A comparison between the Perzyna viscoplastic model and the Consistency viscoplastic model

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Abstract

The elastic–viscoplastic characteristics of the Perzyna model and the Consistency model are considered. Both theories are compared by assessing the evolution of the viscoplastic multiplier, the evolution of the internal variables and the loading/unloading conditions. An explicit expression is derived for the consistency parameter in the Consistency model. Accordingly, it is shown that the Consistency model and the Perzyna model can be treated in a novel, unified algorithmic fashion. Illustrative numerical examples are given to reveal the differences and the similarities between the models, and to elucidate the features of the proposed implicit numerical scheme. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Keywords: Viscoplasticity; Perzyna model; Consistency model; Unified algorithmic framework

1. Introduction

For the analysis of time-dependent failure in materials, such as the propagation of Lüders bands and Portevin–Le Chatelier effects in metals (Wang, 1997), as well as shear banding and creep in geo-materials (Desai and Zhang, 1987; Cristescu, 1994; Samtani et al., 1996; Cristescu and Cazacu, 2000), various viscoplastic material models have been proposed. A widely-used viscoplastic formulation is the Perzyna model (Perzyna, 1966; Olszak and Perzyna, 1969). The main feature of this model is that the rate-independent yield function used for describing the viscoplastic strain can become larger than zero, which effect is known as ‘overstress’. The characteristics of the Perzyna model as well as the numerical discretization have been addressed by various authors (Simo, 1989; Sluys, 1992; Wang, 1997; Simo and Hughes, 1998). Additionally, viscoplasticity can be modeled by incorporation of the time-dependency in a yield function, which, together with the consistency parameter, obeys the classical Kuhn–Tucker relations. In (Wang et al., 1997; Wang, 1997), a so-called Consistency model has been proposed, in which the time-derivative of a rate-dependent yield surface governs the irreversible, viscous deformation behaviour. Furthermore, Mahnken et al. (1998), Johansson et al. (1999) have considered a rate-dependent yield formulation in combination with coupling to damage. Very recently, Ristinmaa and Ottosen (2000) have given a thorough discussion on the main features and implications of modeling rate-dependency within a yield surface concept.

In the current paper, the above-mentioned Consistency model is compared to the Perzyna model. At first sight, the main features of these models appear to be rather different. However, it can be proven that the Perzyna model and the Consistency model to a certain extent are identical. To demonstrate this, we compare the evolution characteristics regarding the viscoplastic multiplier. In the Perzyna model, the rate of the viscoplastic multiplier, also known as the consistency parameter, is explicitly defined via an overstress function, while in the Consistency model it is governed by a non-homogeneous differential equation, i.e. the rate-dependent consistency condition. By recasting the evolution law of the Perzyna model into a format similar to that...
of the Consistency model, it appears that the constitutive parameters of the models can be uniquely related. However, as a result of dissimilarities in the unloading characteristics, the models may respond differently during stress reversals. This is illustrated by means of two numerical examples, which are a single integration point subjected to both a progressive shear loading and a reversible shear loading, and a shear layer subjected to a progressive uni-directional shear deformation.

For carrying out the numerical computations, the Perzyna model and the Consistency model are cast into a novel, unified algorithmic framework. For this purpose, for the Consistency model, an explicit expression for the consistency parameter is derived, similar to the overstress expression in the Perzyna model. Correspondingly, the numerical algorithm is provided with a transparent structure.

2. Theory of viscoplasticity

In the small-strain theory, the total strain rate \( \dot{\varepsilon} \) in an elasto–viscoplastic material point may be additively decomposed into an elastic component \( \dot{\varepsilon}^{el} \) and a viscoplastic component \( \dot{\varepsilon}^{vp} \)

\[
\dot{\varepsilon} = \dot{\varepsilon}^{el} + \dot{\varepsilon}^{vp},
\]

where the superimposed dot denotes the time derivative. The stress rate \( \dot{\sigma} \) is related to the strain rate via the constitutive relation

\[
\dot{\sigma} = D_{el} : \dot{\varepsilon}^{el},
\]

with \( D_{el} \) the fourth-order tensor containing the tangential elastic stiffness moduli, which in this study are assumed to be constant. Double dots "\( : \)" denote a double tensorial contraction. Furthermore, the viscoplastic strain rate evolves via a flow rule,

\[
\dot{\varepsilon}^{vp} = \dot{\lambda} \mathbf{m},
\]

with \( \dot{\lambda} \) a non-negative parameter, known as the consistency parameter (Simo and Hughes, 1998). This parameter specifies the magnitude of \( \dot{\varepsilon}^{vp} \), while \( \mathbf{m}(\sigma, \Phi) \) determines the direction of \( \dot{\varepsilon}^{vp} \). Here, the second-order tensor \( \Phi \) reflects the evolution of the isotropic as well as the anisotropic internal variables. The rate of the internal variables is prescribed by the product of a rate-independent tensor \( \mathbf{p}(\sigma, \Phi) \) and the consistency parameter \( \dot{\lambda} \),

\[
\dot{\Phi} = \mathbf{p} \dot{\lambda}.
\]

