A discontinuous Galerkin method with splitting applied to visco-elastic flow.

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Abstract

In this report a numerical method for viscoelastic flow simulations is developed. It is based on the discontinuous Galerkin (DG) method. In contrast to the standard DG method, which is only applied to the convection terms, the DG method is applied to the full hyperbolic system. In order to do so the characteristic values and directions have to be determined locally and splitted according to their sign. The characteristic values and directions have been determined analytically for multi-mode models with an upper-convected derivative. The numerical method has been applied to stability of a planar Couette flow and the flow around a sphere falling in a tube. The method only seems to be useful in channel/pipe flows. In geometrically complex flows, such as the flow around a sphere, the method is numerically stable only for very small Weissenberg numbers.
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Chapter 1

Introduction

The simulation of viscoelastic fluid flow has become an active research area in recent years (Crochet & Walters 1993). The majority of articles contain simulations of creeping steady flow, motivated by the importance of industrial processing of polymer melts (see, for example, Hulsen & van der Zanden 1991). For polymer solutions however, it has become clear that inertia can have major influence on the flow behaviour (Joseph 1990). At higher flow rates the flow becomes unsteady. In order to study this behaviour time-dependent equations have to be solved.

Phelan et al. (1989) showed that allowing a weak compressibility into the equations of viscoelastic flow makes these equations hyperbolic (see also: Joseph 1990). In that case the equations can formally be written in the following form

\[
\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^{d} A_i(y) \frac{\partial \mathbf{u}}{\partial x_i} + f(y) = 0 \quad \text{in} \quad (0, T) \times \Omega,
\]

where \( \Omega \) is a bounded subdomain of \( \mathbb{R}^d \), \( d = 1, 2 \) or \( 3 \), \( y = (u_1, u_2, \ldots, u_n)^T \), \( f(y) = (f_1, f_2, \ldots, f_n)^T \) is a source term and the matrices \( A_i(y), i = 1, \ldots, d \) are such that any linear combination \( \sum_{i=1}^{d} \xi_i A_i(y) \) has \( n_v \) real eigenvalues and a complete set of eigenvectors. A system having the latter property is called hyperbolic (Whitham 1974).

An important application of hyperbolic systems is in compressible flow. The equations (conservation of mass, momentum and energy), are usually written in the so-called conservation form

\[
\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (F_i(y)) + f(y) = 0,
\]

where \( \mathbf{u} \) is the vector of conserved variables (specific mass, momentum and energy) and \( F_i, i = 1, \ldots, d \) are the flux vectors with respect to all co-ordinate directions. Comparing (1.2) with (1.1) we see that\(^1\)

\[
A_i(y) = \frac{\partial F_i}{\partial \mathbf{u}}, \quad i = 1, \ldots, d.
\]

The majority of numerical methods for hyperbolic systems have been developed with equation (1.2) in mind, including the possibility of shocks, i.e. discontinuous solutions satisfying the conservation laws in integral form (Whitham 1974). Our main application will be visco-elastic fluid flow. Although of course mass, momentum and energy are still conserved, the full system including the stresses cannot be written in conservation form (Hulsen 1986). Whether discontinuous solutions are a physical reality for viscoelastic flows is far from a resolved problem and we will assume that for the general non-conservative system (1.1) the solutions are smooth.

To solve the equations numerically, Phelan et al. (1989) apply a finite difference method that was originally developed for the Euler equations but does not

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\(^1\)We have assumed that \( A_i(y), F_i(y), \) and \( f(y) \) are functions of \( y \) only. An explicit dependence on \( \mathbf{u} \) presents no difficulty in the following as long as the dependence is smooth.
require a conservative form. Unfortunately their method is only a first-order upwinding method, which is known to be much too diffusive. Higher-order methods have been developed (see, for example, Hirsch 1990), but typically within the frame of conservative systems. These schemes achieve to be of higher order by considering the variables in multiple neighbouring cells. A different approach towards higher-order methods is followed by Cockburn & Shu (1989). They use the discontinuous Galerkin method with higher-order polynomials. Since their work is still within the framework of conservative hyperbolic systems, Hulsen (1992) has adapted the method to non-conservative systems with smooth solutions. We should note, however, that the equations for viscoelastic flow in the usual form are not fully hyperbolic because

1. The flow is assumed to be incompressible, i.e. the compression wave speed is assumed to be $\infty$. Only the introduction of a weak (artificial) compressibility resolves this.

2. In many cases a solvent viscosity is present which introduces second order terms (diffusion) in the momentum equation.

Both these problems need special attention and will be treated in this report.

As we will show in chapter 4 the method only seems to be useful in channel/pipe flows. In more complex flows, such as the flow around a sphere, the method is numerically stable only for very small Weissenberg numbers. Since the original DG method (Fortin & Fortin 1989) is much more stable in these cases, further developments will be based on the original DG method. The main reason for documenting the split DG method here that it has been extensively used for transition in channel flows by Draad (1996).
Chapter 2

Governing equations and boundary conditions

The basic equations describing the flow of viscoelastic fluids consist of the basic laws of continuum mechanics and the constitutive equation describing a particular fluid. To simplify the basic equations, we assume that the flow is incompressible and isothermal. The last assumption means that we do not have to consider the energy equation.

2.1 Momentum equation and continuity equation

The balance of linear momentum and the conservation of mass in a fixed bounded space $\Omega$ become

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \text{grad} \mathbf{u} + \text{grad} p = \rho \mathbf{f} + \text{div} \mathbf{t},$$  \hspace{1cm} (2.1)$$

$$\text{div} \mathbf{u} = 0,$$  \hspace{1cm} (2.2)

where $\rho$ is the density, $\mathbf{u}$ the velocity vector, $\mathbf{f}$ a body force vector per unit mass, $p$ the pressure and $\mathbf{t}$ is the extra-stress tensor. The tensor $\mathbf{t}$ is symmetrical and vanishes in equilibrium.

The extra-stress tensor $\mathbf{t}$ is determined by the deformation history of a material particle and has to be specified by the constitutive equation (model) of the particular fluid. Many models have been proposed in the literature. For an overview of various models we refer to the books of Tanner (1985), Bird et al. (1987) and Larson (1988) and the review article by Bird & Wiest (1995).

2.2 Constitutive models

All the models we will consider in the following have a viscous term and are of the form

$$\mathbf{t} = 2\eta_s \mathbf{d} + \mathbf{\tau},$$  \hspace{1cm} (2.3)

where $\mathbf{d} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$, with $\mathbf{L}^T = \text{grad} \mathbf{u}$, is Euler’s rate-of-deformation tensor and $\eta_s$ is the extra-viscosity, which may be zero. The stress tensor $\mathbf{\tau}$ is either given by differential equations (differential model) or by an integral equation (integral model). We will not consider integral models in this report.

The differential models are given by superposition of sub-stresses $\mathbf{\tau}_m$

$$\mathbf{\tau} = \sum_{m=1}^{M} \mathbf{\tau}_m,$$  \hspace{1cm} (2.4)

where $M$ is the number of modes. The sub-stresses $\mathbf{\tau}_m$, $m = 1, \ldots, M$ satisfy the following differential equations

$$\lambda_m \frac{\partial \mathbf{\tau}_m}{\partial t} + f_m(\mathbf{\tau}_m) = 2\eta_m \mathbf{d},$$  \hspace{1cm} (2.5)

where $\lambda_m$ and $\eta_m$ are the relaxation time and viscosity parameter of mode $m$ respectively and $(\stackrel{\circ}{a})$ denotes the upper-convective derivative of a tensor $\mathbf{a}$

$$\stackrel{\circ}{a} = \frac{\partial \mathbf{a}}{\partial t} + \mathbf{u} \cdot \text{grad} \mathbf{a} - \mathbf{L} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{L}^T.$$  \hspace{1cm} (2.6)
The function $f_m$ is an isotropic tensor function
\[
f_m(a) = f_{1m}(I_1, I_2, I_3)1 + f_{2m}(I_1, I_2, I_3)a + f_{3m}(I_1, I_2, I_3)a^2, \tag{2.7}
\]
where $f_{im}$, $i = 1, 2, 3$ are functions of the invariants $I_1, I_2, I_3$ of $a$, given by
\[
I_1 = \text{tr} a, \quad I_2 = \frac{1}{2}[I_1^2 - \text{tr}(a^2)], \quad I_3 = \det a. \tag{2.8}
\]

Examples of differential models are\(^1\) (Larson 1988)
- Upper convected Maxwell: $\eta_s = 0$, $f_{1m} = f_{3m} = 0$, $f_{2m} = 1$,
- Oldroyd-B: $\eta_s \neq 0$, $f_{1m} = f_{3m} = 0$, $f_{2m} = 1$,
- Giesekus: $f_{1m} = 0$, $f_{2m} = 1$, $f_{3m} = \text{constant},$
- Phan-Thien/Tanner: $f_{1m} = f_{3m} = 0$, $f_{2m} = Y(I_1)$.

In Appendix A we have described these and other models in more detail.

The equations (2.1)–(2.5) form a system of partial differential equations in $u, p$ and $\tau_m$, $m = 1, 2, \ldots, M$, which has to be supplemented by proper initial and boundary conditions. Discussions on the type of the equations and whether the system is a properly posed initial value problem can be found elsewhere (van der Zanden et al. 1985; Joseph et al. 1985; Hulsen 1986; van der Zanden & Hulsen 1988; Hulsen 1988a; Joseph 1990). Hulsen (1988b, 1990) has derived some results on constitutive models that are also relevant to this subject. Boundary and initial conditions will be discussed further in section 2.5.

### 2.3 Alternative formulation of differential models

As described by Hulsen (1990) it is possible to identify configuration tensors $b_m$ in the models given by (2.5) that satisfy the conditions

1. $b_m$ is positive definite,
2. $b_m = 1$ in equilibrium.

The relation between $b_m$ and $\tau_m$ is
\[
\tau_m = G_m(b_m - 1), \quad G_m = \frac{\eta_m}{\lambda_m}. \tag{2.9}
\]

The differential models given by (2.5) now become
\[
\lambda_m \overset{\omega}{b_m} + g_m(b_m) = 0, \tag{2.10}
\]
where
\[
g_m(b_m) = \frac{1}{G_m} f_m(G_m(b_m - 1)). \tag{2.11}
\]

For very fast deformation rates, $\|\lambda_m L\| \gg 1$, the tensors $b_m$ approach the Finger deformation tensor. Equation (2.9) means that the stress-strain relation is given by a neo-Hookean model\(^2\). For small and moderate deformation rates the tensors $b_m$ can be interpreted as a ‘reversible strain’ of mode $m$. In this report the formulation in $b_m$ will be preferred over the $\tau_m$ formulation.

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\(^1\)Model having a Gordon-Schowalter derivative instead of the upper-convected derivative cannot be written in the form (2.5). The same is true for models like Larson’s differential model (Larson 1988) and the FENE-type model of Chilcott & Rallison (1988).

\(^2\)Models having a different non-linear stress-strain law are, for example, Larson’s differential model (Larson 1988) and the FENE-type model of Chilcott & Rallison (1988).
2.4 Eigenvalues and eigenvectors

We now try to write our equations into the form (1.1). Combining the equations of sections 2.1 and 2.2 leads to the following system for \((u, p, b_1, b_2, \ldots, b_M)\)

\[
\frac{\partial u}{\partial t} + u \cdot \text{grad} u + \frac{1}{\rho} \text{grad} p - \frac{1}{\rho} \sum_{m} G_m \text{div} b_m - \nu_s \Delta u - f = 0, \\
\text{div} u = 0, \\
\frac{\partial b_m}{\partial t} + u \cdot \text{grad} b_m - L \cdot b_m - b_m \cdot L^T + g_m(b_m) = 0, \quad m = 1, \ldots, M.
\]

where \(\nu_s = \eta_s/\rho\), the kinematic viscosity of the solvent. It is easy to see that we miss a term like \(\partial p/\partial t\) in order to write system (2.12) into the form (1.1). Therefore we introduce the concept of weak compressibility (Phelan et al. 1989; Edwards & Beris 1990; Joseph 1990) where equation (2.12b) is replaced by an equation for weakly compressible flow. The new system now becomes

\[
\frac{\partial u}{\partial t} + u \cdot \text{grad} u + \frac{1}{\rho} \text{grad} p - \frac{1}{\rho} \sum_{m} G_m \text{div} b_m - \nu_s \Delta u - f = 0, \\
\frac{\partial p}{\partial t} + u \cdot \text{grad} p + \kappa \text{div} u = 0, \\
\frac{\partial b_m}{\partial t} + u \cdot \text{grad} b_m - L \cdot b_m - b_m \cdot L^T + b_m \text{div} u + g_m(b_m) = 0, \quad m = 1, \ldots, M.
\]

where \(\kappa \gg \sum_{m} G_m\) is a very large number: the artificial compression modulus. As suggested by Edwards & Beris (1990) we included an extra term \(b_m \text{div} u\) in (2.13c), which vanishes for incompressible flow.

In the following we take \(\eta_s = 0\) and write (2.13) into the following form

\[
\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} + B(u) \frac{\partial u}{\partial y} + C(u) \frac{\partial u}{\partial z} = \text{L.O.T.},
\]

where L.O.T. represents the lower-order terms and the system vector \(u\) is defined by

\[
u^T = (u, v, w, p, b_1^T, \ldots, b_M^T), \quad \text{with} \quad b^T_m = (b_{xx}, b_{yy}, b_{zz}, b_{xy}, b_{xz}, b_{yz}).
\]

The Cartesian co-ordinates are denoted by \((x, y, z)\). Expressions for the coefficient matrices \(A(u), B(u)\) and \(C(u)\) are given in Appendix B. Next we consider small perturbations around a basic solution \(u^0\):

\[
u = u^0 + \delta u.
\]

Substituting this into (2.14) and discarding \(O(u^2)\) terms, we arrive at

\[
\frac{\partial \delta u}{\partial t} + A(u^0) \frac{\partial \delta u}{\partial x} + B(u^0) \frac{\partial \delta u}{\partial y} + C(u^0) \frac{\partial \delta u}{\partial z} = \text{L.O.T.},
\]

where the lower-order terms linear in \(\delta u\) have been moved to the L.O.T.-terms. In the following we simply abbreviate \(A(u^0)\) by \(A\).

