and perpendicular to this a coordinate \(n\).
1.2 External loads

The plate is loaded with forces and moments, which can be concentrated in one point or distributed over a surface or along a line. The following loads are defined and shown in the figure:

- force per unit of area in the mid-plane: \( s_x(x, y), s_y(x, y) \) or \( s_r(r, \theta), s_s(r, \theta) \)
- force per unit of area perpendicular to the mid-plane: \( p(x, y) \) or \( p(r, \theta) \)
- force per unit of length in the mid-plane: \( f_n(s), f_s(s) \)
- force per unit of length perpendicular to the mid-plane: \( f_z(s) \)
- bending moment per unit of length along the edge: \( m_b(s) \)
- torsional moment per unit of length along the edge: \( m_w(s) \)

1.3 Cartesian coordinate system

In the Cartesian coordinate system, three orthogonal coordinate axes with coordinates \( \{x, y, z\} \) are used to identify material and spatial points. As stated before, we assume that the mid-plane of the plate coincides with the plane \( z = 0 \) in the undeformed state. This is not a restrictive assumption, but allows for simplification of the mathematics.
1.3.1 Displacements

The figure shows two points $P$ and $Q$ in the undeformed and in the deformed state. In the undeformed state the point $Q$ is in the mid-plane and has coordinates $(x, y, 0)$. The out-of-plane point $P$ has coordinates $(x, y, z)$. As a result of deformation, the displacement of the mid-plane point $Q$ in the $(x, y, z)$-coordinate directions are $u$, $v$, and $w$, respectively. The displacement components of point $P$, indicated as $u_x$, $u_y$ and $u_z$, can be related to those of point $Q$.

$$
\begin{align*}
  u_x &= u - SR = u - QR \sin(\theta_x) \\
  u_y &= v - ST = v - QT \sin(\theta_y) \\
  u_z &= w + SQ - z \\
  QR &= z^* \cos(\theta_x) \\
  QT &= z^* \cos(\theta_y) \\
  SQ &= z^* \cos(\theta)
\end{align*}
$$

\[
\rightarrow \\
\begin{align*}
  u_x &= u - z^* \sin(\theta_x) \cos(\theta_y) \\
  u_y &= v - z^* \cos(\theta_x) \sin(\theta_y) \\
  u_z &= w + z^* \cos(\theta) - z
\end{align*}
\]
No out-of-plane shear

When it is assumed that there is no out-of-plane shear deformation, the so-called Kirchhoff hypotheses hold:

- straight line elements, initially perpendicular to the mid-plane remain straight,
- straight line elements, initially perpendicular to the mid-plane remain perpendicular to the mid-plane.

The angles $\theta_x$ and $\theta_y$ can be replaced by the rotation angles $\phi_x$ and $\phi_y$ respectively. This is illustrated for one coordinate direction in the figure.

\[ \theta_x = \phi_x ; \quad \theta_y = \phi_y ; \quad \theta = \phi \]

Small rotation

With the assumption that rotations are small, the cosine functions approximately have value 1 and the sine functions can be replaced by the rotation angles. These rotations can be expressed in the derivatives of the \( z \)-displacement \( w \) w.r.t. the coordinates \( x \) and \( y \).

\[
\begin{align*}
\cos(\phi) &= \cos(\phi_y) = \cos(\phi_x) = 1 \\
\sin(\phi_y) &= \phi_y = w_y ; \quad \sin(\phi_x) = \phi_x = w_x \\
\end{align*}
\]

\[
\begin{align*}
u_x &= u - z^* w_y \\
u_y &= v - z^* w_y \\
u_z &= w + z^* - z \\
\end{align*}
\]

\[1\text{ Derivatives are denoted as e.g. } \frac{d}{dx}() = ()_x \text{ and } \frac{d^2}{dx^2}() = ()_{xy}\]
Constant thickness

The thickness of the plate may change due to loading perpendicular to the mid-plane or due to contraction due to in-plane deformation. The difference between \( z \) and \( z^* \) is obviously a function of \( z \). Terms of order higher than \( z^2 \) are neglected in this expression. It is now assumed that the thickness remains constant, which means that \( \zeta(x,y) = 0 \) has to hold.

With the assumption \( z^* \approx z \), which is correct for small deformations and thin plates, the displacement components of the out-of-plane point \( P \) can be expressed in mid-plane displacements.

\[
\Delta z(x,y) = z^*(x,y) - z(x,y) = \zeta(x,y)z + \eta(x,y)z^2 + O(z^3)
\]
\[
\Delta h = \zeta(x,y)\left(\frac{h}{2}\right) - \zeta(x,y)(-\frac{h}{2})
\]
\[
\Delta h = 0 \quad \rightarrow \quad \zeta(x,y) = 0
\]
\[
u_x(x,y,z) = u(x,y) - zw_x
\]
\[
u_y(x,y,z) = v(x,y) - zw_y
\]
\[
u_z(x,y,z) = w(x,y) + \eta(x,y)z^2
\]

1.3.2 Curvatures and strains

The linear strain components in a point out of the mid-plane, can be expressed in the mid-plane strains and curvatures. A sign convention, as is shown in the figure, must be adopted and used consistently. It was assumed that straight line elements, initially perpendicular to the mid-plane, remain perpendicular to the mid-plane, so \( \gamma_{xz} \) and \( \gamma_{yz} \) should be zero. Whether this is true will be evaluated later.

\[
\begin{align*}
\varepsilon_{xx} &= u_{x,x} = u_x - zw_{x,x} = \epsilon_{xx0} - z\kappa_{xx} \\
\varepsilon_{yy} &= u_{y,y} = v_{y,y} - zw_{y,y} = \epsilon_{yy0} - z\kappa_{yy} \\
\gamma_{xy} &= u_{x,y} + u_{y,x} = u_y + v_x - 2zw_{x,y} = \gamma_{xy0} - z\kappa_{xy} \\
\gamma_{xz} &= u_{x,z} &= 2\eta(x,y)z \\
\gamma_{yz} &= u_{y,z} + u_{z,y} &= \eta_{x}z^2 \\
\gamma_{zz} &= u_{z,z} &= \eta_{y}z^2
\end{align*}
\]
1.3.3 Stresses

It is assumed that a plane stress state exists in the plate: \( \sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0 \). Because the plate may be loaded with a distributed load perpendicular to its surface, the first assumption is only approximately true. The second and third assumptions are in consistence with the Kirchhoff hypothesis, but has to be relaxed a little, due to the fact that the plate can be loaded with perpendicular edge loads. However, it will always be true that the stresses \( \sigma_{zz}, \sigma_{zx}, \) and \( \sigma_{zy} \) are much smaller than the in-plane stresses \( \sigma_{xx}, \sigma_{yy}, \) and \( \sigma_{xy} \).