Usually, the second-order tensor \( \mathbf{m} \) in equation (3) is derived from a potential function \( g \), according to

\[
\mathbf{m} = \frac{\partial g}{\partial \sigma}.
\]

Although in general \( g \) may depend on rate-dependent effects \( \dot{\Phi} \), we restrict ourselves to \( g = g(\sigma, \Phi) \), such that rate-effects are not affecting the direction of the viscoplastic flow.

2.1. Perzyna model

In the Perzyna model the evolution of the viscoplastic strain rate is defined as (Perzyna, 1966)

\[
\dot{\varepsilon}^{vp} = \frac{\langle \phi(f) \rangle}{\eta} \mathbf{m},
\]

with \( \eta \) the viscosity parameter, \( \phi \) the overstress function that depends on the rate-independent yield surface \( f(\sigma, \Phi) \), and \( \mathbf{m} \) given by equation (5). When combining equation (3) with equation (6), an explicit expression for the consistency parameter is obtained,

\[
\dot{\lambda} = \frac{\langle \phi(f) \rangle}{\eta},
\]

where "\( \langle \cdot \rangle \)" are the McCauley brackets, such that

\[
\langle \phi(f) \rangle = \begin{cases} 
\phi(f) & \text{if } \phi(f) \geq 0, \\
0 & \text{if } \phi(f) < 0.
\end{cases}
\]

According to Simo (1989), the overstress function \( \phi \) must fulfill the following conditions:

\[
\phi(f) \text{ is continuous in } [0, \infty),
\]

\[
\phi(f) \text{ is convex in } [0, \infty),
\]

\[
\phi(0) = 0,
\]
so that a rate-independent elasto–plastic model is recovered if $\eta \to 0$. The following, widely-used expression for $\phi$ is employed (Desai and Zhang, 1987; Simo, 1989; Sluys, 1992; Wang et al., 1997; Simo and Hughes, 1998):

$$\phi(f) = \left(\frac{f}{\sigma_0}\right)^N. \quad (10)$$

Herein, $\sigma_0$ is commonly chosen as the initial yield stress, and $N$ is a calibration parameter that should satisfy $N \geq 1$ in order to meet condition (9-b).

2.2. Consistency model

Alternatively, viscoplasticity can be modeled by incorporating the rate-dependency in a yield function. This concept has been used to formulate the Consistency model (Wang, 1997; Wang et al., 1997), which appeared to be an efficient and useful tool in the simulation of Lüders bands and Portevin–Le Chatelier effects in metals. In the Consistency model, the rate-dependent yield surface $f_{rd}$ is generally expressed as (Wang, 1997)

$$f_{rd} = f_{rd}(\sigma, \Phi, \dot{\xi}). \quad (11)$$

Similar to classical elasto–plasticity, viscoplastic loading occurs as soon as $f_{rd} = 0$. During viscoplastic loading it is demanded that $\dot{\xi} = \dot{\Phi}/\Phi_1$. If subsequently $f_{rd} < 0$, elastic unloading occurs, with $\dot{\Phi}$ constant. During unloading, as well as reloading with $f_{rd} < 0$, it is required that $\dot{\xi} = \dot{\Phi}/\Phi_1$, the value of $\Phi$ at the onset of elastic unloading.