For computation of eigenvalues and eigenvectors we only consider the first-order terms of (2.17) and write down a new system where the L.O.T.-terms have been discarded:

\[
\frac{\partial \delta u}{\partial t} + A \frac{\partial \delta u}{\partial x} + B \frac{\partial \delta u}{\partial y} + C \frac{\partial \delta u}{\partial z} = 0,
\]
Next we look for local plane wave solutions of the form

$$\delta y = U e^{ik(n \cdot x - c t)};$$  \hspace{1cm} (2.19)

where $k$ is the wavenumber, $n$ is the unit vector in the wave direction and $c$ is the wave speed. With ‘local’ we mean that the coefficient matrices in (2.18) are considered to be constant with respect to the wavelength $2\pi/k$. Substitution of (2.19) into (2.18) leads to the following eigenvalue problem: find $U \neq 0$ such that

$$(K_n - c I)U = 0,$$  \hspace{1cm} (2.20)

with

$$K_n = n_x A + n_y B + n_z C,$$  \hspace{1cm} (2.21)

where $n_x$, $n_y$ and $n_z$ are the components of the vector $n$ in the Cartesian $(x, y, z)$ system. The solution for $c$ can be found from

$$\det(K_n - c I) = 0.$$  \hspace{1cm} (2.22)

Note, that the matrix $K_n$, and thus the eigenvalues and eigenvectors, depend on the wave direction $n$. Developing the determinant leads to the following equation (see Appendix B)

$$\lambda^{6M-2}(\lambda^2 - c_s^2)^2(\lambda^2 - c_c^2) = 0,$$  \hspace{1cm} (2.23)

where $\lambda = u \cdot n - c$ and the shear and compression wave speeds $c_s$ and $c_c$, are given by

$$c_s^2 = \frac{1}{\rho} \sum_m G_m b^{(m)}_{nn},$$  \hspace{1cm} (2.24)

$$c_c^2 = \kappa/\rho + \frac{1}{\rho} \sum_m G_m b^{(m)}_{nn},$$  \hspace{1cm} (2.25)

with $b^{(m)}_{nn} = n \cdot b_m \cdot n$. From equation (2.23) we obtain $6M + 4$ solutions for $c$

$$c = u \cdot n, \quad \text{with multiplicity } 6M - 2,$$  \hspace{1cm} (2.26)

$$c = u \cdot n \pm c_s, \quad \text{with multiplicity } 2,$$  \hspace{1cm} (2.27)

$$c = u \cdot n \pm c_c, \quad \text{with multiplicity } 1.$$  \hspace{1cm} (2.28)

Because $b_m$ is positive definite (see section 2.3), we have $b^{(m)}_{nn} > 0$ and all the solutions for the wave speed $c$ are real for all $n$. This also means that system (2.18) is hyperbolic. The eigenvalues given above correspond to convection, shear waves and compression waves, respectively. In two dimensions the $3M + 3$ solution vector is given by $y = (u, v, p, b_1^T, \ldots, b_M^T)$ with $b_m^T = (b_{xx}, b_{yy}, b_{xy})$; the expressions for the eigenvalues are identical except the multiplicity of the convection, shear wave and compression wave eigenvalues are $3M - 1, 1$, and $1$, respectively. In the following we denote the dimension of $y$ by $n_y$, which is $6M + 4$ in 3D and $3M + 3$ in 2D.

For each of the eigenvalues $c_i = 1, \ldots, n_y$ of the matrix $K_n$ we define the left $l_i$ and right eigenvectors $r_j$ as follows

$$K_n^T l_i = c_i l_i,$$  \hspace{1cm} (2.29)

$$K_n r_i = c_i r_i.$$  \hspace{1cm} (2.30)

The eigenvectors $l_i$ and $r_i$ can be chosen such that (Hirsch 1990)

$$l_i^T r_j = 0 \quad \text{for } i \neq j.$$  \hspace{1cm} (2.31)
See Appendix B for an expression of the eigenvectors.

Any vector \( \mathbf{a} \) can now be decomposed into the base of \( r_i \) or \( l_i \) vectors

\[
\mathbf{a} = \sum_{i=1}^{m} \mathbf{a}^T r_i r_i^T \mathbf{c}_i,
\]

\( (2.32) \)

This is called characteristic decomposition of the vector \( \mathbf{a} \) with respect to the right and left eigenvectors respectively. The right eigenvector decomposition will be the most important for us and the quantity \( w_i = l_i^T \mathbf{a} \) is called the characteristic variable corresponding to eigenvalue \( c_i \).

We have only considered the first-order terms of the system (2.17). If the lower-order terms are taken into account, the system is still called hyperbolic but waves are damped or amplified independent of the wave length. Whenever \( \nu_s \neq 0 \), the type of the system changes to mixed parabolic-hyperbolic type (van der Zanden & Hulsen 1988) and the wave analysis given above is not applicable anymore. However, when \( \nu_s \) is small only short wave lengths are strongly damped and the wave analysis is still useful for the lower wave lengths (Joseph 1990).

### 2.5 Boundary and initial conditions

For the hyperbolic system \( (\nu_s = 0) \) the characteristic components given in (2.32) can be used to say something about the boundary conditions (b.c.) that are required. When we consider (2.18) on a boundary with outward normal \( \mathbf{n} \), the incoming plane waves in the direction \( \mathbf{n} \) are important. Writing (2.18) locally on the boundary we have for the plane waves

\[
\frac{\partial \delta u}{\partial t} + K_n \frac{\partial \delta u}{\partial n} = 0.
\]

(2.34)

After multiplying this equation with \( l_i^T \) we get

\[
\frac{\partial \delta w_i}{\partial t} + c_i \frac{\partial \delta w_i}{\partial n} = 0, \quad i = 1, \ldots, n_v.
\]

(2.35)

with \( \delta w_i = l_i^T \delta u \). These equations become decoupled in the characteristic system. Whenever there is an incoming wave \( c_i < 0 \), an entry conditions has to be known for \( \delta w_i \). This is discussed further in Appendix C. The results are summarised in the following.

Hereafter the \( (x, y, x) \) co-ordinates are transformed such that the positive \( x \) directions corresponds to the direction of \( \mathbf{n} \). We divide the boundary into the following subtypes:

- a. \( u < -c_s \): supercritical inflow,
- b. \( -c_s \leq u \leq c_s, \ u < 0 \): subcritical inflow,
- c. \( -c_s \leq u \leq c_s, \ u \geq 0 \): subcritical outflow,
- d. \( u > c_s \), supercritical outflow.

Correct boundary conditions are as follows

- On the complete boundary, either \( u \) or \( \sigma_{xx} = -p + \tau_{xx} \) can be prescribed.
- For the different boundary types we have to prescribe
a. \( v, w, b_m, m = 1, \ldots, M \), for supercritical inflow,
b. \( b_m, m = 1, \ldots, M \) for subcritical inflow,
c. \( v \) or \( \tau_{xy} \) or \( w \) or \( \tau_{xz} \) for subcritical outflow,
d. none for supercritical outflow.

Other combinations of b.c. are possible but we will not discuss these. On a supercritical inflow boundary all convection and shear waves are going in, i.e. all variables must be prescribed: tangential velocities and the complete fluid memory \( b_m, 1, \ldots, M \). On an subcritical inflow boundary we lose the possibility to prescribe the tangential velocities\(^3\) \( u, w \) and only the fluid memory \( b_m, 1, \ldots, M \) can be prescribed. In two-dimensional flows the b.c. are very similar except that the variables \( w \) and \( \tau_{xz} \) disappear.

For the parabolic-hyperbolic system \((\nu_s = 0)\) less restrictions exist. For example the following b.c. are possible

- \( u \) or \( \sigma = -p + t_{xx} \) on the complete boundary,
- \( v \) or \( t_{xy} \) and \( w \) or \( t_{xz} \) on the complete boundary,
- \( b_m, m = 1, \ldots, M \) on an inflow boundary \( u < 0 \).

If \( \nu_s \) is small these b.c. may lead to steep boundary layers problems in order to adapt to the prescribed variables that are not allowed in the hyperbolic case \( \nu_s = 0 \).

For either \( \nu_s = 0 \) or \( \nu_s \neq 0 \) it is assumed that proper initial conditions are: specification at the initial time of

- \( u \), fulfilling \( \text{div} \ u = 0 \),
- \( b_m, m = 1, \ldots, M \),

for all \( x \in \Omega \).

\(^3\)Characteristic variables of the shear wave consist of a combination of tangential velocities and shear stresses. We could have chosen to prescribe \( v \) and \( w \) but this would mean that the fluid memory \( b_m, 1, \ldots, M \) could not be fully prescribed and be constrained in a rather complicated way.
Chapter 3

Numerical methods

3.1 Introduction


The method has been introduced successfully into the field of viscoelastic flow by Fortin & Fortin (1989) for upwinding the advection term in the constitutive equation using quadratic elements. Others have used the DG method in various forms for solving viscoelastic flow problems (see, for example, Fortin & Fortin 1990, Basombrío et al. 1991, Fortin & Zine 1992, Fortin et al. 1992, Baaijens 1994a).

The DG method together with explicit Runge-Kutta methods for the time-discretisation has been applied to conservative hyperbolic systems by Cockburn and his coworkers (Cockburn & Shu 1989; Cockburn et al. 1989; Cockburn et al. 1990). They also introduce the concepts of characteristic decomposition and splitting of element boundary fluxes and local projection limiters for resolving sharp gradients, such as shocks, without oscillations. Their system of equations is of the conservative form (1.2). Hulsen (1992) has extended the method of splitting for DG methods to non-conservative systems. In this report we will discuss the application to visco-elastic fluid flows. Limiters are not considered here. Since sharp stress gradients are possible in viscoelastic flows (stress boundary layers, sharp corners) it may however appear to be necessary to introduce limiters for these problems too (Baaijens 1994b; Baaijens 1995).

3.2 A DG-method for hyperbolic systems

Before applying the DG method to viscoelastic flow we briefly recall the method introduced by Hulsen (1992) for non-conservative systems.

3.2.1 Discretisation of the weak form

The weak form of the system (1.1) is obtained by multiplication with a test function \( v(x) \), and integration over the domain \( \Omega \)

\[
\int_{\Omega} v^T \left( \frac{\partial u}{\partial t} + \sum_{i=1}^{d} A_i(u) \frac{\partial u}{\partial x_i} + f(u) \right) \, d\Omega = 0 \quad \text{for all} \quad v \in V, \quad (3.1)
\]

where \( V \) is a suitable function space for both \( u \) and \( v \), for example \( (H^1(\Omega))^{n_v} \).

We divide the domain \( \Omega \) into finite elements \( \Omega = \bigcup_{\varepsilon=1}^{n_\varepsilon} \Omega_\varepsilon \). On each element \( \Omega_\varepsilon \), we approximate \( u \) by \( u_h \), which consists of polynomials of the order \( \ell \). The approximation \( u_h \) is allowed to be discontinuous across element boundaries. We will denote this space by \( V_h^{(e)} \). Note that \( V_h^{(e)} \subset (L^2(\Omega))^{n_v} \).

We approximate the space \( V \) by \( V_h^{(e)} \). Note that \( V_h^{(e)} \) is only an approximation
of the space $V$ on element level\(^1\). If the system can be written in a conservative form, it is possible to integrate partially on element level. The boundary integrals can be used then to define weak element boundary conditions based on fluxes of neighbouring elements (Cockburn & Shu 1989; Hulsen 1992). For non-conservative systems this is not possible and we have to proceed differently.

If we substitute the approximations $y_h$, $u_h$ into (3.1) the integral has to be split into a sum over element integrals and a sum over element boundary integrals. The latter terms appear, because on the element boundaries the normal component of $\partial u_h/\partial x_i$, $i = 1, \ldots, d$

$$\frac{\partial u}{\partial n} = \sum_{i=1}^{d} n_i \frac{\partial u_h}{\partial x_i},$$

(3.2)

is infinite\(^2\). We find

$$\sum_{e=1}^{n_e} \left( \int_{\Omega_e} \psi_h^T \frac{\partial u_h}{\partial t} + \sum_{i=1}^{d} A_i(y_h) \frac{\partial u_h}{\partial x_i} - f(y_h) \right) d\Omega_e + \int_{\gamma_h} \psi_h^T \Delta_n d\gamma = 0,$$

(3.3)

where $\gamma_h = \bigcup_{e=1}^{n_e} \partial \Omega_e$, which consists of all element boundaries\(^3\), and $\Delta_n$ is given by

$$\Delta_n = \int_{n^n}^{n^-} \sum_{i=1}^{d} A_i(y_h) \frac{\partial u_h}{\partial x_i} dn = \int_{n^n}^{n^-} \sum_{i=1}^{d} A_i(y_h) n_i \frac{\partial u_h}{\partial n} dn$$

$$= \int_{n^-}^{n^+} \sum_{i=1}^{d} n_i A_i(y) du = \int_{n^-}^{n^+} K_n(y) du,$$

(3.4)

with $n$ a coordinate in the direction of the unit normal vector\(^4\) $n$ on $\gamma_h$; the indices $-$ and $+$ denote at the ‘backside’ and ‘frontside’ of the vector $n$ and the matrix $K_n$ is given by

$$K_n(y) = \sum_{i=1}^{d} n_i A_i(y).$$

(3.5)

Note that in (3.3) $\psi_h$ is still undefined on $\gamma_h$.

If $A_i(y) = \partial F_i(y)/\partial x_i$ (i.e. the system is derivable from a conservative system) we find that

$$\Delta_n = \sum_{i=1}^{d} \int_{n^-}^{n^+} n_i dF_i(y) = \sum_{i=1}^{d} n_i \left( F_i(y^+) - F_i(y^-) \right),$$

(3.6)

For non-conservative systems $\Delta_n$ depends on the path\(^5\) of $y : y^- \to y^+$. However, since we have assumed that the solution $y$ is smooth, and thus $[y] \to 0$ for $h \to 0$, we can always approximate the integral by

$$\Delta_n = H_n(y^-, y^+) [y] + \mathcal{O}([y]^2),$$

(3.7)

where $H_n(a, b) = K_n(a)$ and $[y] = y^+ - y^-$. An obvious choice is

$$H_n(a, b) = K_n \left( \frac{1}{2} (a + b) \right),$$

(3.8)

---

\(^1\) $V_h^{(e)}$ is not a real approximating space of $V$ since $V_h^{(e)}$ is not a subspace of $V$. The space $V_h^{(e)}$ is an approximating space of $(L^2(\Omega))^{n_e}$.

\(^2\) The tangential derivatives of $u$ are finite, although they may jump across the element boundary.

\(^3\) If an element boundary is internal to the domain $\Omega$, i.e. it is connected to an element on both sides, it is considered only once in the integral over $\gamma_h$.

\(^4\) The vector $n$ is not unique. It may point to either side of $\gamma_h$.

\(^5\) We mean the path in $y$-space and not $\mathbb{R}^d$. 
but others are possible. Note that for the conservative system this choice would be $O(|u|^3)$ and exact for linear $K_n(u)$. We drop the term $O(|u|^2)$ and obtain

$$
\sum_{e=1}^{n_e} \left( \int_{\Omega_e} \psi_h^T \left( \frac{\partial u_h}{\partial t} + \sum_{i=1}^{d} A_i(u_h) \frac{\partial u_h}{\partial x_i} + f(u_h) \right) d\Omega \right) + \int_{\gamma_h} \psi_h^T H_n(u^-, u^+) [u] d\gamma = 0. \quad (3.9)
$$

We still have to choose $\psi_h$ on $\gamma_h$. This choice will be based on characteristic decomposition of $H_n(u^-, u^+)$.  

### 3.2.2 Characteristic decomposition and splitting

We define for $i = 1, 2, \ldots, n_v$

- $c_i$: eigenvalues of $H_n(u^-, u^+)$,
- $\xi_i$: right eigenvectors of $H_n(u^-, u^+)$,
- $l_i$: left eigenvectors of $H_n(u^-, u^+)$,

where $H_n(u^-, u^+)$ is given by (3.8). We decompose $[u]$ into characteristic components according to (2.32)

$$
[u] = \sum_{i=1}^{n_v} l_i^T[u] l_i^T \xi_i. \quad (3.10)
$$

Substitution of (3.10) into (3.9) and using $H_n(u^-, u^+) \xi_i = c_i \xi_i$ we obtain for the integral on the boundary

$$
\int_{\gamma_h} \psi_h^T H_n(u^-, u^+) [u] d\gamma = \int_{\gamma_h} \psi_h^T \sum_{i=1}^{n_v} l_i^T[u] l_i^T c_i \xi_i d\gamma = \int_{\gamma_h} \sum_{i=1}^{n_v} v_i h [u] c_i d\gamma, \quad (3.11)
$$

where $v_i h$ is the characteristic component with respect to the $l_i$ base of $\psi_h$: $v_i h = \psi_h^T \xi_i$. Now we take

$$
v_i h = \begin{cases} 
v_i h^-, & \text{if } c_i < 0, \\
v_i h^+, & \text{if } c_i > 0. \end{cases} \quad (3.12)
$$

Reorganising (3.9) with (3.11) and (3.12) into a sum over elements and considering each element independently we obtain the method: find for $e = 1, 2, \ldots, n_e$, $u_h \in V_h^{(e)}$ such that

$$
\int_{\Omega_e} \psi_h^T \left( \frac{\partial u_h}{\partial t} + \sum_{i=1}^{d} A_i(u_h) \frac{\partial u_h}{\partial x_i} - f(u) \right) d\Omega \\
+ \sum_{(in)} \int_{\partial\Omega_e} \psi_h^T l_i [u] l_i^T \xi_i d\gamma = 0 \quad \text{for all } u_h \in V_h^{(e)}, \quad (3.13)
$$

where

$$
\sum_{(in)} = \text{sum over all } i \text{ with } c_i < 0, \quad (3.14)
$$

11
and \( \mathbf{n} \) is the outward\(^6\) normal on \( \partial \Omega \), which is needed to define \([\mathbf{u}]\) and the positive direction for \( K_n \) and \( c_i \).