1.3.4 Isotropic elastic material behavior

For linear elastic material behavior Hooke’s law relates strains to stresses. Material stiffness (\( C \)) and compliance (\( S \)) matrices can be derived for the plane stress state.

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} = \frac{1}{E} \begin{bmatrix}
1 & -\nu & 0 \\
-\nu & 1 & 0 \\
0 & 0 & 2(1+\nu)
\end{bmatrix} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} ; \quad \varepsilon_{zz} = -\frac{\nu}{E} (\sigma_{xx} + \sigma_{yy})
\]

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1}{2}(1-\nu)
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{xy}
\end{bmatrix}
\]

short notation : \( \varepsilon = S\sigma \rightarrow \sigma = S^{-1}\varepsilon = C\varepsilon = C(\varepsilon_0 - z\kappa) \)

Inconsistency with plane stress

The assumption of a plane stress state implies the shear stresses \( \sigma_{yz} \) and \( \sigma_{zx} \) to be zero. From Hooke’s law it then immediately follows that the shear components \( \gamma_{yz} \) and \( \gamma_{zx} \) are also zero. However, the
strain-displacement relations result in non-zero shear. For thin plates, the assumption will be more correct than for thick plates.

\[ \varepsilon_{zz} = 2\eta z \rightarrow \eta(x,y) = \frac{\varepsilon_{zz}}{2z} = \frac{1-v}{E} (\sigma_{xx} + \sigma_{yy}) = \frac{v}{2(1-v)} (w_{,xx} + w_{,yy}) \]

\[ \gamma_{yz} = \frac{vz^2}{2(1-v)} (w_{,xxy} + w_{,yy}) \neq 0 \]

\[ \gamma_{zx} = \frac{vz^2}{2(1-v)} (w_{,xxx} + w_{,xyy}) \neq 0 \]

### 1.3.5 Cross-sectional forces and moments

Cross-sectional forces and moments can be calculated by integration of the in-plane stress components over the plate thickness. A sign convention, as indicated in the figure, must be adopted and used consistently.

The stress components \(\sigma_{xx}\) and \(\sigma_{yy}\) are much smaller than the relevant in-plane components. Integration over the plate thickness will however lead to shear forces, which cannot be neglected.

\[ N = \begin{bmatrix} N_{xx} \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \sigma d\zeta = \int_{-h/2}^{h/2} \{C(\varepsilon_0 - z\kappa)\} d\zeta = hC\varepsilon_0 \]

\[ M = \begin{bmatrix} M_{xx} \\ M_{xy} \end{bmatrix} = -\int_{-h/2}^{h/2} \zeta\sigma d\zeta = -\int_{-h/2}^{h/2} \{C(\varepsilon_0 - z\kappa)\} \zeta d\zeta = \frac{1}{12} h^3 C\kappa \]

\[ D_x = \int_{-h/2}^{h/2} \sigma_{xx} d\zeta \quad ; \quad D_y = \int_{-h/2}^{h/2} \sigma_{yy} d\zeta \]

### Stiffness- and compliance matrix

Integration leads to the stiffness and compliance matrix of the plate.
\[
\begin{bmatrix}
    N \\
    M
\end{bmatrix}
= 
\begin{bmatrix}
    Ch & 0 \\
    0 & Ch^2/12
\end{bmatrix}
\begin{bmatrix}
    \xi_0 \\
    \kappa
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
    \xi_0 \\
    \kappa
\end{bmatrix}
= 
\begin{bmatrix}
    S/h & 0 \\
    0 & 12S/h^3
\end{bmatrix}
\begin{bmatrix}
    N \\
    M
\end{bmatrix}
\]

1.3.6 Equilibrium equations

Consider a small “column” cut out of the plate perpendicular to its plane. Equilibrium requirements of the cross-sectional forces and moments, lead to the equilibrium equations. The cross-sectional shear forces \( D_x \) and \( D_y \) can be eliminated by combining some of the equilibrium equations.

\[
\begin{align*}
N_{xx,x} + N_{xy,y} + s_x &= 0 \\
N_{yy,y} + N_{yx,x} + s_y &= 0 \\
D_{x,x} + D_{y,y} + p &= 0 \\
M_{xx,x} + M_{yy,y} - D_x &= 0 \\
M_{xy,y} + M_{xx,x} - D_y &= 0
\end{align*}
\rightarrow 
M_{xx,x} + M_{yy,y} + 2M_{xy,y} + p &= 0
\]

The equilibrium equation for the cross-sectional bending moments can be transformed into a differential equation for the displacement \( w \). The equation is a fourth-order partial differential equation, which is referred to as a bi-potential equation.

\[
\begin{bmatrix}
    M_{xx} \\
    M_{xy} \\
    M_{yx} \\
    M_{yy}
\end{bmatrix}
= 
\frac{Eh^3}{12(1-v^2)}
\begin{bmatrix}
    1 & v & 0 \\
    v & 1 & 0 \\
    0 & 0 & \frac{1}{2}(1-v) \\
    2w_{,xx} & 2w_{,xy}
\end{bmatrix}
\begin{bmatrix}
    w_{,xx} \\
    w_{,xy}
\end{bmatrix}
\rightarrow 
M_{xx,x} + M_{yy,y} + 2M_{xy,y} + p &= 0
\]

\[
\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{p}{B}
\]

\[
B = \frac{Eh^3}{12(1-v^2)} = \text{plate modulus}
\]

1.3.7 Orthotropic elastic material behavior

A material which properties are in each point symmetric w.r.t. three orthogonal planes, is called orthotropic. The three perpendicular directions in the planes are denoted as the 1-, 2- and 3-coordinate axes and are referred to as the material coordinate directions. Linear elastic behavior of an orthotropic material is characterized by 9 independent material parameters. They constitute the components of the compliance (\( \mathbf{S} \)) and stiffness (\( \mathbf{C} \)) matrix.
In an orthotropic plate, the third material coordinate axis \((3)\) is assumed to be perpendicular to the plane of the plate. The 1- and 2-directions are in its plane and rotated over an angle \(\alpha\) (counterclockwise) w.r.t. the global \(x\)- and \(y\)-coordinate axes.