Employing equation (11) to formulate the consistency condition, $f_{rd} = 0$, yields

$$\dot{f}_{rd} = \frac{\partial f_{rd}}{\partial \sigma} \dot{\sigma} + \frac{\partial f_{rd}}{\partial \Phi} \dot{\Phi} + \frac{\partial f_{rd}}{\partial \dot{\xi}} \dot{\dot{\xi}} = 0. \quad (12)$$

Trivially, the above condition requires a stress point to remain on the rate-dependent yield surface $f_{rd}$ during viscoplastic flow. A more convenient form of equation (12) is (Wang, 1997; Wang et al., 1997)

$$\dot{f}_{rd} = \frac{\partial f_{rd}}{\partial \sigma} \dot{\sigma} - h \dot{\lambda} - y \ddot{\lambda} = 0, \quad (13)$$

in which the hardening modulus $h$ and the strain-rate sensitivity parameter $y$ are given by

$$h = -\frac{\partial f_{rd}}{\partial \Phi} : \dot{\Phi}/\lambda, \quad (14)$$

$$y = -\frac{\partial f_{rd}}{\partial \dot{\xi}} : \ddot{\Phi}/\lambda.$$ 

In the remainder, the consistency condition in the format provided by equation (13) will be used. For distinguishing between loading and unloading, the rate-dependent yield surface and the consistency parameter are subjected to the classical Kuhn–Tucker relations

$$f_{rd} \leq 0, \quad \dot{\lambda} \geq 0, \quad \dot{\lambda} f_{rd} = 0. \quad (15)$$

In correspondence with previous arguments, the above relations describe viscoplasticity when $f_{rd} = 0$, while a purely elastic response is generated when $f_{rd} < 0$.

The direction of the viscoplastic strain rate is determined by equation (5). Because the potential function $g$ in equation (5) generally differs from the rate-dependent yield function, equation (11), the viscoplastic model can be characterized as 'non-associated'.

3. Comparison of the Perzyna model and the Consistency model

3.1. Consistency condition for the Perzyna model

As discussed in the previous section, in the Perzyna model the evolution of the viscoplastic multiplier is explicitly defined in the stress space, equation (7), while in the Consistency model it is governed by a second-order differential equation, equation (13). To compare both evolution laws, we first recall equation (7),

$$\dot{\lambda} = \frac{\phi(f)}{\eta}, \quad (16)$$
in which equation (8-a) is assumed to be valid, i.e. loading is considered. Subsequently, we compute the time derivative of equation (16), yielding
\[
\frac{d\phi}{df} \dot{f} - \eta \ddot{\lambda} = 0.
\] (17)
where the viscosity \(\eta\) is taken as constant. Because the function \(f\) in equation (16) solely depends on the stress and the internal variables, \(\dot{f}\) has the form
\[
\dot{f} = \frac{\partial f}{\partial \sigma} : \dot{\sigma} - \bar{h} \dot{\lambda},
\] (18)
with \(\bar{h}\) given by equation (14-a). Substitution of equation (18) into equation (17), followed by some reordering, leads to a consistency condition for the Perzyna model
\[
\frac{\partial f}{\partial \sigma} : \dot{\sigma} - \bar{h} \dot{\lambda} - \left( \frac{\partial \phi}{\partial f} \right)^{-1} \eta \ddot{\lambda} = 0.
\] (19)
Subsequently, we recall the consistency condition for the Consistency model, equation (13),
\[
\frac{\partial f_{rd}}{\partial \sigma} : \dot{\sigma} - \bar{h} \dot{\lambda} - y \ddot{\lambda} = 0.
\] (20)
Comparing equation (20) with equation (19) shows that both expressions are equal if
\[
\frac{\partial f_{rd}}{\partial \sigma} = \frac{\partial f}{\partial \sigma},
\] (21)
\[y = \eta \left( \frac{\partial \phi(f)}{\partial f} \right)^{-1}.
\]
With the conditions (21), the stress–strain behaviour predicted by the Perzyna model and the Consistency model will be identical, provided that both models are subjected to progressive viscoplastic loading.

3.2. Unloading behaviour

To illustrate that the Consistency model and the Perzyna model yield different responses during unloading, equation (7) is reworked to
\[
f_{rd}^P = f - \phi^{-1}(\eta \dot{\lambda}) = 0.
\] (22)
The expression above can be regarded as a rate-dependent yield condition for the Perzyna model, see also Fig. 1, with the subscript \(rd\) denoting ‘rate-dependence’ and the superscript \(P\) denoting ‘Perzyna’. Unlike the Consistency model, in the Perzyna model the function \(f\) (instead of \(f_{rd}^P\)) is used to check whether viscoplasticity occurs. Viscoplastic deformation is generated as long as overstress is present, \(f > 0\), which can also occur during unloading (\(f_{rd}^P = 0; f = \phi^{-1}(\eta \dot{\lambda})\), area II in Fig. 1). When after sufficient unloading the overstress becomes exhausted, the Perzyna model unloads viscoplastically (\(f < 0; f_{rd}^P < 0\), area I in Fig. 1). Contrarily, the Consistency model always unloads elastically (\(f_{rd} < 0\), both in area I and area II of Fig. 1). The different unloading conditions cause the internal variables \(\Phi\) of the viscoplastic models to evolve differently during stress reversals.