The resulting discretized system can be written in the form

\[
M \dot{\mathbf{U}} + \mathbf{S}(\mathbf{U}, t) = 0, \tag{3.15}
\]

where \( M \) is the mass matrix and \( \mathbf{U} \) is a vector containing all unknowns. The coupling between elements only exists through the boundary integral terms. This means that the mass matrix \( M \) can be inverted on element level and the system may be explicitly written as

\[
\dot{\mathbf{U}} = \mathbf{G}(\mathbf{U}, t). \tag{3.16}
\]

The structure of the system differs from standard finite elements because \([\mathbf{u}]\) on the element boundary contains information from neighbouring elements. According to Cockburn et al. (1990), the volume integrals and boundary integrals have to be evaluated by numerical integration rules that integrate polynomials up to the order \( 2\ell \) and \( 2\ell + 1 \) exactly, respectively, where \( \ell \) is the order of the polynomial used for \( \mathbf{u} \).

### 3.2.3 Initial and boundary conditions

System (1.1) should be supplemented with proper initial and boundary conditions. The initial condition is given by

\[
\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{3.17}
\]

where \( \mathbf{u}_0(\mathbf{x}) \) is a function that makes sense with respect to the physical system investigated. For example if a component of \( \mathbf{u} \) is the density, the corresponding component of \( \mathbf{u}(\mathbf{x}) \) should be positive. For other systems the conditions may not be so obvious.

For \( t \in (0, T) \) boundary conditions have to be given on \( \partial \Omega \). Consider the matrix \( K_n(\mathbf{u}) \) on \( \partial \Omega \) as given by (3.5) with \( \mathbf{n} \) the outward normal. We define again \( c_i, \bar{c}_i, l_i, i = 1, \ldots, n_v \) as the eigenvalues, right and left eigenvectors of \( K_n(\mathbf{u}) \). If at point \( \mathbf{x}_p \) on the boundary

\[
c_1 \leq \cdots \leq c_q < 0 < c_{q+1} \leq \cdots \leq c_{n_v}, \tag{3.18}
\]

then proper boundary conditions at \( \mathbf{x}_p \) are given by (Cockburn \& Shu 1989)

\[
\begin{pmatrix}
  w_1(\mathbf{x}_p, t) \\
  \vdots \\
  w_q(\mathbf{x}_p, t)
\end{pmatrix} = B(\mathbf{x}_p, t) \begin{pmatrix}
  w_{q+1}(\mathbf{x}_p, t) \\
  \vdots \\
  w_{n_v}(\mathbf{x}_p, t)
\end{pmatrix} + \mathbf{g}(\mathbf{x}_p, t), \tag{3.19}
\]

where \( B \) is a \( q \times (m - q) \) matrix, \( \mathbf{g} \) a vector of length \( q \) and the characteristic variables \( w_i \) are given by

\[
w_i = l_i^T \mathbf{u}, \quad i = 1, \ldots, n_v. \tag{3.20}
\]

Implementation of (3.19) in our scheme (3.13) is easy and given by Cockburn et al. (1989) for a one-dimensional system but equally applies to more-dimensional systems as well. We define

\[
w_i^+ = l_i^T \mathbf{u}^+, \quad w_i^- = l_i^T \mathbf{u}^-, \tag{3.21}
\]

\(^6\)In (3.9) \( \mathbf{n} \) is inward to one element and outward to the other element sharing the same side. In the change to the ‘per element’ formulation (3.13), we choose the outward normal and thus \( \mathbf{n} \) is opposite in the two elements.
and implement (3.19) as follows

\[
\begin{pmatrix}
    w_1^+ \\
    \vdots \\
    w_q^+ \\
    w_{q+1}^+ \\
    \vdots \\
    w_{n_v}^+
\end{pmatrix}
= B \begin{pmatrix}
    w_1^{+1} \\
    \vdots \\
    w_q^{+1} \\
    w_{q+1}^{+1} \\
    \vdots \\
    w_{n_v}^{+1}
\end{pmatrix} + q_i,
\] (3.22)

\[w_i^+ = w^-_i, \quad \text{for } i = q + 1, \ldots, n_v.\] (3.23)

Note that (3.23) means that the plus signs in the right-hand side of (3.22) may be changed to a minus sign. From \(f'[u] = [w_i]\) the boundary integral in (3.13) can be computed on the complete boundary.

### 3.2.4 The hyperbolic visco-elastic system

With the results of Appendix B we can directly apply the method given by (3.13) to the hyperbolic visco-elastic system (2.13). From (B.18) we find that the characteristic decomposition of \([u]\) is given by

\[\begin{align*}
[u] &= \sum_{i=1}^{6} l_i [u] r_i + [u]_{\text{conv}}, \quad (3.24)
\end{align*}\]

where expressions for \(l_i, r_i\) can be found in Appendix B. An expression for \([u]_{\text{conv}}\) can be obtained from (B.33) by replacing \(\delta u\) by \([u]\). The eigenvalues are

\[c_1 = u \cdot n + c_c\]
\[c_2 = u \cdot n - c_c\]
\[c_3 = c_5 = u \cdot n + c_s\]
\[c_4 = c_6 = u \cdot n - c_s\]
\[c_{\text{conv}} = u \cdot n\]

where \(c_{\text{conv}}\) is the eigenvalue of \([u]_{\text{conv}}\).

A disadvantage of the method is the compressibility of the flow. The value of \(\kappa\) needs to be very large (\(\kappa \gg \sum m \cdot G_m\)) in order to approximate incompressible flow. The discretised equation (3.16) is usually solved by an explicit time integration method and a large value of \(\kappa\) then leads to severe limitations for the time step \(\Delta t\) due to the compression waves:

\[\Delta t < \frac{f h}{c_c}\] (3.25)

where \(f\) is a factor dependent on the order \(\ell\) of the polynomial interpolation and \(h\) is a typical size of the elements. To resolve this problem we will use special elements that eliminate the compression waves all together. This will be the subject of the next section.

### 3.3 A DG-method for an incompressible visco-elastic system

Application of the DG-method directly to the weak compressible viscoelastic system leads to severe time step restrictions due to compression waves. Therefore we want to develop a method for incompressible flow.

#### 3.3.1 The incompressible limit

We want to take the limit \(\kappa \to \infty\) of the method (3.13) with \([u]\) given by (3.24). To find the irregularity in the limit we have to take the limit of

\[c_i l_i [u] r_i, \quad \text{for } i = 1, \ldots, n_v,\] (3.26)

on the element boundaries. We will discuss the three wave types separately.
**Compression waves**

In the limit $\kappa \to \infty$ we have $c_c \to \infty$. This means that $c_1 = u + c_c \to \infty$ and $c_1 = u - c_c \to -\infty$. Hence, we always have an ‘inflow’ and and ‘outflow’ part. We get from (B.22) and (B.23)

$$c_1 \frac{L^T[u]}{L_1^T L_1} = \frac{(u + c_c) (\rho c_c[u] + [p] - [\tau_{xx}])}{2 \rho c_c^2} \begin{pmatrix} c_c \\ 0 \\ 0 \\ \kappa \\ \vdots \\ 0 \end{pmatrix}$$

$$\frac{c_c \gg |u|}{2} \frac{1}{2} \begin{pmatrix} c_c[u] + \frac{[p] - [\tau_{xx}]}{\rho} \\ 0 \\ 0 \\ \kappa \\ \rho c_c \\ \vdots \\ 0 \end{pmatrix}, \quad (3.27)$$

and

$$c_2 \frac{L^T[u]}{L_2^T L_2} = \frac{(u - c_c) (-\rho c_c[u] + [p] - [\tau_{xx}])}{2 \rho c_c^2} \begin{pmatrix} -c_c \\ 0 \\ 0 \\ \kappa \\ \vdots \\ 0 \end{pmatrix}$$

$$\frac{c_c \gg |u|}{2} \frac{1}{2} \begin{pmatrix} -c_c[u] + \frac{[p] - [\tau_{xx}]}{\rho} \\ 0 \\ 0 \\ \kappa \\ \rho c_c \\ \vdots \\ 0 \end{pmatrix}. \quad (3.28)$$

We conclude from the equations given above that in the limit the irregularity comes from the $c_c[u]$ and $\kappa/\rho c_c$ terms.

**Shear waves**

From (B.22) and (B.23) we find that the shear waves are independent of $\kappa$. Hence there is no irregularity in the incompressible limit. The eigenvalues and eigenvectors are given in Appendix B and are not recalled here.
Convection
The limit for $\kappa \to \infty$ is regular and is easy to compute. We find from (B.41) that

$$\sum_{i=1}^{n_v} \frac{\gamma_i}{\rho_i} \mathbf{r}_i = \mathbf{u} \mathbf{u}_{\text{conv}} c_{\text{c}[u]} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tau_{xx} \end{pmatrix} \frac{d_1}{\rho_1} \frac{d_2}{\rho_2} \cdots \frac{d_M}{\rho_M} \mathbf{r}_i,$$

(3.29)

where $d_m, m = 1, \ldots, M$ are given by

$$d_m = \begin{pmatrix} [b_{yy}^{(m)}] - \frac{[b_{xy}^{(m)}] \tau_{xy}}{\rho c_s^2} \\ [b_{xz}^{(m)}] - \frac{[b_{yz}^{(m)}] \tau_{yz}}{\rho c_s^2} \\ [b_{xz}^{(m)}] - \frac{[b_{xx}^{(m)}] \tau_{xx}}{\rho c_s^2} \end{pmatrix}.$$

(3.30)

3.3.2 Removing the irregularities for the incompressible limit
In the previous subsection we observed two irregularity problems in the incompressible limit. One of them is easily solved. In the limit the pressure is not hyperbolic anymore, but becomes a Lagrange multiplier coupled with the $\text{div } \mathbf{v} = 0$ constraint. Therefore we divide the pressure equation in (3.13) by $\kappa$ and take the limit. In fact, we retain equation (2.12b) instead of (2.13b). This is reflected by the observation that the fourth component in (3.29), $\mathbf{u} \tau_{xx}$, also vanishes after dividing by $\kappa$ and taking the limit.

Removing the second irregularity, corresponding to $c_{\text{c}[u]}$, is more difficult. When taking the limit $\kappa \to \infty$ we have $c_{\text{c}} \to \infty$. If we assume that the solutions remain regular in the limit, it means that $u = 0$. If we put this constraint, i.e. the normal velocity is continuous across the element boundaries, directly into the discretisation space $V_h^{(e)}$, we see that the term $c_{\text{c}[u]}$ vanishes in (3.27) and (3.28). Discretisations that fulfil this constraint are the finite element approximations of the space

$$H(\text{div}; \Omega) = \{ \mathbf{q} | \mathbf{q} \in (L^2(\Omega))^d, \text{div } \mathbf{q} \in L^2(\Omega) \},$$

(3.31)

where $d$ is the dimension of the physical space, e.g. $d = 2$ or 3. Various type of approximations are discussed in the book by Brezzi & Fortin (1991). In this report we will use the discretisation spaces $RT_k$ for triangles and $RT_{[k]}$ for quadrilaterals, introduced by Raviart & Thomas (1977). These spaces have been extended to tetrahedral elements by Nedelec (1980).

The $RT_k$ and $RT_{[k]}$ spaces are like the $V_h^{(e)}$ space defined in section 3.2.1, i.e. vector variables $\mathbf{q}$ may be discontinuous across element boundaries, however with one exception: the normal component $\mathbf{q} \cdot \mathbf{n}$ must be continuous. This means that on the element boundary the degrees of freedom consist of these normal components. Depending on the polynomial order there are internal degrees of freedom as well.
If the velocity is discretised by $RT_k$ and the pressure by polynomials of the order \( k \), i.e. the pressure belongs to the space $P_k$, the combined space $RT_k \times P_k$ is a proper mixed discretisation. The same is true for the $RT_k \times Q_k$ combination for quadrilaterals. In practice we work with the reduced spaces $RT_1$ and $RT_1[k]$ where the divergence is a constant per element. With these elements the number of internal degrees of freedom are reduced and only one pressure variable per element remains in the discretised system. Using the penalty method this pressure variable can be eliminated as well.

For extensive discussion of these and other mixed element spaces we refer to the literature cited above and to Roberts & Thomas (1991). In Appendix D we will discuss some properties of the $RT_k$ and $RT_1[k]$ spaces.

### 3.3.3 Discretisation of the weak form

The DG-method (3.13) with the restrictions to incompressible flow as described in the previous subsections now becomes: find $(u_h, p_h, b_1, \ldots, b_M) \in U_h \times Q_h \times (B_h)^M$ such that for all elements $\Omega_e$

\[
\sum_e \left( \int_{\Omega_e} \mathbf{v}_h \cdot \left( \frac{\partial u_h}{\partial t} + \rho u_h \cdot \nabla u_h + \nabla p - \nabla \cdot \mathbf{f} \right) \, d\Omega \right.
\]

\[
+ \frac{1}{2} \int_{\partial \Omega_e} \mathbf{v}_h \cdot (p - \tau_{nn}) \, d\gamma \right) = \text{B.I.T.} = 0,
\]

\[
- \int_{\Omega_e} q_h \nabla \cdot u_h \, d\Omega = 0,
\]

\[
\int_{\Omega_e} \beta_h \left( \frac{\partial b_m}{\partial t} + u_h \cdot \nabla b_m - L \cdot b_m - b_m \cdot L^T + g_m (b_m) \right) \, d\Omega
\]

\[
+ \text{B.I.T.} = 0, \quad m = 1, \ldots, M.
\]

for all $\mathbf{v}_h \in U_h$, $q_h \in Q_h$, and $\beta_h \in B_h$, where the spaces are given by: $U_h = RT_k(K)$ or $RT_k[k]$, $Q_h = P_k(K)$ and $B_h = (P_l(K))^6$. In (3.32) B.I.T. represents boundary integral terms of the eigenvalues $c \in \{ c_3, c_5, c_6, c_{\text{conv}} \}$:

\[
\text{B.I.T.} = \int_{\partial \Omega_e} \mathbf{v}_h \left( \sum_{i=3}^6 \frac{c_i [\mathbf{u}]}{\zeta_i T_i} + c_{\text{conv}} [\mathbf{u}]_{\text{conv}} \right) \, d\gamma,
\]

where

\[
(\ldots)_{\text{in}} = \text{only terms with } c_i < 0 \text{ or } c_{\text{conv}} < 0.
\]

The eigenvalues are

\[
c_3 = c_5 = u \cdot n + c_s
\]

\[
c_4 = c_6 = u \cdot n - c_s
\]

\[
c_{\text{conv}} = u \cdot n
\]

7Other ways of dealing with the incompressibility constraint are:

- Introduction of a stream function (Raviart 1981; Thomasset 1981). This eliminates all pressures.