For the plane stress state the linear elastic orthotropic material behavior is characterized by 4 independent material parameters: \(E_1, E_2, G_{12}\) and \(v_{12}\). Due to symmetry we have \(v_{21} = \frac{E_{23}}{E_1} v_{12}\).

\[
\epsilon = S \sigma \quad \rightarrow \quad \sigma = S^{-1} \epsilon = C \epsilon
\]

Orthotropic plate

\[
C = \frac{1}{\Delta} \begin{bmatrix} \frac{1-v_{12}v_{21}}{E_2 E_3} & \frac{v_{21}v_{31}+v_{32}}{E_3 E_2} & \frac{v_{21}v_{32}+v_{31}}{E_2 E_3} & 0 & 0 & 0 \\ \frac{v_{12}v_{23}+v_{21}}{E_2 E_3} & \frac{1-v_{12}v_{31}}{E_3 E_2} & \frac{v_{12}v_{32}+v_{31}}{E_2 E_3} & 0 & 0 & 0 \\ \frac{v_{13}v_{21}+v_{23}}{E_2 E_3} & \frac{1-v_{13}v_{31}}{E_3 E_2} & \frac{v_{13}v_{32}+v_{31}}{E_2 E_3} & 0 & 0 & 0 \end{bmatrix}
\]

with \(\Delta = \frac{(1 - v_{12}v_{21} - v_{23}v_{31} - v_{31}v_{13} - v_{12}v_{23}v_{31} - v_{21}v_{32}v_{13})}{E_1 E_2 E_3}\)
Using a transformation matrices $T_ε$ and $T_σ$, for strain and stress components, respectively, the strain and stress components in the material coordinate system (index $^*$) can be related to those in the global coordinate system.

$$c = \cos(α); \quad s = \sin(α)$$

$$\begin{bmatrix}
ε_{xx} \\
ε_{yy} \\
γ_{xy}
\end{bmatrix} =
\begin{bmatrix}
c^2 & s^2 & -cs \\
s^2 & c^2 & cs \\
2cs & -2cs & c^2 - s^2
\end{bmatrix}
\begin{bmatrix}
ε_{11} \\
ε_{22} \\
γ_{12}
\end{bmatrix}
→
ε = T_ε^{-1}ε^*

$$\begin{bmatrix}
σ_{xx} \\
σ_{yy} \\
σ_{xy}
\end{bmatrix} =
\begin{bmatrix}
cc^2 & s^2 & -2cs \\
sc^2 & c^2 & 2cs \\
s^2 & -cs & c^2 - s^2
\end{bmatrix}
\begin{bmatrix}
σ_{11} \\
σ_{22} \\
σ_{12}
\end{bmatrix}
→
σ = T_σ^{-1}σ^*

$$

1.4 Cylindrical coordinate system

In the cylindrical coordinate system, three orthogonal coordinate axes with coordinates \( \{r, θ, z\} \) are used to identify material and spatial points. It is assumed that the mid-plane of the plate coincides with the plan \( z = 0 \) in the undeformed state.

We only consider axisymmetric geometries, loads and deformations. For such axisymmetric problems there is no dependency of the cylindrical coordinate θ. Here we take as an extra assumption that there is no tangential displacement.
1.4.1 Displacements

All assumptions about the deformation, described in the former section, are also applied here:

- no out-of-plane shear (Kirchhoff hypotheses),
- small rotation,
- constant thickness.

The displacement components in radial \( (u_r) \) and axial \( (u_z) \) direction of an out-of-plane point \( P \) can then be expressed in the displacement components \( u \) and \( w \) of the corresponding – same \( x \) and \( y \) coordinates – point \( Q \) in the mid-plane.

\[
\begin{align*}
    u_r(r, \theta, z) &= u(r, \theta) - zw_r \\
    u_z(r, \theta, z) &= w(r, \theta) + \eta(r, \theta)z^2
\end{align*}
\]

1.4.2 Curvatures and strains

The linear strain components in a point out of the mid-plane, can be expressed in the mid-plane strains and curvatures. A sign convention, as indicated in the figure, must be adopted and used consistently. It was assumed that straight line elements, initially perpendicular to the mid-plane, remain perpendicular to the mid-plane, so \( \gamma_z = 0 \).

\[
\begin{align*}
    \varepsilon_{rr} = u_{r,r} &= u_r - zw_{r,r} = \varepsilon_{r0} - z\kappa_r \\
    \varepsilon_{tt} = \frac{1}{r}u_r &= \frac{1}{r}u - \frac{z}{r}w_{r,r} = \varepsilon_{t0} - z\kappa_t \\
    \gamma_{rt} = u_{r,t} + u_{t,r} &= 0 \\
    \varepsilon_{zz} = u_{zz} &= 2\eta(r,z)z \\
    \varepsilon_{rz} = u_{r,z} + u_{z,r} &= \eta_{rz}z^2 \\
    \gamma_{z} = u_{z,z} + u_{z,z} &= 0
\end{align*}
\]

\[\varepsilon = \varepsilon_0 - z\kappa\]
1.4.3 Stresses

A plane stress state is assumed to exist in the plate. The relevant stresses are $\sigma_{rr}$ and $\sigma_{tt}$. Other stress components are much smaller or zero.

1.4.4 Isotropic elastic material behavior

For linear elastic material behavior Hooke’s law relates strains to stresses. Material stiffness ($C$) and compliance ($\mathcal{S}$) matrices can be derived for the plane stress state.
\[
\begin{bmatrix}
\varepsilon_{rr} \\
\varepsilon_{tt}
\end{bmatrix}
= \frac{1}{E} \begin{bmatrix}
1 & -\nu \\
-\nu & 1
\end{bmatrix}
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{tt}
\end{bmatrix}
; \quad \varepsilon_{zz} = -\frac{\nu}{E} (\sigma_{rr} + \sigma_{tt})
\]
\[
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{tt}
\end{bmatrix}
= \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu \\
\nu & 1
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{rr} \\
\varepsilon_{tt}
\end{bmatrix}
\]

Short notation: \(\varepsilon = S\sigma\rightarrow \sigma = S^{-1}\varepsilon = C\varepsilon = C(\varepsilon_0 - z\kappa)\)

### 1.4.5 Cross-sectional forces and moments

Cross-sectional forces and moments can be calculated by integration of the in-plane stress components over the plate thickness. A sign convention, as indicated in the figure, must be adopted and used consistently.