![Fig. 1. Two-dimensional graphical representation of coinciding yield surfaces for the Perzyna model and the Consistency model, see equations (22) and (11), respectively. Area I: Perzyna model unloads elastically, Area II: Perzyna model unloads viscoplastically. In both area I and area II the Consistency model unloads elastically.](image)
4. Algorithmic treatment

According to equation (13), in the Consistency model the consistency parameter $\dot{\lambda}$ is defined implicitly via a differential equation. Instead of solving the differential equation in a discretized manner (Wang, 1997), in this section it is demonstrated how to obtain an explicit expression for the consistency parameter. This expression can be directly transformed into a numerical scheme, such that the Perzyna model and the Consistency model fit within a unified algorithmic framework.

4.1. Derivation of the consistency parameter in the Consistency model

For deriving an expression for the consistency parameter $\dot{\lambda}$, we rewrite the consistency condition, equation (13), as

$$\dot{\lambda} + \frac{h}{y} \lambda = \frac{n}{y} \phi,$$

(23)
in which $n = \bar{f}_{d\lambda}/\bar{\sigma}$. Assigning to $n$, $h$ and $y$ their values at the end of the considered discrete time step, the differential equation (23) becomes incrementally-linear. In order to solve this non-homogeneous linearized differential equation, Laplace transformation can be used,

$$\mathcal{L}[k](s) = \int_0^\infty e^{-st} k(t) \, dt,$$

(24)
in which $k$ is an arbitrary function of time $t$, $s$ is the Laplace transform parameter, and $\mathcal{L}[k](s)$ is the Laplacian of $k$. Thus, the Laplace transform of equation (23) becomes

$$s^2 \mathcal{L}[\lambda](s) - s\dot{\lambda}(0) - \dot{\lambda}(0) + \frac{h}{y} \left( s \mathcal{L}[\lambda](s) - \lambda(0) \right) = \frac{1}{y} n \left( s \mathcal{L}[\sigma](s) - \sigma(0) \right),$$

(25)
with $\mathcal{L}[\lambda](s)$ the Laplacian of $\lambda$, and $\mathcal{L}[\sigma](s)$ the Laplacian of $\sigma$. Additionally, $\lambda(0)$, $\dot{\lambda}(0)$ and $\sigma(0)$ are the initial conditions, referring to the beginning of the considered discrete time step. Subsequently, equation (25) is reworked to

$$\mathcal{L}[\lambda](s) = \frac{\dot{\lambda}(0)(s + \frac{h}{y}) + \dot{\lambda}(0)}{s^2 + \frac{h}{y}s} + \frac{s n \mathcal{L}[\sigma](s) - n \sigma(0)}{y(s^2 + \frac{h}{y}s)},$$

(26)
which can be further expanded into partial fractions as

$$\mathcal{L}[\lambda](s) = \frac{\dot{\lambda}(0)}{s} + \frac{\dot{\lambda}(0)}{y} \left( \frac{y}{hs} - \frac{y}{h(s + \frac{h}{y})} \right) + \frac{n \mathcal{L}[\sigma](s)}{y(s^2 + \frac{h}{y}s)} - \frac{1}{h} \frac{n \sigma(0)}{y(s^2 + \frac{h}{y}s)}.$$

(27)

For $s > -h/y$ and $s > 0$, the inverse transform of equation (27) can be found by invoking the theorem for convolution integrals (Boyce and di Prima, 1992), yielding

$$\lambda(t) = \lambda(0) + \frac{y}{h} \dot{\lambda}(0) (1 - e^{-\frac{h}{y}t}) + \int_0^t \frac{1}{y} e^{-\frac{h}{y}(t-\tau)} n : \sigma(\tau) \, d\tau - \frac{1}{h} \frac{n \sigma(0)}{y} (1 - e^{-\frac{h}{y}t}).$$

(28)

Differentiation of equation (28) with respect to time provides the following solution for $\dot{\lambda}$:

$$\dot{\lambda}(t) = \dot{\lambda}(0) e^{-\frac{h}{y}t} - \int_0^t \frac{h}{y^2} e^{-\frac{h}{y}(t-\tau)} n : \sigma(\tau) \, d\tau + \frac{1}{y} \left( n : \sigma(t) - n : \sigma(0) e^{-\frac{h}{y}t} \right).$$