- Introduction of Lagrange multipliers to impose the continuity of the normal velocities (Brezzi & Fortin 1991; Roberts & Thomas 1991). With these so-called hybrid methods, the velocities and pressures can be eliminated on element level.
and

\[
\begin{align*}
\frac{\mathcal{L}_3[y]}{\mathcal{L}_3} &= \frac{1}{2} \left( \frac{[v]}{c_s} - \frac{[\tau_{xy}]}{\rho c_s^2} \right), \\
\frac{\mathcal{L}_4[y]}{\mathcal{L}_4} &= \frac{1}{2} \left( \frac{[v]}{c_s} - \frac{[\tau_{xy}]}{\rho c_s^2} \right), \\
\frac{\mathcal{L}_5[y]}{\mathcal{L}_5} &= \frac{1}{2} \left( \frac{[w]}{c_s} - \frac{[\tau_{xz}]}{\rho c_s^2} \right), \\
\frac{\mathcal{L}_6[y]}{\mathcal{L}_6} &= \frac{1}{2} \left( -\frac{[w]}{c_s} - \frac{[\tau_{xz}]}{\rho c_s^2} \right).
\end{align*}
\]

(3.35) (3.36) (3.37) (3.38)

The matrix \( R = [r_3, r_4, r_5, r_6] \) is given by

\[
R = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\rho c_s & -\rho c_s & 0 & 0 \\
0 & 0 & \rho c_s & -\rho c_s \\
0 & 0 & 0 & 0 \\
H_1^{xy} & H_1^{xy} & H_1^{xz} & H_1^{xz} \\
H_2^{xy} & H_2^{xy} & H_2^{xz} & H_2^{xz} \\
\vdots & \vdots & \vdots & \vdots \\
H_M^{xy} & H_M^{xy} & H_M^{xz} & H_M^{xz}
\end{pmatrix}
\]

(3.39)

where \( H_m^{xy} \) and \( H_m^{xz} \), \( m = 1, \ldots, M \) are defined by

\[
H_m^{xy} = \begin{pmatrix}
0 \\
-2b_{xy}^{(m)} \\
0 \\
-b_{xx}^{(m)} \\
0
\end{pmatrix}, \quad H_m^{xz} = \begin{pmatrix}
0 \\
-2b_{xz}^{(m)} \\
0 \\
-b_{xx}^{(m)} \\
-b_{xy}^{(m)}
\end{pmatrix}
\]

(3.40)

Furthermore, we have

\[
[u]_{\text{conv}} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
d_1 \\
d_2 \\
\vdots \\
d_M
\end{pmatrix}
\]

(3.41)

where \( d_m \), \( m = 1, \ldots, M \) are given by

\[
d_m = \begin{pmatrix}
[b_{yz}^{(m)}] - b_{xy}^{(m)} [\tau_{xy}] \\
[b_{yz}^{(m)}] - b_{xz}^{(m)} [\tau_{xz}] \\
[b_{yz}^{(m)}] - b_{xx}^{(m)} [\tau_{xy}] \\
[b_{yz}^{(m)}] - b_{xx}^{(m)} [\tau_{xz}]
\end{pmatrix}
\]

(3.42)
In order to obtain a symmetric \((v, p)\) system we partially integrate the \(\text{grad} \, p\) term in (3.32a) on element level and obtain for the pressure terms

\[
\cdots - \sum_e \int_{\Omega_e} p \, \text{div} \, v_h \, d\Omega + \sum_e \int_{\partial\Omega_e} v_h \cdot n \, \left(\frac{1}{2} (p^+ + p^-)\right) \, d\gamma + \cdots \tag{3.43}
\]

Since the normal velocities are continuous across element boundaries the internal boundary integrals cancel out and the pressure terms become

\[
\cdots - \sum_e \int_{\Omega_e} p \, \text{div} \, v_h \, d\Omega + \int_{\partial\Omega} v_h \cdot n \, p^b \, d\gamma + \cdots \tag{3.44}
\]

where \(p^b = \frac{1}{2} (p^+ + p^-)\) is the pressure on the boundary. A similar partial integration can be carried out for the \(\text{div} \, \tau\) term. The terms involving the stress \(\tau\) become

\[
\cdots + \sum_e \int_{\Omega_e} \tau : (\text{grad} \, v_h)^T \, d\Omega + \sum_e \int_{\partial\Omega_e} n \cdot \tau \cdot (v_h - (n \cdot v_h) n) \, d\gamma
\]

\[
- \int_{\partial\Omega} v_h \cdot n \, \tau^b_{nn} \, d\gamma + \cdots, \tag{3.45}
\]

where we have defined the boundary normal stress by \(\tau^b_{nn} = \frac{1}{2} (\tau^+_{nn} + \tau^-_{nn})\). Contrary to \(\text{div} \, v_h\), computing \(\text{grad} \, v_h\) for curved elements is not easy (see Appendix D). Therefore, we integrate back partially and find for the stress terms:

\[
\cdots \sum_e \left( - \int_{\Omega_e} v_h \cdot \text{div} \, \tau \, d\Omega - \frac{1}{2} \int_{\partial\Omega_e}^{\text{int}} v_h \cdot n \, \tau_{nn} \, d\gamma \right)
\]

\[
- \int_{\Omega_e} v_h \cdot n \, (\tau^b_{nn} - \tau^-_{nn}) \, d\gamma + \cdots \tag{3.46}
\]

where \(\partial\Omega_e^{\text{int}}\) denotes internal element boundaries. The weak system of equation now becomes: find \((u_h, p_h, b_1, \ldots, b_M) \in U_h \times Q_h \times (B_h)^M\) such that for all elements \(\Omega_e\)

\[
\sum_e \left( \int_{\Omega_e} v_h \cdot \left( \rho \frac{\partial u_h}{\partial t} + \rho u_h \cdot \text{grad} \, u_h - \text{div} \, \tau - \rho f \right) \, d\Omega - \int_{\Omega_e} p_h \, \text{div} \, v_h \, d\Omega \right.
\]

\[
- \frac{1}{2} \int_{\partial\Omega_e} v_h \cdot n \, (\tau_{nn} \, d\gamma + \text{b.i.t.}) \right) - \int_{\partial\Omega} v_h \cdot n \, (-p^b + \tau^b_{nn} - \tau^-_{nn}) \, d\gamma = 0, \tag{3.47a}
\]

\[
- \int_{\Omega_e} q_h \, \text{div} \, u_h \, d\Omega = 0, \tag{3.47b}
\]

\[
\int_{\Omega} \beta_h : \left( \frac{\partial b_m}{\partial t} + u_h \cdot \text{grad} \, b_m - L \cdot b_m - b_m \cdot L^T + g_m(b_m) \right) \, d\Omega
\]

\[
+ \text{b.i.t.} = 0, \quad m = 1, \ldots, M. \tag{3.47c}
\]

for all \(v_h \in U_h, q_h \in Q_h\), and \(\beta_h \in B_h\), where the spaces are given by: \(U_h = RT_k(K)\) or \(RT^k(K)\), \(Q_h = P_h(K)\) and \(B_h = (P_k(K))^6\). The polynomial order \(\ell\) of the approximation space \(B_h\) for \(b_m\) can be taken independently from \(m\). However, we generally use \(\ell = k\).

If we reduce the space from \(RT_k(K)\) to \(RT^k_k(K)\) for \(u_h\) and \(v_h\) we see that the pressure term in (3.47a) becomes

\[
- \int_{\Omega_e} p_h \, \text{div} \, v_h \, d\Omega = - \text{div} \, v_h \int_{\Omega_e} p_h \, d\Omega \tag{3.48}
\]
which means that of all pressure degrees of freedom, only one per element remains, for example

\[
\bar{p}_h = \frac{\int_{\Omega_h} p_h \, d\Omega}{A_e}, \quad A_e = \int_{\Omega_e} \, d\Omega.
\]  

Equation (3.47b) also reduces to one equation per element, which is a constraint on the normal velocities on the boundary.

After numerical integration of the integrals we obtain a system of the following form

\[
M \ddot{u} + L^T p = R_u(u, b),
\]

\[
L \dot{u} = 0,
\]

\[
\dot{b} = R_b(u, b),
\]

where \( u, p \) and \( b \) are the vectors of unknowns of \( u_h, p_h \) and \( b_m \), respectively. The matrices \( M \) and \( L \) are constant. The coefficient matrices for \( \dot{b} \) have already been inverted on element level. The vector of pressures \( p \) contains the averaged element pressures given by (3.49). These pressures can be eliminated by the penalty method as follows. We penalise the pressure \( \bar{p}_h \) by

\[
\Delta t \epsilon_p (\bar{p}_h - \bar{p}_h^r) + \text{div} \, u_h = 0,
\]

where \( \epsilon_p \) is a small number, \( \Delta t \) is the time step in the time integration scheme (see section 3.6) and \( \bar{p}_h^r \) is a reference pressure, for which we will usually take the pressure at the previous time step. From (3.51) we find that

\[
\Delta t \epsilon_p \int_{\Omega_e} (\bar{p}_h - \bar{p}_h^r) \, d\Omega + \int_{\Omega_e} \text{div} \, u_h \, d\Omega = 0,
\]

which leads to

\[
\Delta t (p - p^r) = \frac{1}{\epsilon_p A_e} \, L \dot{u},
\]

and the system now becomes

\[
M \ddot{u} + \frac{1}{\Delta t} C \dot{u} = R_u(u, \dot{b}) + L^T p^r,
\]

\[
\dot{b} = R_b(u, b),
\]

where \( C \) is defined by

\[
C = \frac{1}{\epsilon_p A_e} L^T L.
\]

### 3.3.4 A remark on boundary conditions

The boundary conditions corresponding to the shear waves and the convection as discussed in section 2.5 and appendix C are quite naturally implemented by the method for the full hyperbolic system given in section 3.2.3. However, the normal velocity/pressure part does not have characteristics anymore and can be handled by the traditional way of treating boundary conditions in finite elements.
3.4 A mixed method for viscous terms

The viscous term \(-\nu \Delta u\) in (2.12a) cannot be discretised with RT\(_k\) or RT\(_{k[\cdot]}\) elements in the usual way. The reason is the discontinuity of tangential velocities across element boundaries: div \(u_h\) is integrable but grad \(u_h\) is not. Therefore we have to use a mixed method as described by Raviart (1981).

The mixed method will be illustrated for the Stokes problem:

\[
\begin{align*}
\text{grad } p - \nu \Delta u &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega,
\end{align*}
\]

with boundary conditions

\[
u u = \bar{u} \quad \text{on } \partial \Omega.
\]

Using
\[
\Delta u = \text{grad}(\text{div } u) - \text{curl curl } u,
\]
and introducing
\[
\omega = \nu \text{curl } u,
\]
we can write (3.56) as follows:

\[
\begin{align*}
\text{grad } p + \text{curl } \omega &= f, \\
\text{div } u &= 0, \\
-\nu^{-1} \omega &= 0.
\end{align*}
\]

The weak form of (3.60) is: find \((u, p, \omega) \in U \times Q \times \Theta\) such that

\[
- \int_{\Omega} p \text{div } v \, dx + \int_{\Omega} v \cdot \omega \, dx = - \int_{\partial \Omega} p v \cdot n \, d\gamma + \int_{\Omega} f \cdot v \, dx, \\
- \int_{\Omega} q \text{div } u \, dx &= 0, \\
\int_{\Omega} u \cdot \text{curl } \theta \, dx - \int_{\Omega} \nu^{-1} \omega \cdot \theta \, dx = - \int_{\partial \Omega} \theta \cdot (n \times u) \, d\gamma,
\]

for all \((v, q, \theta) \in U \times Q \times \Theta\), where \(U = H(\text{div}; \Omega), Q = L^2(\Omega)\) and \(\Theta = (H^1(\Omega))^3\). Note that

1. for the boundary conditions (3.57) the pressure boundary integral in (3.61a) vanishes.
2. in (3.61c) the boundary integral only involves the tangential velocity components. This means that tangential velocities are prescribed weakly, contrary to the normal velocity.

Good mixed methods are given by the following approximation spaces (Raviart 1981)

\[
U_h = RT_k(K), \quad Q_h = P_k, \quad \Theta_h = P_{k+1},
\]

or

\[
U_h = RT_{k[\cdot]}(K), \quad Q_h = Q_k, \quad \Theta_h = Q_{k+1},
\]

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The structure of the discretised systems becomes
\[
\begin{pmatrix}
0 & L^T & B \\
L & 0 & 0 \\
B^T & 0 & -A
\end{pmatrix}
\begin{pmatrix}
u \\ p \\ \omega
\end{pmatrix} =
\begin{pmatrix}
R_u \\
0 \\
R_\omega
\end{pmatrix}
\] (3.62)

If we include viscous terms in the viscoelastic system (3.50) as described above for the Stokes system, we get system of the following form
\[
M \dot{u} + L^T p + B \omega = R_u(u, b),
\] (3.63a)
\[
L \dot{u} = 0,
\] (3.63b)
\[
B^T u - A \omega = R_\omega,
\] (3.63c)
\[
\dot{b} = R_b(u, b).
\] (3.63d)

If we apply the penalty method to eliminate the pressure we get
\[
M \dot{u} + \frac{1}{\Delta t} C u + B \omega = R_u(u, b) + L^T p^*,
\] (3.64a)
\[
B^T u - A \omega = R_\omega,
\] (3.64b)
\[
\dot{b} = R_b(u, b).
\] (3.64c)

Although the mixed method for viscous terms works well, there are some major disadvantages compared to standard Galerkin methods for viscous flow:

a. Using traction boundary conditions is difficult.

b. The number of unknowns is larger, especially in 3D.

c. Even if the viscous terms are treated by explicit time integration a matrix system must be solved for obtaining \( \omega \).

d. It is very difficult to treat viscosities that are not constant, e.g. generalised Newtonian models and temperature dependence. Implementation of other viscous models, such as anisotropic fluids, is also difficult.

Unfortunately however, it is not possible to use standard Galerkin methods with RT elements.

For computing forces and couples on submerged bodies we can use the equations from appendix E.

### 3.5 Numerical quadrature

The integrals (3.47) and (3.61) are integrated numerically by Gaussian quadrature. According to Cockburn et al. (1990) all volume integrals in the DG-method need quadrature that integrates polynomials up to the order \(2\ell\) exactly and boundary integrals up to \(2\ell + 1\). We apply this rule of thumb with \(\ell = k\) for the \(RT_k\) space as well. Therefore we apply it to all integrals in (3.47) and (3.61), except for the integral that leads to the matrix \(A\) in (3.62). The latter integral is integrated with a quadrature rule that integrates polynomials up to \(2(k+1)\) exactly. The quadrature rules that have been used are summarised in table 3.1. For the \(RT_{[k]}\) space we use \((k+1) \times (k+1)\) and \((k+2) \times (k+2)\) point Gauss-Legendre integration respectively.

### 3.6 Time integration

To solve the systems (3.50) and (3.63) we need a discretisation in time. We use explicit methods (Runge-Kutta) and mixed explicit/implicit methods (Karniadakis et al. 1991).
Table 3.1: Number of integration point $n_g$ for the volume integrals of triangular elements. The co-ordinates and weights can be found elsewhere (Dunavant 1985).

<table>
<thead>
<tr>
<th>$k$</th>
<th>$n_g$</th>
<th>$n_g$ for $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>16</td>
</tr>
</tbody>
</table>

3.6.1 Explicit methods

Following Cockburn & Shu (1989) we can solve our systems using explicit Runge-Kutta (RK) methods. In order to do so the systems (3.50) and (3.63) needs to be written in the form (3.16). This can be done for both systems, at least in principle, by elimination of the pressure vector $p$ and, for (3.63), the vorticity vector $\omega$. We will use the following standard RK methods (Ralston & Rabinowitz 1978) for a system of the form (3.16).