The stress component \(\sigma_{zr}\) is much smaller than the relevant in-plane components. Integration over the plate thickness will however lead to a shear force, which cannot be neglected.

\[
N = \begin{bmatrix}
N_r \\
N_t
\end{bmatrix} = \frac{h}{2} \begin{bmatrix}
\sigma_{rr} \\
\sigma_{tt}
\end{bmatrix} dz = \frac{h}{2} \int_{-h/2}^{h/2} C(\varepsilon_0 - z\kappa) dz = hC\varepsilon_0
\]

\[
M = \begin{bmatrix}
M_r \\
M_t
\end{bmatrix} = -\frac{h}{2} \begin{bmatrix}
\sigma_{rr} \\
\sigma_{tt}
\end{bmatrix} z dz = -\frac{h}{2} \int_{-h/2}^{h/2} C(\varepsilon_0 - z\kappa) z dz = \frac{1}{12} h^3 C\kappa
\]

\[
D = \frac{h}{2} \sigma_{zr} dz
\]

Stiffness- and compliance matrix

Integration leads to the stiffness and compliance matrix of the plate.

\[
\begin{bmatrix}
N \\
M
\end{bmatrix} = \begin{bmatrix}
C_1 h & 0 \\
0 & C_1 h^3 / 12
\end{bmatrix}
\begin{bmatrix}
\varepsilon_0 \\
\kappa
\end{bmatrix} \rightarrow
\begin{bmatrix}
\varepsilon_0 \\
\kappa
\end{bmatrix} = \begin{bmatrix}
S/h & 0 \\
0 & 12S/h^3
\end{bmatrix}
\begin{bmatrix}
N \\
M
\end{bmatrix}
\]
1.4.6 Equilibrium equations

Consider a small “column” cut out of the plate perpendicular to its plane. Equilibrium requirements of the cross-sectional forces and moments, lead to the equilibrium equations. The cross-sectional shear force $D$ can be eliminated by combining some of the equilibrium equations.

$$
N_{r} + \frac{1}{r} (N_{r} - N_{t}) + s_{r} = 0
$$
$$
\begin{align*}
D_{r} + D + rp &= 0 \\
rM_{r} + M_{r} - M_{t} - rD &= 0
\end{align*}
$$

The equilibrium equation for the cross-sectional bending moments can be transformed into a differential equation for the displacement $w$. The equation is a fourth-order partial differential equation, which is referred to as a bi-potential equation.

$$
\begin{align*}
\left[ \begin{array}{c}
M_{r} \\
M_{t}
\end{array} \right] &= \frac{Eh^{3}}{12(1 - \nu^{2})} \left[ \begin{array}{cc}
1 & \nu \\
\nu & 1
\end{array} \right] \left[ \begin{array}{c}
-w_{rr} \\
-\frac{1}{r}w_{r}
\end{array} \right] \\
rM_{r,rr} + 2M_{r} - M_{t} + rp &= 0
\end{align*}
$$

$$
\begin{align*}
\frac{d^{4}w}{dr^{4}} + \frac{2 d^{3}w}{r dr^{3}} - \frac{1 d^{2}w}{r^{2} dr^{2}} + \frac{1 dw}{r^{3} dr} &= \frac{p}{B} \\
\left( \frac{d^{2}w}{dr^{2}} + \frac{1}{r} \frac{dw}{dr} \right) \left( \frac{d^{2}w}{dr^{2}} + \frac{1}{r} \frac{dw}{dr} \right) &= \frac{p}{B}
\end{align*}
$$

$$
B = \frac{Eh^{3}}{12(1 - \nu^{2})} = \text{plate modulus}
$$
1.4.7 Solution

The differential equation has a general solution comprising four integration constants. Depending on the external load, a particular solution \( w_p \) also remains to be specified. The integration constants have to be determined from the available boundary conditions, i.e. from the prescribed displacements and loads.

The cross-sectional forces and moments can be expressed in the displacement \( w \) and/or the strain components.

\[
w = a_1 + a_2 r^2 + a_3 \ln(r) + a_4 r^2 \ln(r) + w_p
\]

\[
M_r = -B \left( \frac{d^2 w}{dr^2} + \frac{v}{r} \frac{dw}{dr} \right)
\]

\[
M_t = -B \left( \frac{1}{r} \frac{dw}{dr} + \frac{v}{r^2} \frac{d^2 w}{dr^2} \right)
\]

\[
D = \frac{dM_r}{dr} + \frac{1}{r} (M_r - M_t) = -B \left( \frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right)
\]

1.4.8 Examples

This section contains some examples of axisymmetric plate bending problems.

- Plate with central hole
- Solid plate with distributed load
- Solid plate with vertical point load

Plate with central hole

A circular plate with radius \( 2R \) has a central hole with radius \( R \). The plate is clamped at its outer edge and is loaded by a distributed load \( q \) perpendicular to its plane in negative \( z \)-direction. The equilibrium equation can be solved with proper boundary conditions, which are listed below.
boundary conditions

- \( w(r = 2R) = 0 \)
- \( \left( \frac{dw}{dr} \right)(r = 2R) = 0 \)
- \( m_b(r = R) = M_r(r = R) = 0 \)
- \( f_z(r = R) = D(r = R) = 0 \)

differential equation

\[
\frac{d^4w}{dr^4} + \frac{2}{r} \frac{d^3w}{dr^3} - \frac{1}{r^2} \frac{d^2w}{dr^2} + \frac{1}{r^3} \frac{dw}{dr} = -\frac{q}{B}
\]

general solution

\[ w = a_1 + a_2 r^2 + a_3 \ln(r) + a_4 r^2 \ln(r) + w_p \]

particulate solution

\[ w_p = \alpha r^4 \rightarrow \text{substitution} \rightarrow 24\alpha + 48\alpha - 12\alpha + 4\alpha = -\frac{q}{B} \rightarrow \alpha = -\frac{q}{64B} \]

solution

\[ w = a_1 + a_2 r^2 + a_3 \ln(r) + a_4 r^2 \ln(r) - \frac{q}{64B} r^4 \]

The 4 constants in the general solution can be solved using the boundary conditions. For this purpose these have to be elaborated to formulate the proper equations.

\[
\frac{dw}{dr} = 2a_2 + \frac{a_3}{r} + a_4(2r \ln(r) + r) - \frac{q}{16B} r^3
\]

\[
M_r = -B \left( \frac{d^2w}{dr^2} + \frac{v}{r} \frac{dw}{dr} \right)
\]

\[
= -B \left\{ 2(1 + v)a_2 - (1 - v) \frac{1}{r^2} a_3 + 2(1 + v) \ln(r)a_4 + (3 + v)a_4 - (3 + v) \frac{q}{16B} r^2 \right\}
\]

\[
M_t = -B \left( \frac{d^2w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right)
\]

\[
= -B \left\{ 2(1 + v)a_2 + (1 - v) \frac{1}{r^2} a_3 + 2(1 + v) \ln(r)a_4 + (3 + v + 1)a_4 - (3v + 1) \frac{q}{16B} r^2 \right\}
\]

\[
D = \frac{dM_r}{dr} + \frac{1}{r} (M_t - M_r)
\]

\[
= B \left( -\frac{4}{r} a_4 + \frac{q}{2B} r \right)
\]

The equations are summarized below. The 4 constants can be solved.

After solving the constants, the vertical displacement at the inner edge of the hole can be calculated.
Solid plate with distributed load

A solid circular plate has radius $R$ and uniform thickness. It is clamped at its outer edge and loaded with a distributed force per unit of area $q$ in negative $z$-direction. The equilibrium equation can be solved with proper boundary conditions, which are listed below.