(29)
The integral expression in the right-hand side of equation (29) can be easily evaluated in a numerical manner, which makes this expression certainly suitable for incorporation in a numerical algorithm. Alternatively, equation (28) may be combined with equation (29), which yields the closed-form expression

$$\dot{\lambda}(t) = \dot{\lambda}(0) + \frac{1}{y} \left( n : \sigma(t) - n : \sigma(0) \right) - \frac{h}{y} (\dot{\lambda}(t) - \dot{\lambda}(0)).$$

(30)

Although the structure of equation (30) is simpler than that of equation (29), the consistency parameter $\dot{\lambda}$ now not only depends on the stress, but also on the viscoplastic multiplier $\lambda$. This, however, does not generate complications in a numerical procedure, as both parameters can be evaluated at each discrete time step.
4.2. Implicit integration scheme

The governing equations for the considered viscoplastic models can be divided into three categories, viz. a category in which the evolution of the stress is described, a category regarding the consistency parameter, and a category involving the evolution of internal variables. The equations in the first category are obtained upon combining equations (1), (2) and (3), leading to

\[ \dot{\sigma} = D^i : (\dot{e} - \dot{\lambda} \mathbf{m}). \]  

(31)

Additionally, in the Perzyna model the viscoplastic multiplier evolves according to equation (7), whereas in the Consistency model it evolves according to equation (30). Lastly, the evolution of the internal variables is specified in equation (4). Driven by the strain increment \( \Delta \varepsilon_{n+1} \), the following update has to be carried out

\[ \{\sigma_n, \Phi_n, \lambda_n, \varepsilon_n\} \rightarrow \{\sigma_{n+1}, \Phi_{n+1}, \lambda_{n+1}, \varepsilon_{n+1}\}, \]  

(32)

with the subscript \( n \) referring to the beginning of the considered time step, and the subscript \( n+1 \) referring to the end of the time step. To perform this update, the equations mentioned above need to be discretized and cast into a residual format (Borja, 1991; Groen, 1997; Suiker, 1998; de Borst and Heeres, 2000). In this fashion, we obtain for the stresses

\[ r_\sigma = \sigma_{n+1} - \sigma_n - D^i : \Delta \varepsilon_{n+1} + \Delta \dot{\lambda}_{n+1} D^i : \mathbf{m}(\sigma_{n+1}, \Phi_{n+1}). \]  

(33)

Furthermore, in correspondence with equation (7), for the Perzyna model we can formulate the following residual regarding the consistency parameter,

\[ r_\lambda = \Delta \lambda_{n+1} - \frac{\phi(f(\sigma_{n+1}, \Phi_{n+1}))}{\eta} \Delta r_{n+1}, \]  

(34)

where we have used the relation \( \Delta \lambda_{n+1} = \dot{\lambda}_{n+1} \Delta t_{n+1} \). Making use of equation (30) followed by some reordering, in the Consistency model the residual for the consistency parameter becomes

\[ r_\lambda = \Delta \lambda_{n+1} - \frac{\Delta \lambda_{n+1}}{\eta \lambda_{n+1}} \left( \frac{\Delta \varepsilon_{n+1}}{\Delta \lambda_{n+1}} \right) \Delta \varepsilon_n - \frac{\Delta \varepsilon_{n+1}}{\eta \lambda_{n+1}} (\Phi_{n+1} : \sigma_{n+1} - \Phi_n : \sigma_{n+1} + \Phi_{n+1} : \varepsilon_{n+1} - \Phi_n : \varepsilon_{n+1}). \]  

(35)

The residual equation (35) is computed for the first time at the onset of viscoplasticity, where \( \sigma_n \) then should reflect the initial yield stress. Obviously, the incremental plastic multiplier and the time increment at the previous time-step, \( \Delta \lambda_n \) and \( \Delta t_n \), need to be evaluated in order to update the consistency parameter, equation (35). This is not the case for the update of the consistency parameter in the Perzyna model, equation (34). Subsequently, the evolution of the internal variables is described by discretizing equation (4) as

\[ r_\Phi = \Phi_{n+1} - \Phi_n - p(\sigma_{n+1}, \Phi_{n+1}) \Delta \lambda_{n+1}. \]  

(36)

The residual problem characterized by equations (33)–(36) can be solved by requiring

\[ r(a_{n+1}) = 0 \]  

(37)

to hold within some prescribed tolerance, where \( a_{n+1} \) contains the variables \( \sigma_{n+1}, \lambda_{n+1} \) and \( \Phi_{n+1} \). Generally, the residuals are interrelated, and therefore the solution has to be obtained in an iterative manner. Accordingly, a Newton–Raphson scheme can be used, in which the iterative update \( da_{n+1}^i \) is given by

\[ da_{n+1}^i = - \left( \frac{\partial r_{n+1}}{\partial a_{n+1}^i} \right)^{-1} r_{n+1}^i. \]  

(38)

where the subscripts \( i \) and \( i + 1 \) refer to the previous iteration and to the current iteration, respectively.