**first-order**

$$U_{n+1} = U_n + \Delta t G(U_n, t_n) \quad (=\text{Euler Forward}),$$

(3.65)

**second-order**

$$k_1 = \Delta t G(U_n, t_n),$$

$$k_2 = \Delta t G(U_n + \frac{1}{2}k_1, t_n + \frac{1}{2}\Delta t),$$

$$U_{n+1} = U_n + \frac{1}{2}(k_1 + k_2) \quad (=\text{Heun’s method}),$$

(3.66)

**third-order**

$$k_1 = \Delta t G(U_n, t_n),$$

$$k_2 = \Delta t G(U_n + \frac{1}{2}k_1, t_n + \frac{1}{2}\Delta t),$$

$$k_3 = \Delta t G(U_n + k_2, t_n + \Delta t),$$

$$U_{n+1} = U_n + \frac{1}{2}(k_1 + 2k_2 + 3k_3),$$

(3.67)

**fourth-order**

$$k_1 = \Delta t G(U_n, t_n),$$

$$k_2 = \Delta t G(U_n + \frac{1}{2}k_1, t_n + \frac{1}{2}\Delta t),$$

$$k_3 = \Delta t G(U_n + \frac{1}{2}k_2, t_n + \frac{1}{2}\Delta t),$$

$$k_4 = \Delta t G(U_n + k_3, t_n + \Delta t),$$

$$U_{n+1} = U_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

(3.68)

where $\Delta t$ is the time-step. These are not the only RK methods, many others are possible, but if the system is linear all RK methods of the same order are identical. We will use an RK-method of the same order in time as the order of the space discretisation (= polynomial order +1), i.e. RK-order = $k + 1$.

Since we use the penalty method to eliminate the pressure vector, we do not solve for $\mathbf{u}$, but solve the vector $\mathbf{d} = \mathbf{u}_n + \Delta t \mathbf{u}$ and write (3.50) in the form

$$Md + \Delta t L^T p = \Delta t R_u(u, \mathbf{b}) + My_n,$$

(3.69a)

$$Ld = 0,$$

(3.69b)

$$\mathbf{b} = R_b(u, \mathbf{b}).$$

(3.69c)
Table 3.2: Factors for the critical eigenvalue.

<table>
<thead>
<tr>
<th>k</th>
<th>f_{wave}^k</th>
<th>f_{relax}^k</th>
<th>f_{visc}^k</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>1</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td>12.56</td>
<td>1</td>
<td>170.1</td>
</tr>
<tr>
<td>3</td>
<td>19.50</td>
<td>1</td>
<td>380.2</td>
</tr>
</tbody>
</table>

Now penalising $\text{div} \, d_h$ similar to (3.51), we get

$$\epsilon_p \Delta t (\bar{p}_h - \bar{p}_h') + \text{div} \, d_h = 0,$$

and

$$\Delta t (p - p') = \frac{1}{\epsilon_p A_e \Lambda_\delta},$$

and the system now becomes

$$(M + C) \dot{\delta} = \Delta t R_u(u, \dot{b}) + M \dot{y}_n + \Delta t L^T p',$$

where $C$ is defined by

$$C = \frac{1}{\epsilon_p A_e} L^T L.$$  

For the reference pressure $\bar{p}'$, we use the last computed value of $\bar{p}$. The matrix system is solved by direct matrix methods.

Since the RK methods are explicit, the time step $\Delta t$ has to satisfy a CFL (Courant-Friedrich-Lewy) condition for stability. Following Hulsen (1992), we find that a satisfactory estimate of the critical eigenvalue $\mu_{crit}$ for 2D flows is

$$\mu_{crit} \approx \max_\Omega \mu,$$

with

$$\mu = f_{wave}^k \left( \frac{|u| + c_x}{\delta} \right) + f_{relax}^k \frac{1}{\lambda_{\text{min}}} + f_{visc}^k \left( \frac{1}{h_x^2} \right),$$

where $c_x$ and $c_y$ are the (local) shear wave speeds in $x$ and $y$ direction respectively, $h_x$ and $h_y$ are typical mesh sizes in $x$ and $y$ direction and $\lambda_{\text{min}}$ is the minimum effective relaxation time of all modes:

$$\lambda_{\text{min}} = \min_m \lambda_{m}^{\text{eff}},$$

where $\lambda_{m}^{\text{eff}}$ is given by

$$\lambda_{m}^{\text{eff}} = \frac{\lambda_m}{\text{maximum eigenvalue of } \partial g_m(\dot{\delta})/\partial \dot{\delta}_m}.$$  

and $g_m$ is the ‘vector of components’ of the corresponding tensor $g_m$ in (2.10). The factors $f_{wave}^k$, $f_{relax}^k$ and $f_{visc}^k$ depend on the polynomial order $k$ and are given in the table 3.2. The CFL condition is now given by

$$\Delta t \leq \frac{z_{\text{real}}}{\mu_{crit}},$$

where $-z_{\text{real}}$ is the intersection point of the stability diagram of the RK method with the negative real axis. The values of $z_{\text{real}}$ for the four RK-methods are given in table 3.3. We usually multiply the upper bound in (3.78) by a factor of 0.9 to be on the save side. It is possible to adapt $\Delta t$ at the start of each time step.
Table 3.3: Intersection points of the stability diagram with the negative real axis for RK-methods.

<table>
<thead>
<tr>
<th>z_{real}</th>
</tr>
</thead>
<tbody>
<tr>
<td>RK1</td>
</tr>
<tr>
<td>RK2</td>
</tr>
<tr>
<td>RK3</td>
</tr>
<tr>
<td>RK4</td>
</tr>
</tbody>
</table>

### 3.6.2 Mixed explicit/implicit methods

From (3.74) we see that the stability is determined by the largest eigenvalue. Sometimes this corresponds to a very fast transient behaviour that we’re not interested in. The equations are called stiff. In particular the viscous terms can severely limit the time step. To avoid this it is possible to apply implicit integration of the full system, but this increases the computation time considerably, since all non-linear terms need an iteration process. Furthermore, implementing this is not an easy task for the DG-method, because the structure of the Newton matrices is different from standard finite elements.

A different approach is given by Karniadakis et al. (1991), where some terms are treated explicitly and others, typically the linear viscous terms, implicitly. If the system (3.16) is written as

\[
\begin{align*}
\hat{U} &= G_e(\hat{U}, t) + G_i(\hat{U}, t),
\end{align*}
\]

the \( G_e \) term can be treated explicitly and the \( G_i \) term implicitly. The multi-step method is given by

\[
\begin{align*}
\gamma_0 U_{n+1} - \frac{1}{\Delta t} \sum_{k=0}^{J-1} \alpha_k U_{n-k} &= \sum_{k=0}^{J-1} \beta_k G_e(U_{n-k}) + G_i(U_{n+1}).
\end{align*}
\]

where \( J \) is the order of the method. The coefficients can be found elsewhere (Karniadakis et al. 1991; Hulsen 1996).

The method can be directly applied to the system (3.64), while treating the penalty term and the viscous term implicitly. The left-hand side of the system becomes

\[
\begin{align*}
\begin{pmatrix}
\frac{1}{\Delta t}(\gamma_0 M + C) & B \\
B^T & -A
\end{pmatrix}
\begin{pmatrix}
U_{n+1} \\
\phi_{n+1}
\end{pmatrix} &= \cdots, \quad \gamma_0 b_{n+1} = \cdots,
\end{align*}
\]

where the right-hand side depends on the solutions at former time steps. The matrix system is solved by direct matrix methods.

The stability of the method has been analysed for a one-dimensional linear equation by Hulsen (1996), because the analysis by Karniadakis et al. (1991) is incorrect. By using (3.74) we can extend the stability analysis to our system. It is however not possible to have an adaptive time step strategy for multi-step methods.

### 3.7 Limit behaviour

In this section we consider the following limit behaviour

a. No visco-elasticity: \( G_m \rightarrow 0 \). This leads to either the incompressible Euler or Navier-Stokes equations.

b. Creeping flow: \( \rho \rightarrow 0 \).
c. Small relaxation times: $\lambda_m \to 0$.

d. Large relaxation times: $\lambda_m \to \infty$. This leads to an elastic neo-Hookian solid.

In order to study these limits we write out the following

$$
c_3 \frac{\partial T}{\partial T} \frac{[u]}{[c_3]} = \frac{1}{2} (u + c_s) \left( \frac{[v]}{c_s} - \frac{[\tau_{xy}]}{\rho c_s^2} \right)
= \frac{1}{2} \left( u \frac{[v]}{c_s} - u \frac{[\tau_{xy}]}{\rho c_s^2} + [v] - \frac{[\tau_{xy}]}{\rho c_s} \right),
$$

(3.83)

$$
c_4 \frac{\partial T}{\partial T} \frac{[u]}{[c_4]} = \frac{1}{2} (u - c_s) \left( - \frac{[v]}{c_s} + \frac{[\tau_{xy}]}{\rho c_s^2} \right)
= \frac{1}{2} \left( -u \frac{[v]}{c_s} + u \frac{[\tau_{xy}]}{\rho c_s^2} + [v] + \frac{[\tau_{xy}]}{\rho c_s} \right),
$$

(3.84)

$$
c_5 \frac{\partial T}{\partial T} \frac{[u]}{[c_5]} = \frac{1}{2} (u + c_s) \left( \frac{[w]}{c_s} - \frac{[\tau_{xz}]}{\rho c_s^2} \right)
= \frac{1}{2} \left( u \frac{[w]}{c_s} - u \frac{[\tau_{xz}]}{\rho c_s^2} + [w] - \frac{[\tau_{xz}]}{\rho c_s} \right),
$$

(3.85)

$$
c_6 \frac{\partial T}{\partial T} \frac{[u]}{[c_6]} = \frac{1}{2} (u - c_s) \left( - \frac{[w]}{c_s} - \frac{[\tau_{xz}]}{\rho c_s^2} \right)
= \frac{1}{2} \left( -u \frac{[w]}{c_s} - u \frac{[\tau_{xz}]}{\rho c_s^2} + [w] + \frac{[\tau_{xz}]}{\rho c_s} \right),
$$

(3.86)

and the split vectors

$$
c_3 \frac{\partial T}{\partial T} \frac{[u]}{[c_3]} r_3 = \begin{pmatrix}
\frac{1}{2} \left( \rho u \frac{[v]}{c_s} - u \frac{[\tau_{xy}]}{\rho c_s^2} + \rho c_s [v] - [\tau_{xy}] \right) \\
0 \\
0 \\
0 \\
\frac{1}{2} \left( u \frac{[v]}{c_s} - u \frac{[\tau_{xy}]}{\rho c_s^2} + [v] - \frac{[\tau_{xy}]}{\rho c_s} \right) H^{xy}
\end{pmatrix},
$$

(3.87)

$$
c_4 \frac{\partial T}{\partial T} \frac{[u]}{[c_4]} r_4 = \begin{pmatrix}
\frac{1}{2} \left( \rho u \frac{[v]}{c_s} + u \frac{[\tau_{xy}]}{\rho c_s^2} - \rho c_s [v] - [\tau_{xy}] \right) \\
0 \\
0 \\
0 \\
\frac{1}{2} \left( -u \frac{[v]}{c_s} - u \frac{[\tau_{xy}]}{\rho c_s^2} + [v] + \frac{[\tau_{xy}]}{\rho c_s} \right) H^{xy}
\end{pmatrix},
$$

(3.88)

$$
c_5 \frac{\partial T}{\partial T} \frac{[u]}{[c_5]} r_5 = \begin{pmatrix}
\frac{1}{2} \left( \rho u \frac{[w]}{c_s} - u \frac{[\tau_{xz}]}{\rho c_s^2} + \rho c_s [w] - [\tau_{xz}] \right) \\
0 \\
0 \\
0 \\
\frac{1}{2} \left( u \frac{[w]}{c_s} - u \frac{[\tau_{xz}]}{\rho c_s^2} + [w] - \frac{[\tau_{xz}]}{\rho c_s} \right) H^{xz}
\end{pmatrix},
$$

(3.89)

$$
c_6 \frac{\partial T}{\partial T} \frac{[u]}{[c_6]} r_6 = \begin{pmatrix}
\frac{1}{2} \left( \rho u \frac{[w]}{c_s} + u \frac{[\tau_{xz}]}{\rho c_s^2} - \rho c_s [w] - [\tau_{xz}] \right) \\
0 \\
0 \\
0 \\
\frac{1}{2} \left( -u \frac{[w]}{c_s} + u \frac{[\tau_{xz}]}{\rho c_s^2} + [w] + \frac{[\tau_{xz}]}{\rho c_s} \right) H^{xz}
\end{pmatrix},
$$

(3.90)
where
\[
H^{xy} = \begin{pmatrix}
H_{11}^{xy} \\
H_{12}^{xy} \\
\vdots \\
H_{MM}^{xy}
\end{pmatrix}, \quad H^{xz} = \begin{pmatrix}
H_{11}^{xz} \\
H_{22}^{xz} \\
\vdots \\
H_{MM}^{xz}
\end{pmatrix},
\] (3.91)

The remaining split vector, corresponding to the convection, is \( u|u|_{\text{conv}} \) as given by (3.41) and (3.42).

In supercritical flows, i.e. \(|u| > c_s\), all split vectors are ‘one-sided’ in the upstream direction. This means that all vectors in (3.33) contribute to the same side. Then, we find that
\[
\sum_{i=3}^{6} c_i \frac{T^T[u]}{T^i} c_i + c_{\text{conv}}[u]_{\text{conv}} = \begin{pmatrix}
0 \\
\rho[u] - [\tau_{xy}] \\
\rho[u] - [\tau_{xz}] \\
0
\end{pmatrix}.
\] (3.92)

### 3.7.1 No viscoelasticity

In this limit we have \( G_m \to 0 \). From (2.24) we obtain
\[
c_s \sim \left( \sum_m G_m \right)^\frac{1}{2} \to 0.
\] (3.93)

Furthermore, since \([\tau_{xy}]\) and \([\tau_{xz}]\) scale like \( G_m \), we get
\[
[\tau_{xy}] \to 0, \quad [\tau_{xz}] \to 0, \quad \frac{[\tau_{xy}]}{\rho c_s} \to 0, \quad \frac{[\tau_{xz}]}{\rho c_s} \to 0.
\] (3.94)

We distinguish two different cases:

- **Zero normal velocity**: \( u = 0 \), exactly, for example on a fixed wall or a plane of symmetry. We always have subcritical flow: \( c_s > |u| = 0 \). On an internal boundary with \( u = 0 \) we find that
\[
\sum_{i=3}^{6} c_i \frac{T^T[u]}{T^i} c_i + c_{\text{conv}}[u]_{\text{conv}} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{1}{2}[v] H^{xy} + \frac{1}{2}[w] H^{xz}
\end{pmatrix},
\] (3.95)

On an external boundary with \( u = 0 \) we have only one incoming wave:
\[
\begin{pmatrix}
0 \\
0 \\
0 \\
[\rho c_s][v] H^{xy} + [\rho c_s][w] H^{xz}
\end{pmatrix},
\] (3.96)

where we have used \( \rho c_s[v] = [\tau_{xy}] \) and \( \rho c_s[w] = [\tau_{xz}] \) according the outgoing boundary conditions as discussed in section 3.2.3.