Besides the obvious boundary conditions, there are also two symmetry conditions, one for the deformation and the other for the load.

boundary conditions

- $w(r = R) = 0$
- $\left(\frac{dw}{dr}\right) (r = R) = 0$
- $\left(\frac{d^2w}{dr^2}\right) (r = 0) = 0$
- $D(r = 0) = 0$

differential equation

$$\frac{d^4w}{dr^4} + \frac{2}{r} \frac{d^3w}{dr^3} - \frac{1}{r^2} \frac{d^2w}{dr^2} + \frac{1}{r^3} \frac{dw}{dr} = -\frac{q}{B}$$

general solution

$$w = a_1 + a_2 r^2 + a_3 \ln(r) + a_4 r^2 \ln(r) + w_p$$

particulate solution

$$w_p = \alpha r^4 \quad \rightarrow \quad \text{substitution} \quad \rightarrow$$
\[
24\alpha + 48\alpha - 12\alpha + 4\alpha = -\frac{p}{B} \quad \Rightarrow \quad \alpha = -\frac{q}{64B}
\]

solution
\[
w = a_1 + a_2 r^2 + a_3 \ln(r) + a_4 r^2 \ln(r) - \frac{q}{64B} r^4
\]

The 4 constants in the general solution can be solved using the boundary and symmetry conditions. For this purpose these have to be elaborated to formulate the proper equations.

The two symmetry conditions result in two constants to be zero. The other two constants can be determined from the boundary conditions.

After solving the constants, the vertical displacement at \( r = 0 \) can be calculated.

\[
\frac{dw}{dr} = 2a_2 r + \frac{a_3}{r} + a_4 (2r \ln(r) + r) - \frac{q}{16B} r^3
\]

\[
D = \frac{dM_r}{dr} + \frac{1}{r} (M_r - M_t)
\]

\[
= B \left( \frac{4}{r} a_4 + \frac{q}{2B} r \right)
\]

boundary conditions

\[
\left( \frac{dw}{dr} \right) (r = 0) = 0 \quad \Rightarrow \quad a_3 = 0
\]

\[
D(r = 0) = 0 \quad \Rightarrow \quad a_4 = 0
\]

\[
\left( \frac{dw}{dr} \right) (r = R) = 2a_2 R - \frac{q}{16B} R^3 = 0 \quad \Rightarrow \quad a_2 = \frac{q}{32B} R^2
\]

\[
w(r = R) = a_1 + a_2 R^2 - \frac{q}{64B} R^4 = 0 \quad \Rightarrow \quad a_1 = -\frac{q}{64B} R^4
\]

displacement in the center

\[
w(r = 0) = -\frac{q}{64B} R^4
\]

**Solid plate with vertical point load**

A solid circular plate has radius \( R \) and uniform thickness \( d \). The plate is clamped at its edge. It is loaded in its center \((r = 0)\) by a point load \( F \) perpendicular to its plane in the positive \( z \)-direction. The equilibrium equation can be solved using appropriate boundary conditions, listed below.

Besides the obvious boundary conditions, there is also the symmetry condition that in the center of the plate the bending angle must be zero. In the same point the total cross-sectional force \( K = 2\pi D \) must equilibrate the external force \( F \).
boundary conditions

- \( w(r = R) = 0 \)
- \( \left( \frac{dw}{dr} \right)(r = R) = 0 \)
- \( \left( \frac{dw}{dr} \right)(r = 0) = 0 \)
- \( K(r = 0) = -F \)

differential equation

\[
\frac{d^4w}{dr^4} + \frac{2}{r} \frac{d^3w}{dr^3} - \frac{1}{r^2} \frac{d^2w}{dr^2} + \frac{1}{r^3} \frac{dw}{dr} = 0
\]

general solution

\[
w = a_1 + a_2 r^2 + a_3 \ln(r) + a_4 r^2 \ln(r)
\]

The derivative of \( w \) and the cross-sectional load \( D \) can be written as a function of \( r \).

The symmetry condition results in one integration constant to be zero. Because the derivative of \( w \) is not defined for \( r = 0 \), a limit has to be taken. The cross-sectional equilibrium results in a known value for another constant. The remaining constants can be determined from the boundary conditions at \( r = R \).

After solving the constants, the vertical displacement at \( r = 0 \) can be calculated.

\[
\left( \frac{dw}{dr} \right)(r = R) = 2R a_2 + \frac{F}{8\pi B} (2\ln(R) + R) = 0 \quad \rightarrow \quad a_2 = -\frac{F}{16\pi B} (2\ln R + 1)
\]

displacement at the center

\[
w(r = 0) = \lim_{r \to 0} \frac{FR^2}{16\pi B} \left\{ 1 - \left( \frac{r}{R} \right)^2 + 2 \left( \frac{r}{R} \right)^2 \ln \left( \frac{r}{R} \right) \right\} = \frac{FR^2}{16\pi B}
\]
The stresses can also be calculated. They are a function of both coordinates $r$ and $z$. We can conclude that at the center, where the force is applied, the stresses become infinite, due to the singularity, provoked by the point load.

\[
\sigma_{rr} = -\frac{3Fz}{\pi d^3} \left\{ 1 + (1 + \nu) \ln \left( \frac{r}{R} \right) \right\}
\]

\[
\sigma_{tt} = -\frac{3Fz}{\pi d^3} \left\{ \nu + (1 + \nu) \ln \left( \frac{r}{R} \right) \right\}
\]
2 Laminates

Laminates are plates, which are made by stacking a number of layers, also called plies or lamina’s. Each ply can be fabricated from various materials, having different mechanical properties within a wide range. The properties may be isotropic, completely anisotropic, or orthotropic. The latter occurs when a ply is a composite material, consisting of a matrix which is enforced by long fibers in one direction. The material directions are denoted as 1 (fiber, longitudinal) and 2 (transversal).

To increase the bending stiffness of the laminate, the thickness can be enlarged, by applying a layer of filler material. e.g. a foam, which is considered to be isotropic. The figure shows an exploded view of a laminate with fiber reinforced plies. Also a foam laminate is shown.

The mechanical properties of the laminate are determined by its stiffness matrix, which can be calculated from the properties of the plies, their thicknesses and their stacking sequence. The behavior of the laminate is based on linear plate bending theory.