In order to obtain quadratic convergence at the system level, a tangent matrix must be formulated that is consistent with the update algorithm, equation (37) (Simo and Taylor, 1985; Simo and Govindjee, 1988). To derive this matrix, the stresses are differentiated as follows:

\[ \frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} = \frac{\partial \sigma_{n+1}}{\partial a_{n+1}} + \frac{\partial \sigma_{n+1}}{\partial \alpha_{n+1}} \cdot \frac{\partial \alpha_{n+1}}{\partial \varepsilon_{n+1}}. \]  

(39)

For a converged solution a variation in strain does not cause a variation of the residuals, which is expressed by the condition

\[ \frac{dr_{n+1}}{\partial \varepsilon_{n+1}} = \frac{\partial r_{n+1}}{\partial a_{n+1}} + \frac{\partial r_{n+1}}{\partial \alpha_{n+1}} \cdot \frac{\partial \alpha_{n+1}}{\partial \varepsilon_{n+1}} = 0. \]  

(40)
which can be rewritten as
\[
\frac{\partial \mathbf{a}_{n+1}}{\partial \mathbf{e}_{n+1}} = - \left( \frac{\partial \mathbf{r}_{n+1}}{\partial \mathbf{a}_{n+1}} \right)^{-1} \frac{\partial \mathbf{r}_{n+1}}{\partial \mathbf{e}_{n+1}}.
\] (41)

By substituting the above equation into equation (39), the consistent tangent matrix becomes
\[
\frac{\partial \sigma_{n+1}}{\partial \mathbf{e}_{n+1}} = \frac{\partial \sigma_{n+1}}{\partial \mathbf{a}_{n+1}} \left( \frac{\partial \mathbf{r}_{n+1}}{\partial \mathbf{a}_{n+1}} \right)^{-1} \frac{\partial \mathbf{r}_{n+1}}{\partial \mathbf{e}_{n+1}}.
\] (42)

Note that the gradients \( \partial \mathbf{r}/\partial \mathbf{a} \) have already been computed in equation (38). The matrix \( \partial \mathbf{r}/\partial \mathbf{a} \) in equation (42) is generally non-symmetric, and additionally the matrices \( \partial \mathbf{a}/\partial \mathbf{a} \) and \( \partial \mathbf{a}/\partial \mathbf{e} \) do not commute. These facts cause the consistent tangent stiffness matrix to be generally non-symmetric (Simo and Hughes, 1998).

5. Numerical examples

In this section, the behaviour of the viscoplastic models is illustrated via numerical examples. For this purpose, von Mises plasticity is incorporated in the Perzyna model and in the Consistency model. The numerical examples that are studied are a single integration point subjected to shear load (reversals) and a shear layer subjected to a progressive, uni-directional shear deformation with a constant deformation rate.

5.1. Von Mises plasticity

The yield function \( f \) for a von Mises plasticity formulation is given by
\[
f = q - \bar{\sigma},
\] (43)
where \( q \) is the deviatoric stress invariant according to
\[
q = \sqrt{3} J_2^{\sigma},
\] (44)
with \( J_2 = \mathbf{\sigma}' : \mathbf{\sigma}' \). Here, the deviatoric stress reads \( \sigma' = \sigma - p \pi \), with the hydrostatic stress \( p = \frac{1}{3} \pi : \sigma \), where \( \pi \) is the second-order unity tensor. The rate-independent yield stress \( \bar{\sigma} \) is formulated as
\[
\bar{\sigma} = \sigma_0^\prime + h \kappa,
\] (45)
in which \( \sigma_0^\prime \) is the initial yield stress, and \( h \) is the hardening modulus, being positive for hardening and negative for softening, see Fig. 2. In this study, we only consider softening problems, where for simplicity reasons we have adopted a linear softening law. Nevertheless, it should be realized that the proposed numerical algorithm allows for employing a non-linear softening law as well. In equation (45), \( \kappa \) is a history parameter that can be obtained upon time integration along the loading path,
\[
\kappa = \int \dot{\kappa} \, dt,
\] (46)
with
\[
\dot{\kappa} = \sqrt{\frac{2}{3}} \dot{\epsilon}^{\text{vp}} : \dot{\epsilon}^{\text{vp}}.
\] (47)