- **Non-zero normal velocity**: \( u \neq 0 \). In this case we have supercritical flow in the limit \( c_s \to 0 \). Hence, in this limit we can use equation (3.92) but now with the results of (3.94):
\[
\sum_{i=3}^{6} c_i \frac{T^T[u]}{T^i} c_i + c_{\text{conv}}[u]_{\text{conv}} \to \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}.
\] (3.97)
We see that the limit $G_m \to 0$ is regular and that the equations for $b_m$ are decoupled from the $(u, p)$ equations. The discretisation of the $(u, p)$ system corresponds to the incompressible Euler equation or to the Navier-Stokes equation if viscous terms are taken into account (see section 3.4). The method for the Navier-Stokes equation is identical to the method discussed by Raviart (1981, pages 108–109) and Thomasset (1981, pages 104–105).

### 3.7.2 Creeping flow

In this limit we have: $\rho \to 0$. Since $G_m$ is finite we find

$$\rho c_s^2 \text{ is finite, } \rho c_s \to 0, \quad c_s \to \infty. \quad (3.98)$$

In the limit we have subcritical flow ($c_s > |u|$), hence on every (internal or external) boundary the shear waves split to both sides. The splitting of the vectors according to (3.87)–(3.90) is singular due to the terms $[\tau_{xy}]/\rho c_s$ and $[\tau_{xz}]/\rho c_s$:

$$c_4 \frac{\partial [u]}{\partial t} \frac{\tau_4 + c_6 \frac{\partial [u]}{\partial t} \tau_6}{\tau_4} \to \begin{pmatrix} 0 \\ -\frac{1}{2} \tau_{xy} \\ -\frac{1}{2} \tau_{xz} \\ 0 \\ \frac{1}{2} (u [\tau_{xy}] \rho c_s^{-1} + [v] + [\tau_{xy}] \rho c_s^{-1}) H^{xy} + \frac{1}{2} (u [\tau_{xz}] \rho c_s^{-1} + [w] + [\tau_{xz}] \rho c_s^{-1}) H^{xz} \end{pmatrix}. \quad (3.99)$$

Due to this singular behaviour it not possible with the split DG-method as used here to describe the limit of creeping flows. A possibility to resolve the singular behaviour is using mixed methods with $[v] = [w] = 0$.

### 3.7.3 Small relaxation times

In this limit we have: $\lambda_m \to 0$. We expect Newtonian behaviour and take $\eta_m$ finite in the limit and find

$$G_m = \frac{\eta_m}{\lambda_m} \to \infty, \quad \rho c_s \to \infty, \quad c_s \to \infty, \quad \rho c_s^2 \to \infty. \quad (3.100)$$

In the limit we have subcritical flow ($c_s > |u|$), hence on every (internal or external) boundary the shear waves split to both sides. The splitting of the vectors according to (3.87)–(3.90) is singular due to the terms $\rho c_s[v]$ and $\rho c_s[w]$:

$$c_4 \frac{\partial [u]}{\partial t} \frac{\tau_4 + c_6 \frac{\partial [u]}{\partial t} \tau_6}{\tau_4} \to \begin{pmatrix} 0 \\ \frac{1}{2} \left( (\rho u[v] - \rho c_s[v] - \tau_{xy}) \right) \\ 0 \\ \frac{1}{2} \left( (\rho u[w] - \rho c_s[w] - \tau_{xz}) \right) \\ \frac{1}{2} [v] H^{xy} + \frac{1}{2} [w] H^{xz} \end{pmatrix}. \quad (3.101)$$

Due to this singular behaviour it not possible with the split DG-method as used here to describe the limit of infinitely small relaxation times. A possibility to resolve the singular behaviour is using standard finite elements with $[v] = [w] = 0$.

### 3.7.4 Large relaxation times

For large relaxation times $\lambda_m \to \infty$ the behaviour becomes equal to an elastic neo-Hookean solid. From equation (2.10) we find that the evolution equation for $b_m$ becomes

$$\ddot{b}_m = -\frac{1}{\lambda_m} g_m(b_m) \to 0, \quad (3.102)$$
which means that

\[
b_1 = b_2 = \cdots = b_M = b = F \cdot F^T,
\]

where \( b \) is the (macroscopic) Finger deformation tensor and \( F \) the deformation gradient. The distinction between modes vanishes and the stress-strain relation is given by (2.4) and (2.9):

\[
\tau = \sum_{m=1}^{M} G_m(b_m - 1) = G(b - 1),
\]

with \( G = \sum_m G_m \).

3.8 Handling negative values of \( c_s^2 \)

Due to numerical errors it is possible that

\[
c_s^2 = \frac{1}{\rho} \sum_m G_m b_m^{(m)} \leq 0.
\]

Within the split DG-method it is not possible to continue, since we cannot compute \( c_s \). If the numerical errors remain very local in space or time, for example a single integration point or time step, we may possibly continue with our computations by taking \( c_s \rightarrow 0 \). Note that in this limit we do not have \( G_m \rightarrow 0 \). We distinguish two different cases

- Zero normal velocity: \( u = 0 \), exactly, for example on a fixed wall or a plane of symmetry. We always have subcritical flow: \( c_s > |u| = 0 \). On an internal boundary with \( u = 0 \) we find that

\[
\begin{align*}
\gamma_3 & \quad \frac{\partial [u]}{\partial x_3} \\
\gamma_4 & \quad \frac{\partial [u]}{\partial x_4} \\
\gamma_5 & \quad \frac{\partial [u]}{\partial x_5} \\
\gamma_6 & \quad \frac{\partial [u]}{\partial x_6}
\end{align*}
\]

\[
\begin{pmatrix}
0 \\
-\frac{1}{2} [\tau_{xy}] \\
0 \\
\frac{1}{2} [v] - \frac{[\tau_{xy}]}{\rho c_s} H_{xy}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
-\frac{1}{2} [\tau_{xy}] \\
0 \\
\frac{1}{2} [v] + \frac{[\tau_{xy}]}{\rho c_s} H_{xy}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
-\frac{1}{2} [\tau_{xz}] \\
0 \\
\frac{1}{2} [w] - \frac{[\tau_{xz}]}{\rho c_s} H_{xz}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
-\frac{1}{2} [\tau_{xz}] \\
0 \\
\frac{1}{2} [w] + \frac{[\tau_{xz}]}{\rho c_s} H_{xz}
\end{pmatrix}
\]

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The terms like \([\tau_{xy}] / \rho c_s\) are singular. Therefore, we do not split these terms and sum the split vectors for these terms so that they vanish. So we end up with

\[
\begin{align*}
&c_4 \frac{[T][u]}{[T]_4} r_4 + c_6 \frac{[T][u]}{[T]_6} r_6 \rightarrow \begin{pmatrix} 0 \\ -\frac{1}{2}[\tau_{xy}] \\ -\frac{1}{2}[\tau_{xz}] \\ 0 \\ \frac{1}{2}[v]H^{xy} + \frac{1}{2}[w]H^{xz} \end{pmatrix}.
\end{align*}
\tag{3.110}
\]

On an external boundary with \(u = 0\) the limit is regular because of the way the boundary conditions are handled

\[
\begin{align*}
&c_4 \frac{[T][u]}{[T]_4} r_4 + c_6 \frac{[T][u]}{[T]_6} r_6 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ [v]H^{xy} + [w]H^{xz} \end{pmatrix},
\end{align*}
\tag{3.111}
\]

where we have used \(\rho c_s[v] = [\tau_{xy}]\) and \(\rho c_s[w] = [\tau_{xz}]\) according the outgoing boundary conditions.

- Non-zero normal velocity: \(u \neq 0\). In this case we have supercritical flow in the limit \(c_s \to 0\). Hence, in this limit all split vectors are one-sided

\[
\sum_{i=3}^{6} c_i \frac{[T][u]}{[T]_i} r_i + c_{\text{conv}}[u]_{\text{conv}} = \begin{pmatrix} 0 \\ \rho u[v] - [\tau_{xy}] \\ \rho u[w] - [\tau_{xz}] \\ 0 \end{pmatrix}. \tag{3.112}
\]

There is a problem with this formulation: the splitting is discontinuous at a change of sign for the normal velocity. This can lead to jumps in the solution. Therefore, we split as follows

\[
\sum_{i=3}^{6} c_i \frac{[T][u]}{[T]_i} r_i + c_{\text{conv}}[u]_{\text{conv}} = \begin{pmatrix} 0 \\ \rho u[v] \\ \rho u[w] \\ u[b] \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2}[\tau_{xy}] \\ -\frac{1}{2}[\tau_{xz}] \\ \frac{1}{2}[v]H^{xy} + \frac{1}{2}[w]H^{xz} \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2}[\tau_{xy}] \\ -\frac{1}{2}[\tau_{xz}] \\ \frac{1}{2}[v]H^{xy} + \frac{1}{2}[w]H^{xz} \end{pmatrix}. \tag{3.113}
\]

The first terms is upwinded in the \(u\)-direction. The last two terms are distributed on either side of an internal boundary. We keep (3.112) for the external boundaries. Note that (3.110) is obtained for \(u = 0\) on an internal boundary.
Chapter 4

Some examples and conclusions

In this chapter we will shortly discuss two flow examples: stability of planar Couette flow and the flow around a sphere. Based on these examples we will make some conclusions about the suitability of the DG method with splitting for computation of viscoelastic flows.

4.1 Stability of planar Couette flow

This test problem has been put forward by Brown et al. (1993). The base flow and the dimensions have been depicted in figure 4.1. The viscoelastic model is the upper-convected Maxwell model (UCM). The dimensionless numbers governing the flow are the Weissenberg number \( We = \lambda \dot{\gamma} \) and the Reynolds number \( Re = \rho U H / \eta \), where \( U = \dot{\gamma} H \).

The time-dependent flow calculations are started from the base flow perturbed with small \((O(10^{-3}))\) random disturbances. It is assumed that the damping of these disturbances is governed by the linear stability. The base flow is linearly stable for any \( We \) if \( Re = 0 \) (Gorodtsov & Leonov 1967) and most probably also for \( Re \neq 0 \) (Renardy & Renardy 1986). Hence any instabilities observed in computing the initial value problem are of a numerical origin.

We have computed the initial value problem on a square domain \( H \times H \) and equal number of quadrilateral elements \( N \times N \) in both space dimensions on an equidistant mesh. We use periodical boundary conditions in \( x \)-direction. The time discretisation is an explicit Runge-Kutta method.

We find a critical Weissenberg number \( We_{\text{crit}} \) above which the flow becomes unstable. In figure 4.2 we have depicted the results for \( We_{\text{crit}} \) as a function of \( N \) and for two values of the polynomial order\(^1\) \( p \). The elasticity number \( E = Re / We = 1 \). Note that the number of unknowns is proportional to approximately \((Np)^2\).

We observe that

- \( We_{\text{crit}} \) is similar to the EVSS method (Brown et al. 1993).
- \( We_{\text{crit}} \) is more or less independent of mesh size (\( h \)-refinement).
- \( p \)-refinement, i.e. increasing \( p \), decreases \( We_{\text{crit}} \). This is unexpected and somewhat disappointing.

The value of \( We_{\text{crit}} \) can be increased by stretching the elements in the direction of the flow \((x)\), which has also been found by Keiller (1992) for a finite difference method. For example using a computational area of \( 10H \times H \) for a \( 3 \times 3 \) mesh increases \( We_{\text{crit}} \) from 3.7 to 7.0. This can be used to good effect in channel type flows as demonstrated by Draad (1996), who has used the methods developed in this report to compute transition in two-dimensional channel flows.

\(^1\)In this chapter we denote the polynomial order by \( p \), which is the notation used in the literature.
\[ u = \gamma y \]

Figure 4.1: Planar Couette base flow

Figure 4.2: Critical Weissenberg number DG-split

The elasticity number \( E = Re / We = 1 \).
4.2 Flow around a sphere falling a tube

The flow around a sphere falling along the axis of a long cylindrical tube is a well-known benchmark (see, for example, Bodart & Crochet 1994). In figure 4.3 we have depicted the geometry for a sphere to tube ratio of 2:1.

We consider the steady flow of an Oldroyd-B model with \( \eta_s/(\eta_s + \eta_1) = 0.1 \). The dimensionless numbers characterising the flow are the Weissenberg number \( We = \lambda U/a \) and the Reynolds number \( Re = \rho U a/(\eta_s + \eta_1) \). In our method it is not possible to use \( Re = 0 \) and therefore have used \( Re = 1 \) in the computations.

In order to compute the flow we use a frame that moves with the sphere. We have used periodical boundary conditions with a period of 30a instead of the usual uniform inflow and outflow boundary conditions. The mesh that is used has a typical radial resolution of 0.03a on the surface of the sphere. The steady flow is found as a limit of a time-dependent calculation.

We find that the flow becomes numerically unstable at relatively low values of \( We \), i.e. \( We_{crit} = 0.1 \) for \( p = 1 \) and \( We_{crit} = 0.2 \) for \( p = 5 \). For these values the elasticity has only a small effect on the flow. We were not able to obtain higher values. With other methods stable values for \( We \) have been obtained that are an order of magnitude higher. For example, we have been able to obtain a \( We_{crit} = 1.5 \) for \( p = 5 \) with the standard DG method of Fortin & Fortin (1989) using the same mesh.

4.3 Conclusions

The DG method based on splitting, as developed in the previous chapter, has only an acceptable stability for channel type flows, but there is no improvement compared to other methods such as EVSS. For flows with complex geometries the method in its
current form is far inferior to existing methods and therefore not a good candidate to compute general flows with viscoelastic fluids. For example, the standard DG method is much better for complex flow geometries. At this moment it is unclear why the splitted method fails for flows with complex geometries.
Appendix A

Visco-elastic fluid models

In this appendix we give some models that satisfy the neo-Hookean form (2.9) and (2.10). For an overview of various models we refer to the books of Tanner (1985), Bird et al. (1987) and Larson (1988) and the review article by Bird & Wiest (1995).

Upper-convected Maxwell ($\eta_s = 0$) and Oldroyd-B ($\eta_s \neq 0$)

\[ \lambda_m b_m + b_m - 1 = 0 \]  
(A.1)

Giesekus

\[ \lambda_m b_m + b_m - 1 + \alpha_m (b_m - 1)^2 = 0 \]  
(A.2)

Leonov 2D

\[ \lambda_m b_m + \frac{1}{2} (b_m^2 - 1) = 0 \]  
(A.3)

Leonov 3D

\[ \lambda_m b_m + \frac{1}{2} \left( b_m^2 - 1 - \frac{1}{3} (I_{1m} - I_{2m}) b_m \right) = 0 \]  
(A.4)

where

\[ I_{1m} = \text{tr} b_m \]  
(A.5)

\[ I_{2m} = \frac{1}{2} \left[ I_{1m}^2 - \text{tr}(b_m^2) \right] = \text{tr} b_m^{-1} \]  
(A.6)

Phan-Thien/Tanner

\[ \lambda_m b_m + Y(\text{tr} b_m)(b_m - 1) = 0 \]  
(A.7)

where

\[ Y(\text{tr} b_m) = \begin{cases} 1 + \epsilon_m (\text{tr} b_m - 3) & \text{linear form} \\ e^{\epsilon_m (\text{tr} b_m - 3)} & \text{exponential form} \end{cases} \]  
(A.8)

Modified Leonov 2D

\[ \lambda_m b_m + \frac{1}{2} \phi(2I_{1m})(b_m^2 - 1) = 0 \]  
(A.9)

where

\[ I_{1m} = \text{tr} b_m \]  
(A.10)

\[ \phi(x) = \left( 1 + \frac{2\alpha_m}{\pi} \arctan \left[ \frac{\beta_m}{4} (x - 6) \right] \right)^{-1} \]  
(A.11)
Modified Leonov 3D

\[ \lambda_m b_m + \frac{1}{2} \phi(I_{1m} + I_{2m}) \left( b_m^2 - 1 - \frac{1}{3}(I_{1m} - I_{2m})b_m \right) = 0 \]  
(A.12)

where

\[ I_{1m} = \text{tr} b_m \]  
(A.13)

\[ I_{2m} = \frac{1}{2} \left[ I_{1m}^2 - \text{tr}(b_m^2) \right] = \text{tr} b_m^{-1} \]  
(A.14)

\[ \phi(x) = \left\{ 1 + \frac{2\alpha_m}{\pi} \text{arctan} \left[ \frac{\beta_m}{4} (x - 6) \right] \right\}^{-1} \]  
(A.15)
Appendix B

Eigenvalues and eigenvectors of the hyperbolic viscoelastic system

In this appendix we will derive the eigenvalues and eigenvectors of the system of equations given by (2.18).