2.1 Lamina strain

For one ply \((k)\) the strain components in the global coordinate system can be related to the strain in the mid-plane and the curvature of the mid-plane. The strain components in the material coordinate system – indicated with superscript \(\ast\) – are calculated with the transformation matrix \(T_\varepsilon\).

\[
\varepsilon_k(z) = \varepsilon_0 - z\kappa \quad \rightarrow \quad \varepsilon_k^*(z) = T_\varepsilon \varepsilon_k(z)
\]

with

\[
T_\varepsilon = \begin{bmatrix}
    c^2 & s^2 & cs \\
    s^2 & c^2 & -cs \\
   -2cs & 2cs & c^2 - s^2
\end{bmatrix}
\]
2.2 Lamina stress

Assuming linearly elastic material behavior according to Hooke’s law, the stress components in the material directions can be determined, using the material stiffness matrix $C$. Stress components in the global coordinate system are calculated with the transformation matrix $T^{-1}$.

$$\sigma_k^i = C^i_k \varepsilon_k^i \quad \Rightarrow \quad \sigma_k = T^{-1}_\sigma C^i_k T_{\sigma_k} \varepsilon_k = C_k \varepsilon_k = C_k (\varepsilon_0 - z \kappa)$$

with

$$T_\sigma = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} ; \quad T^{-1}_\sigma = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix}$$

2.3 ABD-matrix

The cross-sectional forces and moments can be determined by summation of the integrated stress components over each individual ply. The result is the so-called ABD-matrix, which relates cross-sectional forces and moments to mid-plane strains and curvatures.

$$N_k = \int_{z_{k-1}}^{z_k} \sigma_k dz = (z_k - z_{k-1}) C_k \varepsilon_0 - \frac{1}{2} (z_k^2 - z_{k-1}^2) C_k \kappa = A_k \varepsilon_0 + B_k \kappa$$

$$M_k = -\int_{z_{k-1}}^{z_k} \sigma_k z dz = -\frac{1}{2} (z_k^2 - z_{k-1}^2) C_k \varepsilon_0 + \frac{1}{3} (z_k^3 - z_{k-1}^3) C_k \kappa = B_k \varepsilon_0 + D_k \kappa$$

summation over all plies
The ABD-matrix characterizes the mechanical behavior of the laminate, as it relates the cross-sectional loads to strains and curvatures of the mid-plane. The figure shows the influence of some components of the ABD-matrix.

\[ N = \sum_{k=1}^{n} N_k = A\varepsilon_0 + B\kappa \quad ; \quad M = \sum_{k=1}^{n} M_k = B\varepsilon_0 + D\kappa \rightarrow \]

\[
\begin{bmatrix}
N \\
M
\end{bmatrix} = \begin{bmatrix}
A & B \\
B & D
\end{bmatrix} \begin{bmatrix}
\varepsilon_0 \\
\kappa
\end{bmatrix} \rightarrow \begin{bmatrix}
\varepsilon_0 \\
\kappa
\end{bmatrix} = \begin{bmatrix}
a & b \\
b & d
\end{bmatrix} \begin{bmatrix}
N \\
M
\end{bmatrix}
\]

2.4 Thermal and humidity loading

When the laminate is subjected to a change in temperature, thermal strains will occur due to thermal expansion. In each ply these strains will generally be different, leading to deformation. Analogously, the laminate can be placed in a humid environment, which also causes strains caused by absorption.

The thermal and humidity strains can be calculated in each ply, given its coefficients of thermal expansion (\(\alpha_1\) and \(\alpha_2\)) and humid driven expansion (\(\beta_1\) and \(\beta_2\)). To calculate stresses these strains \(\hat{\varepsilon}_k\) must be subtracted from the mechanical strains \(\varepsilon_k\).
The cross-sectional forces and moments are related to the mid-plane strains and curvatures by the ABD-matrix. The forces caused by thermal and humidity expansion are simply added.

Strains in ply $k$ due to temperature change $\Delta T_k$ and absorption $c_k$:

\[
\begin{bmatrix}
\hat{\varepsilon}_{11} \\
\hat{\varepsilon}_{22} \\
\hat{\gamma}_{12}
\end{bmatrix} =
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
0
\end{bmatrix} \Delta T_k +
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
0
\end{bmatrix} c_k \rightarrow \hat{\varepsilon}_k^* = \alpha_k \Delta T_k + \beta_k c_k
\]

Stresses due to thermal (absorption) and mechanical strains $\varepsilon_k^*$:

\[
\sigma_k^* = C_k (\varepsilon_k^* - \hat{\varepsilon}_k^*) \rightarrow \sigma_k = C_k (\varepsilon_k - \hat{\varepsilon}_k) = C_k (\varepsilon_0 - z \kappa - \hat{\varepsilon}_k)
\]

Forces and moments in ply $k$:

\[
N_k = \int_{z_{k-1}}^{z_k} \sigma_k dz = A_k \varepsilon_0 + B_k \kappa - A_k \hat{\varepsilon}_k \rightarrow N = \sum_{k=1}^{n} N_k
\]

\[
M_k = -\int_{z_{k-1}}^{z_k} \sigma_k z dz = B_k \varepsilon_0 + D_k \kappa - B_k \hat{\varepsilon}_k \rightarrow M = \sum_{k=1}^{n} M_k
\]

\[
\begin{bmatrix}
N \\
M
\end{bmatrix} =
\begin{bmatrix}
A & B \\
B & D
\end{bmatrix}
\begin{bmatrix}
\varepsilon_0 \\
\kappa
\end{bmatrix} -
\begin{bmatrix}
\hat{N} \\
\hat{M}
\end{bmatrix} \rightarrow
\]

\[
\begin{bmatrix}
\varepsilon_0 \\
\kappa
\end{bmatrix} =
\begin{bmatrix}
a & b \\
b & d
\end{bmatrix}
\begin{bmatrix}
N \\
M
\end{bmatrix} +
\begin{bmatrix}
\hat{N} \\
\hat{M}
\end{bmatrix}
\]

### 2.5 Stacking

The fibers in the plies can be oriented randomly, however, mostly a certain orientation relative to the orientation in other plies is chosen, resulting in cross-ply, angle-ply and regular angle-ply laminates. Fiber orientation in plies above and under the mid-plane can be symmetric and anti-symmetric. The resulting mechanical behavior is related to certain components in the ABD-matrix.