Adopting \( \mathbf{m} = \mathbf{n} = \partial f/\partial \sigma \), and invoking equations (43) and (44), the flow rule, equation (3), specifies to
\[
\dot{\epsilon}^{\text{vp}} = \dot{\kappa} \frac{3}{2} \frac{\sigma'}{q}.
\] (48)
Upon substitution of equation (48) into equation (47) we obtain \( \dot{\kappa} = \dot{\lambda} \). When invoking equations (43) and (45), the rate-dependent yield function for the Consistency model, equation (11), may be formulated as

\[
\dot{f}_{rd} = q - \sigma_0 - h \lambda - y \dot{\lambda}.
\]  

(49)

At the beginning of a loading step, the above expression is used to check whether or not viscoplastic loading occurs. For the Perzyna model, this check is carried out by using the overstress function given in equation (10) with the yield function, equation (43), being substituted. The material parameters thereby have been chosen as \( N = 1 \) and \( \alpha = \sigma_0 \). With these choices, condition (21-b) reduces to \( y = \eta \sigma_0 \).

5.2. Discretization aspects

The residual expression involving the update of the stresses is given by equation (33). Incorporation of von Mises plasticity, equation (43), into equation (10), and using equation (45) and the above-mentioned values for \( N \) and \( \alpha \), the residual for the consistency parameter in the Perzyna model, equation (34), becomes

\[
r_{\lambda} = \Delta \lambda_{n+1} - \frac{\eta_{n+1} - \sigma_0 - h \lambda_{n+1}}{\eta \sigma_0} \Delta \lambda_{n+1}.
\]  

(50)

Alternatively, for the Consistency model the residual for the consistency parameter, equation (35), turns into

\[
r_{\lambda} = \Delta \lambda_{n+1} - \frac{\Delta \lambda_{n+1} (\eta \sigma_0 + h \Delta \lambda_{n+1})}{\eta \sigma_0 + h \Delta \lambda_{n+1}} \Delta \lambda_{n} - \frac{\Delta \lambda_{n+1}}{\eta \sigma_0 + h \Delta \lambda_{n+1}} \left( n_{n+1} : \sigma_{n+1} - n_{n+1} : \sigma_{n} \right). 
\]  

(51)

Lastly, with equation (45) the residual involving the update of the internal variable, \( \sigma_{n+1} \), for both models specifies to

\[
r_{\sigma} = \sigma_{n+1} - h \Delta \lambda_{n+1}.
\]  

(52)

Note that the format of the above residual corresponds to that of the general expression, equation (36).

5.3. Single integration point loaded in shear

In Fig. 3, the response of a single integration point is depicted during stress reversals. The stress reversals are characterized by an initial loading phase, which nearby the peak strength changes into unloading towards zero shear stress, followed by reloading until complete failure is achieved. The loading/unloading rate corresponds to \( |\dot{\epsilon}_{xy}| = 1.0 \text{ s}^{-1} \), and the constitutive parameters are given in Table 1. Apparently, Fig. 3 shows a different behaviour during unloading, which is in correspondence with the discussion in Section 3. The different unloading behaviour also influences the subsequent reloading phase. During reloading, Consistency viscoplasticity is reactivated at the stress level at which unloading was initiated. Conversely, Perzyna

\[
\begin{array}{cccc}
E \text{ [Nm}^{-2}] & \nu & \eta \text{ [s]} & \sigma_0 \text{ [Nm}^{-2}] & h \text{ [Nm}^{-2}] \\
2.0E7 & 0.2 & 1.0 & 2.0E3 & -5.0E6
\end{array}
\]

Table 1

The constitutive parameters for the Perzyna model and the Consistency model.
viscoplasticity is reactivated at the stress level at which overstress became exhausted during the previous unloading phase. Despite these differences, in the softening branch the models tend to give a similar response again.

In Table 2, the convergence behavior just before the peak load (step 10) has been given. The norm of the initial residual for the Perzyna model is smaller than that for the Consistency model, which is due to the different formulation of the consistency parameter, cf. equations (34) and (35). Nevertheless, this does not affect the convergence rate; both algorithms converge quadratically.