Writing (2.13) into a Cartesian \((x, y, z)\) co-ordinate system we arrive at the following expressions for the coefficient matrices \(A, B\) and \(C\):

\[
A = \begin{pmatrix}
\vdots & : 1/\rho : \\
uI & : 0 & D_1^x & D_2^x & \ldots & D_M^x \\
\vdots & : 0 : \\
kappa & 0 & 0 & \vdots \\
E_1^x & : \\
E_2^x & : uI \\
\vdots & : \\
E_M^x & : 
\end{pmatrix}, \quad \text{ (B.1)}
\]

\[
B = \begin{pmatrix}
\vdots & : 0 : \\
vI & : 1/\rho & D_1^y & D_2^y & \ldots & D_M^y \\
\vdots & : 0 : \\
0 & \kappa & 0 & \vdots \\
E_1^y & : \\
E_2^y & : vI \\
\vdots & : \\
E_M^y & : 
\end{pmatrix}, \quad \text{ (B.2)}
\]
where the matrices $D_m^i$ and $E_m^i$ are given by

\[
D_m^x = -\frac{G_m}{\rho} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}, \quad (B.4)
\]

\[
D_m^y = -\frac{G_m}{\rho} \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}, \quad (B.5)
\]

\[
D_m^z = -\frac{G_m}{\rho} \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}, \quad (B.6)
\]

\[
E_m^x = \begin{pmatrix}
-b_{xx} & 0 & 0 & 0 & 0 & 0 \\
b_{yy} & -2b_{xy} & 0 & 0 & 0 & 0 \\
b_{zz} & 0 & -2b_{xz} & 0 & 0 & 0 \\
0 & -b_{xx} & 0 & 0 & 0 & 0 \\
b_{yz} & -b_{xz} & -b_{yz} & 0 & 0 & 0 \\
\end{pmatrix}_m, \quad (B.7)
\]

\[
E_m^y = \begin{pmatrix}
-2b_{xy} & b_{xx} & 0 & 0 & 0 & 0 \\
0 & -b_{yy} & 0 & 0 & 0 & 0 \\
0 & b_{zz} & -2b_{yz} & 0 & 0 & 0 \\
-b_{yy} & 0 & 0 & 0 & 0 & 0 \\
-b_{yz} & b_{xz} & -b_{xy} & 0 & 0 & 0 \\
\end{pmatrix}_m, \quad (B.8)
\]

\[
E_m^z = \begin{pmatrix}
-2b_{xz} & 0 & b_{xx} & 0 & 0 & 0 \\
0 & -2b_{yz} & 0 & 0 & 0 & 0 \\
-b_{yy} & 0 & 0 & -b_{zz} & 0 & 0 \\
-b_{yz} & b_{xz} & -b_{xy} & 0 & 0 & 0 \\
0 & 0 & b_{yz} & 0 & 0 & 0 \\
\end{pmatrix}_m, \quad (B.9)
\]

Next we want to develop the determinant equation (2.22). For this we need the
matrix \( K_n \) given by (2.21), which leads to the matrix equation

\[
K_n - c I = \begin{pmatrix}
(\mathbf{u} \cdot \mathbf{n} - c) I & \vdots & n_x / \rho & \vdots \\
\vdots & \ddots & D_2 & \cdots & D_M \\
\vdots & \vdots & n_z / \rho & \vdots \\
\kappa n_x & \kappa n_y & \kappa n_z & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
E_1 & \vdots & \vdots & (\mathbf{u} \cdot \mathbf{n} - c) I \\
E_2 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
E_M & \vdots & \vdots & \vdots 
\end{pmatrix},
\]

where

\[
D_m = n_x D_m^x + n_y D_m^y + n_z D_m^z, \\
E_m = n_x E_m^x + n_y E_m^y + n_z E_m^z.
\]

The strategy to develop the determinant of \( K_n - c I \) is

1. Multiply the first 3 rows of (B.10) with \( \lambda = \mathbf{u} \cdot \mathbf{n} - c \), which gives a factor of \( \lambda^{-3} \) in front of the determinant.

2. Use the right-lower diagonal matrix in (B.10) to eliminate the upper-right matrices \( \lambda(n_x / \rho, n_y / \rho, n_z / \rho)^T \) and \( \lambda D_m, \ m = 1, \ldots, M \).

3. Develop the determinant from the lower-right corner which leads to the following 3 \( \times \) 3 determinant

\[
\lambda^{6M-2} \det \left( (\lambda^2 - c_s^2) I - \frac{\kappa}{\rho} \eta n y^T \right),
\]

where \( \eta = (n_x, n_y, n_z)^T \) and \( c_s \) is given by

\[
c_s^2 = \frac{1}{\rho} \sum_m G_m b^{(m)}_{nm},
\]

with \( b^{(m)}_{nm} = \mathbf{n} \cdot \mathbf{b}_m \cdot \mathbf{n} \).

4. Develop the 3 \( \times \) 3 determinant. This leads to

\[
\lambda^{6M-2}(\lambda^2 - c_s^2)^2 (\lambda^2 - c_c^2) = 0,
\]

where \( c_c \) is given by

\[
c_c^2 = \frac{\kappa}{\rho} + \frac{1}{\rho} \sum_m G_m b^{(m)}_{nm}.
\]

Next we want to discuss the eigenvectors and characteristic decomposition. We order the 6\( M + 4 \) eigenvectors \( c_i \) as follows

\[
\mathbf{u} \cdot \mathbf{n} + c_c, \mathbf{u} \cdot \mathbf{n} - c_c, \mathbf{u} \cdot \mathbf{n} + c_s, \mathbf{u} \cdot \mathbf{n} - c_s, \mathbf{u} \cdot \mathbf{n} + c_s, \mathbf{u} \cdot \mathbf{n} - c_s, \mathbf{u} \cdot \mathbf{n}, \ldots, \mathbf{u} \cdot \mathbf{n}.
\]

(B.17)
The characteristic decomposition of a vector $\delta u$ is given by

$$
\delta u = \sum_{i=1}^{m} \frac{l_i^T \delta u}{l_i^T t_i} r_i,
$$

$$
= \sum_{i=1}^{6} \frac{l_i^T \delta u}{l_i^T t_i} r_i + \delta u_{\text{conv}},
$$

where

$$
\delta u_{\text{conv}} = \sum_{i=1}^{m} \frac{l_i^T \delta u}{l_i^T t_i} r_i,
$$

is the part corresponding to the convection. From (B.18) we find that

$$
\delta u_{\text{conv}} = \delta u - \sum_{i=1}^{6} \frac{l_i^T \delta u}{l_i^T t_i} r_i.
$$

Since we are not interested in the individual eigenvectors of the convection, we will only compute $\delta u_{\text{conv}}$ from (B.20). Furthermore, without loss of generality we position the positive $x$-axis in the direction $n$, i.e. we take $K_n = A$, which makes it more easy to compute the eigenvectors. The eigenvectors $l_i, r_i, i = 1, \ldots, 6$ are written in $m \times 6$ matrices:

$$
L = [l_1, \ldots, l_6] \quad \text{and} \quad R = [r_1, \ldots, r_6].
$$

Substitution of the eigenvalues $c_1, \ldots, c_6$ into the eigensystem (2.22) leads to the following matrices $L$ and $R$

$$
L = \begin{pmatrix}
\rho c_c & -\rho c_c & 0 & 0 & 0 & 0 \\
0 & 0 & \rho c_s & -\rho c_s & 0 & 0 \\
0 & 0 & 0 & 0 & \rho c_s & -\rho c_s \\
1 & 1 & 0 & 0 & 0 & 0 \\
F_1 & F_2 & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix},
$$

$$
R = \begin{pmatrix}
c_c & -c_c & 0 & 0 & 0 & 0 \\
0 & 0 & c_s & -c_s & 0 & 0 \\
0 & 0 & 0 & c_s & -c_s & 0 \\
\kappa & \kappa & 0 & 0 & 0 & 0 \\
H_1 & H_2 & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix},
$$

where $F_m$ and $H_m$, $m = 1, \ldots, M$ are defined by

$$
F_m = \begin{pmatrix}
-G_m & -G_m & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -G_m & -G_m & 0 & 0 \\
0 & 0 & 0 & 0 & -G_m & -G_m \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

39
\[ H_m = \begin{pmatrix}
-\frac{b_{xx}^{(m)}}{\rho y_x^{(m)}} & -\frac{b_{xx}^{(m)}}{\rho y_x^{(m)}} & 0 & 0 & 0 & 0 \\
\frac{b_{yy}^{(m)}}{\rho y_x^{(m)}} & \frac{b_{yy}^{(m)}}{\rho y_x^{(m)}} & -2\frac{b_{xy}^{(m)}}{\rho y_x^{(m)}} & -2\frac{b_{xy}^{(m)}}{\rho y_x^{(m)}} & 0 & 0 \\
0 & 0 & -\frac{b_{xx}^{(m)}}{\rho y_x^{(m)}} & -\frac{b_{xx}^{(m)}}{\rho y_x^{(m)}} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{b_{xx}^{(m)}}{\rho y_x^{(m)}} & -\frac{b_{xx}^{(m)}}{\rho y_x^{(m)}} \\
\frac{b_{yy}^{(m)}}{\rho y_x^{(m)}} & \frac{b_{yy}^{(m)}}{\rho y_x^{(m)}} & -\frac{b_{yy}^{(m)}}{\rho y_x^{(m)}} & -\frac{b_{yy}^{(m)}}{\rho y_x^{(m)}} & 0 & 0 \\
\end{pmatrix}. \quad \text{(B.25)}
\]

It is easily verified that the matrix \( L^T R \) is a diagonal matrix. The diagonal components are the inproducts \( l_{i}^{T} r_{i}, \ i = 1, \ldots, 6 \), which are given by

\[
\begin{align*}
\frac{b_{xx}^{(m)}}{\rho y_x^{(m)}} &= \begin{dcases} 
2\rho c_x^2, & \text{for } i = 1, 2 \\
2\rho c_x^2, & \text{for } i = 3, 4, 5, 6 
\end{dcases} \quad \text{(B.26)}
\end{align*}
\]

The characteristic variables \( \delta w_i = l_{i}^{T} \delta u, \ i = 1, \ldots, 6 \) of a vector \( \delta u \) are

\[
\begin{align*}
\delta w_1 &= \rho c_x \delta u + \delta p - \sum_{m=1}^{M} G_m \delta b_{xx}^{(m)}, \\
\delta w_2 &= -\rho c_x \delta u + \delta p - \sum_{m=1}^{M} G_m \delta b_{xx}^{(m)}, \\
\delta w_3 &= \rho c_x \delta v - \sum_{m=1}^{M} G_m \delta b_{xy}^{(m)}, \\
\delta w_4 &= -\rho c_x \delta v - \sum_{m=1}^{M} G_m \delta b_{xy}^{(m)}, \\
\delta w_5 &= \rho c_w \delta w - \sum_{m=1}^{M} G_m \delta b_{xz}^{(m)}, \\
\delta w_6 &= -\rho c_w \delta w - \sum_{m=1}^{M} G_m \delta b_{xz}^{(m)}. \\
\end{align*}
\]

The vector \( \delta u_{\text{conv}} \) can now be calculated from \( \text{(B.20)} \)

\[
\delta u_{\text{conv}} = \begin{pmatrix}
\frac{\delta p - \kappa (\delta p - \sum_{m=1}^{M} G_m \delta b_{xx}^{(m)})}{\rho c_x^2} \\
\delta d_1 \\
\delta d_2 \\
\vdots \\
\delta d_M
\end{pmatrix}, \quad \text{(B.33)}
\]
where $d_i, i = 1, \ldots, M$ are given by

$$d_i = \begin{pmatrix}
\delta b^{(i)}_{xx} + \frac{b_{xx}^{(i)}(\delta p - \sum_m G_m \delta b^{(m)}_{xx})}{\rho c^2_p} \\
\delta b_{yy}^{(i)} - \frac{b_{yy}^{(i)}(\delta p - \sum_m G_m \delta b^{(m)}_{xx})}{\rho c^2_p} - \frac{b_{xy}^{(i)} \sum_m G_m \delta b^{(m)}_{xy}}{\rho c^2_s} \\
\delta b_{zz}^{(i)} - \frac{b_{zz}^{(i)}(\delta p - \sum_m G_m \delta b^{(m)}_{xx})}{\rho c^2_p} - \frac{b_{xz}^{(i)} \sum_m G_m \delta b^{(m)}_{xz}}{\rho c^2_s} \\
\delta b_{xy}^{(i)} - \frac{b_{xy}^{(i)} \sum_m G_m \delta b^{(m)}_{xx}}{\rho c^2_s} \\
\delta b_{xz}^{(i)} - \frac{b_{xz}^{(i)} \sum_m G_m \delta b^{(m)}_{xx}}{\rho c^2_s} \\
\delta b_{yz}^{(i)} - \frac{b_{yz}^{(i)}(\delta p - \sum_m G_m \delta b^{(m)}_{xx})}{\rho c^2_p}
\end{pmatrix}.$$

(B.34)

With help of (2.4) and (2.9) we can rewrite the equations (B.27)–(B.34) as follows

$$\delta w_1 = \rho c_c \delta u + \delta p - \delta \tau_{xx}, \quad (B.35)$$
$$\delta w_2 = -\rho c_c \delta u + \delta p - \delta \tau_{xx}, \quad (B.36)$$
$$\delta w_3 = \rho c_s \delta v - \delta \tau_{yx}, \quad (B.37)$$
$$\delta w_4 = -\rho c_s \delta v - \delta \tau_{yx}, \quad (B.38)$$
$$\delta w_5 = \rho c_s \delta w - \delta \tau_{xz}, \quad (B.39)$$
$$\delta w_6 = -\rho c_s \delta w - \delta \tau_{xz}, \quad (B.40)$$

and

$$\delta y_{\text{conv}} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\kappa(\delta p - \delta \tau_{xx}) \\
\ldots \\
\delta M
\end{pmatrix}, \quad (B.41)$$

where $d_i, i = 1, \ldots, M$ are given by

$$d_i = \begin{pmatrix}
\delta b^{(i)}_{xx} + \frac{b_{xx}^{(i)}(\delta p - \delta \tau_{xx})}{\rho c^2_p} \\
\delta b_{yy}^{(i)} - \frac{b_{yy}^{(i)}(\delta p - \delta \tau_{xx})}{\rho c^2_p} - \frac{b_{xy}^{(i)} \delta \tau_{xy}}{\rho c^2_s} \\
\delta b_{zz}^{(i)} - \frac{b_{zz}^{(i)}(\delta p - \delta \tau_{xx})}{\rho c^2_p} - \frac{b_{xz}^{(i)} \delta \tau_{xz}}{\rho c^2_s} \\
\delta b_{xy}^{(i)} - \frac{b_{xy}^{(i)} \delta \tau_{xy}}{\rho c^2_s} \\
\delta b_{xz}^{(i)} - \frac{b_{xz}^{(i)} \delta \tau_{xz}}{\rho c^2_s} \\
\delta b_{yz}^{(i)} - \frac{b_{yz}^{(i)}(\delta p - \delta \tau_{xx})}{\rho c^2_p}
\end{pmatrix}.$$

(B.42)
Appendix C

Boundary conditions for the hyperbolic viscoelastic system

Before discussing the boundary conditions of the viscoelastic system we will first consider a simple example.