The notation of the ply-orientations is illustrated with the next examples:

- $[0/0/90/90/90/45/..]$: total stacking and orientation sequence is given
- $[0_3/90_2/45/..]$: total stacking and orientation sequence given; index gives number of plies
- $[0_3/90_2/45/-45_3]_s$: symmetric laminate
cross-ply  
- orthotropic plies
- material directions = global directions.
- \( A_{13} = A_{23} = 0 \)

angle-ply  
- orthotropic plies
- each ply material direction 1 is rotated over \( \alpha^o \) w.r.t. global direction \( x \).

regular angle-ply  
- orthotropic plies
- subsequent plies have material direction 1 rotated alternatively over \( \alpha^o \) and \(-\alpha^o\) w.r.t. the global \( x \)-axis.
- even number of plies \( \rightarrow A_{13} = A_{23} = 0 \)

symmetric  
- symmetric stacking w.r.t. mid-plane
- \( B = 0 \)

anti-symm.  
- anti-symmetric stacking w.r.t. mid-plane
- \( D_{13} = D_{23} = 0 \)

quasi-isotropic  
- \( \alpha_k = k \frac{\pi}{n} \) with \( k = 1, \ldots, n \) (\( n \) = number of plies)

2.5.1 Recommendations for laminate stacking

There are no general rules for the number of plies in a laminate, their (mutual) fiber orientation and their stacking sequence. For most applications, however, there are some recommendations.

- Choose a symmetric laminate.
  Because \( B = 0 \), there is no coupling between forces and curvatures and between strains and moments.

- Minimize material direction differences in subsequent plies.
  Large differences lead to high inter-laminar shear stresses, which may cause delamination.

- Minimize stiffness differences of subsequent plies.
  Large difference lead to high inter-laminar shear stresses.

- Avoid neighboring plies with the same orientation.
  The same orientation results in high inter-laminar shear stresses under thermal loading.

- Ensure negative inter-laminar normal stresses (pressure).
  Negative stresses (tensile) may lead to delamination.

2.6 Damage

Several damage phenomena can occur in a laminate:

- fibre rupture
- fibre buckling
- matrix cracking
• fibre-matrix de-adhesion
• interlaminar delamination

The occurrence of these damage phenomena can be monitored with a damage criterion.

2.6.1 Damage : Tsai-Hill

For anisotropic materials the yield criterion of Tsai-Hill is often used. Either tensile \( (T_i) \) or compressive \( (C_i) \) yield limits are used. The shear limit is denoted as \( S \). The yield criterion is used in each ply of the laminate in the material coordinate system.

Failure can also occur by exceeding the maximum strain in longitudinal \( (\varepsilon_{fl}) \) or transversal \( (\varepsilon_{ft}) \) direction.

Tsai-Hill yield criterion for tensile loading

\[
\left( \frac{\sigma_{11}^2}{T_i^2} \right) - \left( \frac{\sigma_{11} \sigma_{22}}{T_i^2} \right) + \left( \frac{\sigma_{22}^2}{T_i^2} \right) + \left( \frac{\sigma_{12}^2}{S^2} \right) = 1
\]

Tsai-Hill yield criterion for compressive loading

\[
\left( \frac{\sigma_{11}^2}{C_i^2} \right) - \left( \frac{\sigma_{11} \sigma_{22}}{C_i^2} \right) + \left( \frac{\sigma_{22}^2}{C_i^2} \right) + \left( \frac{\sigma_{12}^2}{S^2} \right) = 1
\]

2.6.2 Damage : ILSS

An important failure mode in laminates is the loss of adhesion between plies. This occurs when the interlaminar shear stress exceeds a limit value: the interlaminar shear strength (ILSS).

The interlaminar shear stresses \( (ils) \) are calculated as the difference between the global stress component values between two adjacent plies, indicated here as \( (\ )_t \) (top) and \( (\ )_b \) (bottom).

\[
ils_{xx} = |\sigma_{xxb} - \sigma_{xxt}| ; \quad ils_{yy} = |\sigma_{yyb} - \sigma_{yyt}| ; \quad ils_{xy} = |\sigma_{xyb} - \sigma_{xyt}|
\]

2.7 Material parameters for some materials

Material parameters for Carbon fibre, HP-PE fibre and epoxy are listed in the table\(^2\). Material parameters for fibre-matrix composites are also listed in the table. HP-PE fibres can be plasma treated (pl.tr.) to improve adhesion to the epoxy matrix. The fibre volume fraction is \( V_f = 0.5 \).

\(^2\)from PhD thesis T.Peijs
2.8 Examples

Some laminates have been modeled and analyzed with Matlab. For each example the laminate build-up and the loading is presented. The resulting stiffness matrix is shown. The loading results in strains and curvatures of the mid-plane. The deformation is visualized and the ply-strains and stresses are plotted, both in global directions and in fiber directions.

- Random 4-ply laminate
- Symmetric 8-ply laminate
- Anti-symmetric 8-ply laminate

2.8.1 Random 4-ply laminate

<table>
<thead>
<tr>
<th>Laminate build-up (lam)</th>
</tr>
</thead>
<tbody>
<tr>
<td>z-</td>
</tr>
<tr>
<td>2.000</td>
</tr>
<tr>
<td>1.000</td>
</tr>
<tr>
<td>0.000</td>
</tr>
</tbody>
</table>
Mechanical load \( \{l_d\} \)
\[
[N_{xx} N_{yy} N_{xy} M_{xx} M_{yy} M_{xy}] = \begin{bmatrix} 100.00 & 0.00 & 0.00 & 100.00 & 0.00 & 0.00 \end{bmatrix}
\]

Stiffness matrix
\[
\begin{bmatrix}
2.57 \times 10^8 & 4.45 \times 10^7 & 4.15 \times 10^7 & -1.83 \times 10^5 & -3.98 \times 10^4 & -1.71 \times 10^4 \\
4.45 \times 10^7 & 2.52 \times 10^8 & 2.83 \times 10^7 & -3.98 \times 10^4 & -4.62 \times 10^5 & -2.37 \times 10^4 \\
4.15 \times 10^7 & 2.83 \times 10^7 & 7.55 \times 10^7 & -1.71 \times 10^4 & -2.37 \times 10^4 & -6.53 \times 10^4 \\
-1.83 \times 10^5 & -3.98 \times 10^4 & -1.71 \times 10^4 & 3.77 \times 10^2 & 9.80 \times 10^1 & 5.17 \times 10^1 \\
-3.98 \times 10^4 & -4.62 \times 10^5 & -2.37 \times 10^4 & 9.80 \times 10^1 & 1.11 \times 10^3 & 4.73 \times 10^1 \\
-1.71 \times 10^4 & -2.37 \times 10^4 & -6.53 \times 10^4 & 5.17 \times 10^1 & 4.73 \times 10^1 & 1.43 \times 10^2 \\
\end{bmatrix}
\]

Strains in the mid-plane \( \{e_0\} \)
\[
[ e_{xx} e_{yy} e_{xy} ] = \begin{bmatrix} 3.810 \times 10^{-4} & -1.079 \times 10^{-4} & -3.102 \times 10^{-4} \end{bmatrix}
\]

Curvatures of the mid-plane \( \{k_{r}\} \)
\[
[k_{xx} k_{yy} k_{xy}] = \begin{bmatrix} 4.784 \times 10^{-1} & -6.892 \times 10^{-2} & -2.637 \times 10^{-1} \end{bmatrix}
\]