Subsequently, we have subjected the integration point to a progressive, uni-directional shear deformation, again with \( \dot{\varepsilon}_{xy} = 1.0 \text{ s}^{-1} \). Two different viscosity parameters are considered; \( \eta = 1.0 \text{ s} \) and \( \eta = 5.0 \text{ s} \). The resulting stress–strain diagrams in Fig. 4 illustrate that for both viscosities the models respond identically.

5.4. Shear layer

Fig. 5 depicts a shear layer subjected to a progressive shear deformation with a rate \( \dot{u}_{\text{top}} = 1.0 \text{ mm/s}^{-1} \). The shear layer is modeled by 200 six-noded triangular elements with a three-point Gauss integration scheme. The constitutive parameters are again given in Table 1. This model has also been used by other investigators to analyze shear localization (Sluys, 1992; de Borst, 1993; Gutiérrez and de Borst, 1999). In order to activate the regularizing properties of the viscoplastic models, the initial yield stress presented in equation (45) is assumed to vary smoothly along the layer height \( H \), according to

\[
\sigma_0(x) = \sigma_0 - a \sin \left( \frac{\pi x}{H} \right) \tag{53}
\]

Here, \( x \) is the distance with respect to the layer bottom and \( a = 0.4 \sigma_0 \) is the amplitude of the imperfection. In the analysis, the height of the shear layer has been taken as \( H = 100 \text{ mm} \). In Table 3 the (global) convergence behavior at the system level has been depicted at different loading stages. Remarkably, the Consistency model reveals a somewhat faster convergence than the Perzyna model, which is mainly due to elastic (instead of viscoplastic) unloading of the Consistency model for material points outside the shear band. However, the different unloading behaviour hardly affects the overall load-displacement behaviour, see Fig. 6. Also, the displacement profile nearby the end of softening (step 23) is almost equivalent for both models, see Fig. 7. The local deformation pattern in the center of the shear layer is governed by the regularizing properties of the viscosity parameter \( \eta \) (Gutiérrez and de Borst, 1999).

**Table 2**

<table>
<thead>
<tr>
<th>Step</th>
<th>Iteration</th>
<th>Consistency model</th>
<th>Perzyna model</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4.566E+02</td>
<td>3.169E−02</td>
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<tr>
<td></td>
<td>4</td>
<td>1.137E−13</td>
<td>1.136E−14</td>
</tr>
</tbody>
</table>

Fig. 4. Stress–strain response for the Perzyna model and the Consistency model, obtained under progressive deformation. The viscosity parameters are \( \eta = 1.0 \text{ s} \) and \( \eta = 5.0 \text{ s} \). The models respond equally for both values of \( \eta \).
\[
\mathbf{u}^{\text{top}} = 1.0\text{mm/s}
\]

**Fig. 5.** The shear layer: finite element mesh and applied loading.

Table 3

<table>
<thead>
<tr>
<th>Step</th>
<th>Iteration</th>
<th>Consistency model</th>
<th>Perzyna model</th>
</tr>
</thead>
<tbody>
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<td>3</td>
<td>1.735E−15</td>
<td>2.141E−14</td>
</tr>
</tbody>
</table>

Global convergence: norm of the iterative displacement update \( \|\mathbf{u}_{i+1}^{n+1}\|/\|\Delta \mathbf{u}_{1}^{n+1}\| \), where \( \|\mathbf{u}_{i+1}^{n+1}\| \) is the norm of the incremental displacement vector in iteration \( i + 1 \), and \( \|\Delta \mathbf{u}_{1}^{n+1}\| \) is the norm of the incremental displacement vector in the first iteration. The iterative procedure is considered to be converged if the norm of the residual vector becomes smaller than 1.0E−12.

**Fig. 6.** The shear load versus the horizontal displacement at the top of the shear layer. With the Consistency model (solid line) and the Perzyna model (dashed line) virtually the same result is obtained.
Fig. 7. The final horizontal displacement profile of the shear layer, where $u_{\text{top}} = 2.3 \times 10^{-5}$ m (step 23 in Fig. 6). The displacement profile for the Perzyna model and the Consistency model are nearly identical.

6. Conclusions

The basic notions of the Perzyna model and the Consistency model have been presented. By comparing these models, it has been proven that the Perzyna model and the Consistency model give an identical response during progressive loading, while differences appear during unloading. This has been further illustrated by means of two numerical examples. On the system level, the new implicit numerical scheme derived for the Consistency model yields a somewhat higher convergence rate than that derived for the Perzyna model, which is mainly due to the different unloading characteristics of the models.

References