### 2.8.2 Symmetric 8-ply laminate

Laminate build-up (lam)
\[
\begin{array}{cccccccc}
z_{-} & z_{+} & \text{ang} & E_{l} & E_{t} & n_{ult} & G_{l} \\
0.400 & 0.500 & 90.000 & 100.000 & 50.000 & 0.300 & 35.000 \\
0.200 & 0.400 & 0.000 & 120.000 & 70.000 & 0.300 & 40.000 \\
0.100 & 0.200 & 30.000 & 100.000 & 50.000 & 0.300 & 35.000 \\
0.000 & 0.100 & -30.000 & 200.000 & 100.000 & 0.300 & 60.000 \\
-0.100 & -0.000 & -30.000 & 200.000 & 100.000 & 0.300 & 60.000 \\
-0.200 & -0.100 & 30.000 & 100.000 & 50.000 & 0.300 & 35.000 \\
-0.400 & -0.200 & 0.000 & 120.000 & 70.000 & 0.300 & 40.000 \\
\end{array}
\]
29

-0.500 -0.400 90.000 100.000 50.000 0.300 35.000

-----------------------------
Mechanical load \(\{ld\}\)
\[
\begin{bmatrix}
N_{xx} & N_{yy} & N_{xy} & M_{xx} & M_{yy} & M_{xy}
\end{bmatrix} =
\begin{bmatrix}
100.00 & 0.00 & 100.00 & 100.00 & 100.00 & 100.00
\end{bmatrix}
\]
-----------------------------

Stiffness matrix
\[
\begin{bmatrix}
1.16e+08 & 2.13e+07 & -2.82e+06 & -2.12e-12 & 1.06e-13 & 1.25e-14 \\
2.13e+07 & 8.99e+07 & -1.71e+06 & 1.06e-13 & -8.47e-13 & -5.71e-16 \\
-2.82e+06 & -1.71e+06 & 4.19e+07 & 1.25e-14 & -5.71e-16 & -4.24e-13
\end{bmatrix}
\]

Strains in the mid-plane \(e_0\)
\[
\begin{bmatrix}
exx & eyy & exy
\end{bmatrix} =
\begin{bmatrix}
9.527e-07 & -1.794e-07 & 2.444e-06
\end{bmatrix}
\]
Curvatures of the mid-plane \(kr\)
\[
\begin{bmatrix}
kxx & kyy & kxy
\end{bmatrix} =
\begin{bmatrix}
1.106e+01 & 1.097e+01 & 3.190e+01
\end{bmatrix}
\]

---

2.8.3 Anti-symmetric 8-ply laminate

---

Laminate build-up \(\text{lam}\)
\[
\begin{array}{cccccccc}
z_- & z_+ & \text{ang} & E_l & E_t & \nu_{1l} & G_l \\
0.400 & 0.500 & 90.000 & 100.000 & 50.000 & 0.300 & 35.000 \\
0.200 & 0.400 & 0.000 & 120.000 & 70.000 & 0.300 & 40.000 \\
0.100 & 0.200 & 30.000 & 100.000 & 50.000 & 0.300 & 35.000 \\
0.000 & 0.100 & -30.000 & 200.000 & 100.000 & 0.300 & 60.000 \\
-0.100 & -0.000 & 30.000 & 200.000 & 100.000 & 0.300 & 60.000 \\
-0.200 & -0.100 & -30.000 & 100.000 & 50.000 & 0.300 & 35.000 \\
-0.400 & -0.200 & -0.000 & 120.000 & 70.000 & 0.300 & 40.000
\end{array}
\]
-0.500 -0.400 -0.300 0.000 0.300 0.400 0.500
-90.000 -50.000 50.000 100.000 100.000 100.000

Mechanical load (ld)
\[
[Nxx \ Nyy \ Nxy \ Mxx \ Myy \ Mxy] = \begin{bmatrix}
 100.00 & 0.00 & 100.00 & 100.00 & 100.00 & 100.00 \\
\end{bmatrix}
\]

Stiffness matrix
\[
\begin{bmatrix}
 1.16e+08 & 2.13e+07 & 1.26e-11 & -2.12e-12 & 1.06e-13 & -5.45e+01 \\
 2.13e+07 & 8.99e+07 & 1.00e-10 & 1.06e-13 & -8.47e-13 & -1.72e+02 \\
 1.26e-11 & 1.00e-10 & 4.19e+07 & -5.45e+01 & -1.72e+02 & -4.24e-13 \\
 -2.12e-12 & 1.06e-13 & -5.45e+01 & 7.42e+00 & 1.55e+00 & 4.82e-19 \\
 1.06e-13 & -8.47e-13 & -1.72e+02 & 1.55e+00 & 7.42e+00 & 2.19e-18 \\
 -5.45e+01 & -1.72e+02 & -4.24e-13 & 3.71e-18 & 2.19e-18 & 3.11e+00
\end{bmatrix}
\]

Strains in the mid-plane (e0)
\[
\begin{bmatrix}
  exx & eyy & exy \\
\end{bmatrix} = \begin{bmatrix}
 4.859e-06 & 6.048e-05 & 6.274e-05 \\
\end{bmatrix}
\]

Curvatures of the mid-plane (kr)
\[
\begin{bmatrix}
  kxx & kyy & kxy \\
\end{bmatrix} = \begin{bmatrix}
 1.115e+01 & 1.115e+01 & 3.217e+01 \\
\end{bmatrix}
\]

---

\[
\begin{array}{llll}
 r & \sigma \quad [\text{Pa}] & \epsilon \quad [\text{\%}] & w \quad [\text{mm}] \\
-0.500 & -0.400 & -0.300 & 0.000 & 0.300 & 0.400 & 0.500 \\
-90.000 & -50.000 & 50.000 & 100.000 & 100.000 & 100.000
\end{array}
\]

\[
\begin{array}{llll}
 x & \sigma \quad [\text{Pa}] & \epsilon \quad [\text{\%}] & w \quad [\text{mm}] \\
0.000 & 0.010 & 0.020 & \ldots & \ldots & \ldots & \ldots \\
50.000 & 100.000 & 100.000 & 100.000 & 100.000 & 100.000 & 100.000
\end{array}
\]

---

\[
\begin{array}{llll}
 y & \sigma \quad [\text{Pa}] & \epsilon \quad [\text{\%}] & w \quad [\text{mm}] \\
0.000 & 0.010 & 0.020 & \ldots & \ldots & \ldots & \ldots \\
50.000 & 100.000 & 100.000 & 100.000 & 100.000 & 100.000 & 100.000
\end{array}
\]