On an interval \([a, b]\) we have the prototype system

\[
\begin{align*}
\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial \rho}{\partial x} &= 0, \quad \rho > 0, \\
\frac{\partial p}{\partial t} + U \frac{\partial p}{\partial x} + G \frac{\partial u}{\partial x} &= 0, \quad G > 0,
\end{align*}
\]

which may also be written as

\[
\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = 0, \quad \text{with} \quad A = \begin{pmatrix} U & 1/\rho \\ G & U \end{pmatrix}, \quad u = \begin{pmatrix} u \\ \tau \end{pmatrix}.
\]

The eigenvalues \(c_i\) and eigenvectors \(l_i\) and \(r_i\) of \(A\) are

\[
\begin{align*}
c_1 &= U + c, \quad l_1 = \begin{pmatrix} \rho c \\ 1 \end{pmatrix}, \quad r_1 = \begin{pmatrix} c \\ G \end{pmatrix}, \\
c_2 &= U - c, \quad l_2 = \begin{pmatrix} -\rho c \\ 1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} -c \\ G \end{pmatrix},
\end{align*}
\]

where \(c = \sqrt{G/\rho}\) is the wave speed. The characteristic variables are

\[
\begin{align*}
w_1 &= l_1^T u = \rho cu + \tau, \\
w_2 &= l_2^T u = -\rho cu + \tau.
\end{align*}
\]

Multiplying the system (C.3) by \(l_i^T, \ i = 1, 2\) we obtain

\[
\begin{align*}
\frac{\partial w_1}{\partial t} + c_1 \frac{\partial w_1}{\partial x} &= 0, \\
\frac{\partial w_2}{\partial t} + c_2 \frac{\partial w_2}{\partial x} &= 0.
\end{align*}
\]

The variable \(w_1\) is transported with speed \(c_1 = U + c\) and \(w_2\) with speed \(c_2 = U - c\).

We distinguish three ‘flow’-conditions:

a. \(U < -c\): supercritical flow. Both the variables \(w_1\) and \(w_2\) are transported in negative \(x\)-direction.

b. \(-c \leq U < c\): subcritical flow. The variables \(w_1\) and \(w_2\) are transported in positive and negative \(x\)-direction, respectively.

c. \(U \geq c\): supercritical flow. Both the variables \(w_1\) and \(w_2\) are transported in positive \(x\)-direction.
If we consider boundary conditions at \( x = b \), waves travelling in positive \( x \)-direction are outgoing and waves travelling in negative \( x \)-direction are in-going. Characteristic variables that are outgoing cannot be prescribed: they are determined from the ‘inside’. Characteristic variables that are in-going have to be prescribed directly or should be derivable from the outgoing characteristic variables. For the three flow conditions given above we have for the b.c. at \( x = b \):

a. \( U < -c \); supercritical inflow: Both the variables \( w_1 \) and \( w_2 \) are in-going and should be fully prescribed. From (C.6) and (C.7) we find that this is equivalent to prescribing both \( u \) and \( \tau \).

b. \(-c \leq U < c \); sub-critical flow: The variable \( w_1 \) is outgoing and cannot be prescribed. The variable \( w_2 \) is in-going and has to be prescribed directly or can possibly depend \( w_1 \). From (C.6) and (C.7) we find for example

\[
\begin{align*}
  w_2 &= w_1 - 2\rho cu, \quad (C.10) \\
  w_2 &= -w_1 + 2\tau, \quad (C.11)
\end{align*}
\]

but others are possible. The first equation can be used when we prescribe \( u = \bar{u} \):

\[
  w_2 = w_1 - 2\rho \bar{c}u, \quad (C.12)
\]

and the second can be used when we prescribe \( \tau = \bar{\tau} \):

\[
  w_2 = -w_1 + 2\bar{\tau}. \quad (C.13)
\]

Both boundary conditions are of the ‘reflecting’ type: the outgoing wave \( w_1 \) comes back via \( w_2 \). Prescribing \( w_2 \) directly, independent of \( w_1 \), is a non-reflecting boundary condition. However, this type of b.c. will not be discussed here.

c. \( U \geq c \); supercritical outflow. Both the variables \( w_1 \) and \( w_2 \) are outgoing and cannot be prescribed. This means that neither \( u \) nor \( \tau \) can be prescribed.

Proper initial conditions for (C.6) and (C.7) are specification of \( w_1 \) and \( w_2 \) at the initial time for all \( x \in (a, b) \). This is equivalent with specification of both \( u \) and \( \tau \) at the initial time for all \( x \in (a, b) \).

The results of the linear system can be used to analyse the boundary conditions of the nonlinear hyperbolic visco-elastic system. When we consider (2.18) on a boundary with outward normal \( n \), the plane waves in the direction \( n \) are important. Writing (2.18) locally on the boundary we have for the plane waves

\[
  \frac{\partial \delta u}{\partial t} + K_n \frac{\partial \delta u}{\partial n} = 0. \quad (C.14)
\]

After multiplying this equation with \( l_i^T \) we get

\[
  \frac{\partial \delta w_i}{\partial t} + c_i \frac{\partial \delta w_i}{\partial n} = 0, \quad i = 1, \ldots, n_v, \quad (C.15)
\]

with \( \delta w_i = l_i^T \delta u \). These equations become decoupled in the characteristic system. These decoupled systems can now be used to discuss b.c. similarly to the linear system. The results are for the perturbed system in \( \delta u \), but we assume that they are valid for the non-linear system in \( u \) as well.

The introduction of weak compressibility is artificial. The coefficient \( \kappa \) is a very large number: \( \kappa \gg \sum G_m \). We are really only interested in the limit \( \kappa \to \infty \).

Upon taking this limit we make the following observations
i. The wave speed \( c \) becomes infinite and the set of eigenvalues and vectors corresponding to the compression: \( c_1, \rho_1, \tau_1, c_2, \rho_2, \tau_2 \), and the characteristic variables \( \delta w_1 \) and \( \delta w_2 \) degenerate, as expected. In this limit the pressure becomes a Lagrangian multiplier corresponding to the constraint \( \text{div}\, u = 0 \). The wave analysis for the compression is not valid anymore. For finite, but large, \( \kappa \) we conclude that we have subcritical flow \( (|u| < c_c) \) on the complete boundary. This means that the characteristic variables \( \delta w_1 \) and \( \delta w_2 \), as given by (B.35) and (B.36), are going out and going in, respectively. Possible b.c. are: prescribing either \( \delta u \), the normal velocity, or \( \delta(-p + \tau_{xx}) = \delta\sigma_{xx} \), the normal stress. Since these types of b.c. do not degenerate in the limit \( \kappa \to \infty \) we assume that they are valid in this limit as well.

ii. The eigenvalues, eigenvectors and characteristic variables corresponding to shear waves remain regular in the incompressible limit. In fact, they are independent of \( \kappa \). Hence, it seems legitimate to assume that the b.c. for \( \delta w_3, \ldots, \delta w_6 \), as given by (B.37)–(B.40), are still valid for \( \kappa \to 1 \). This leads to the following classification

a. supercritical inflow, \( u < -c_s \): \( \delta v, \delta \tau_{xy}, \delta w, \delta \tau_{xz} \) must all be prescribed.

b. sub-critical flow, \( -c_s \leq u < c_s \): prescribing either \( \delta v \) or \( \delta \tau_{xy} \) and either \( \delta w \) or \( \delta \tau_{xz} \) are possible boundary conditions.

c. supercritical outflow, \( u \geq c_s \): \( \delta v, \delta \tau_{xy}, \delta w, \delta \tau_{xz} \) cannot be prescribed.

iii. The eigenvalues, eigenvectors and characteristic variables corresponding to convection remain regular in the incompressible limit. Using \( \rho c_s^2 \sim \kappa \) for \( \kappa \to \infty \) we find from the equations (B.41) and (B.42) in the limit

\[
\delta u_{\text{conv}} = \left( \begin{array}{c} 0 \\ \delta\tau_{xx} \\ \delta\tau_{xy} \\ \delta\tau_{xz} \\ \vdots \\ \delta d_M \end{array} \right),
\]

where \( d_i, i = 1, \ldots, M \) are given by

\[
d_m = \left( \begin{array}{c} \delta b_{2x} \\ \delta b_{xy} \\ \delta b_{yx} \\ \delta b_{2y} \\ \delta b_{xz} \\ \delta b_{yz} \end{array} \right) = \left( \begin{array}{c} b_{xz}^{(m)} \delta\tau_{xy} \\ \frac{b_{xy}^{(m)} \delta\tau_{xy}}{\rho c_s^2} \\ \frac{b_{yx}^{(m)} \delta\tau_{xy}}{\rho c_s^2} \\ \frac{b_{zz}^{(m)} \delta\tau_{xx}}{\rho c_s^2} \\ \frac{b_{xz}^{(m)} \delta\tau_{xy}}{\rho c_s^2} \\ \frac{b_{yz}^{(m)} \delta\tau_{xy}}{\rho c_s^2} \end{array} \right).
\]

It is not difficult to see that \( \delta u_{\text{conv}} \) only depends on \( \delta b_m, m = 1, \ldots, M \), and not on \( \delta u \) and \( \delta p \). Furthermore, it is easy to verify that

\[
\sum_{m=1}^{M} G_m d_m(4) = \sum_{m=1}^{M} G_m \delta b_{xy}^{(m)} - \frac{\sum_{m=1}^{M} G_m b_{xx}^{(m)}}{\rho c_s^2} \delta\tau_{xy} = \delta\tau_{xy} - \delta\tau_{xy} = 0,
\]

\[
\text{(C.18)}
\]
\[
\sum_{m=1}^{M} G_m \delta b_m^{(m)}(5) = \sum_{m=1}^{M} G_m \delta b_{xx}^{(m)} - \sum_{m=1}^{M} G_m b_{xx}^{(m)} \frac{\rho c_s^2}{\rho c_s^2} \delta \tau_{xx} = \delta \tau_{xx} - \delta \tau_{xx} = 0,
\]

where we have used that \( \rho c_s^2 = \sum_{m=1}^{M} G_m b_{xx}^{(m)} \). We conclude from (C.18) and (C.19) that it is not possible to derive \( \delta \tau_{xy} \) and \( \delta \tau_{xz} \) from \( \delta u_{\text{conv}} \). These shear stresses are not convected with speed \( u \). This was expected because they are involved and determined by the shear waves. All other variables of \( b_m \), \( m = 1, \ldots, M \) are convected with speed \( u \) and have to be prescribed at an inflow boundary \( (u < 0) \).

The results derived above can be summarised as follows. We have assumed that the results for \( \delta u \) can be extended to \( u \). We divide the boundary into the following sub types:

a. \( u < -c_s \): supercritical inflow,

b. \(-c_s \leq u \leq c_s , u < 0 \): subcritical inflow,

c. \(-c_s \leq u \leq c_s , u \geq 0 \): subcritical outflow,

d. \( u > c_s \), supercritical outflow.

Correct boundary conditions are as follows

- On the complete boundary, either \( u \) or \( \sigma_{xx} = -p + \tau_{xx} \) can be prescribed.

- For the different boundary types we have to prescribe

  a. supercritical inflow: \( v, w, b_m, m = 1, \ldots, M \)
  
  b. subcritical inflow: there are various possibilities, for example:

      1. full fluid memory: \( b_m, m = 1, \ldots, M \)
      2. \( v, w \) and a subset of \( b_m, m = 1, \ldots, M \) such that \( \delta u_{\text{conv}} \) is fully specified.

      In the latter case \( \tau_{xy} \) and \( \tau_{xz} \) cannot be specified.
  
  c. subcritical outflow: \( v \) or \( \tau_{xy} \) and \( w \) or \( \tau_{xz} \)

  d. none.

On a supercritical inflow boundary all convection and shear waves are going in, i.e. all variables must be prescribed: tangential velocities and the complete fluid memory \( b_m, 1, \ldots, M \). On an subcritical inflow boundary we lose the possibility to prescribe either the tangential velocities or the shear stresses. Prescribing shear stresses is much easier, because they are determined from \( b_m \). This means that the fluid memory \( b_m, 1, \ldots, M \) can then be fully prescribed.
Appendix D

Raviart-Thomas finite element spaces

D.1 The $RT_k$ spaces

The $RT_k$ space can be introduced as follows (Brezzi & Fortin 1991; Roberts & Thomas 1991):

$$RT_k(K) = (P_k(K))^d + xP_k(K),$$  \hspace{1cm} \text{(D.1)}

on a triangle or tetrahedron $K \equiv \Omega_e$ and $P_k(K)$ is given by

$$P_k(K) : \text{the space of polynomials of degree } \leq k.$$ \hspace{1cm} \text{(D.2)}

Note that the space is defined per element $K$. The dimension of the space (per element) is

$$\dim RT_k(K) = \begin{cases} (k+1)(k+3) & \text{for } d = 2, \\ \frac{1}{2}(k+1)(k+2)(k+4) & \text{for } d = 3. \end{cases} \hspace{1cm} \text{(D.3)}$$

A vector field $q$ that is an element of the $RT_k$ space has the following properties

i. $\text{div } q \in P_k(K)$.

ii. the normal component $q \cdot n$ is a polynomial of degree $k$ on a line ($d = 2$) or a surface ($d = 3$) with normal vector $n$ ($x \cdot n = \text{constant}$).

iii. the normal components $q \cdot n$ are continuous across element boundaries.

The properties ii. and iii. means that we need to define the following number of normal components $q \cdot n$ on the boundaries of an element

$$n_{\text{boun}} = \begin{cases} 3(k+1) & \text{for } d = 2, \\ 2(k+1)(k+2) & \text{for } d = 3. \end{cases} \hspace{1cm} \text{(D.4)}$$

Note that a triangle has three sides and a tetrahedron has four. From (D.3) and (D.4) we find that the number of internal degrees of freedom is

$$n_{\text{internal}} = \dim RT_k(K) - n_{\text{boun}} = \begin{cases} k(k+1) & \text{for } d = 2, \\ \frac{1}{2}k(k+1)(k+2) & \text{for } d = 3. \end{cases} \hspace{1cm} \text{(D.5)}$$

The internal degrees of freedom that can be used are of the form

$$\int_K p_m q \, dx, \quad p_m \in P_k(K).$$ \hspace{1cm} \text{(D.6)}

Note that indeed $n_{\text{internal}}$ independent degrees of freedom, as given by (D.5), can be defined. An $RT_k$ triangle has been depicted in figure D.1.

The $RT_k$ space can be used to solve the second-order elliptic problem

\begin{align*}
\Delta u + f &= 0 & \text{in } \Omega, \hspace{1cm} \text{(D.7)} \\
u &= 0 & \text{on } \partial \Omega, \hspace{1cm} \text{(D.8)}
\end{align*}
Good approximation spaces for $Q \times U$ are: $Q_h = RT_k(K)$ and $U_h = P_k(K)$, where $U_h$ is discontinuous across element boundaries. According to the resemblance with the velocity-pressure problem in fluid flow we can use identical discretisations for that problem: $RT_k$ for the discretised velocity $u_h$ and $P_k$ for the discretised pressure $p_h$.

**D.2 The $RT_0^0$ and $RT_1^1$ spaces**

When we use $RT_k \times P_k$ for the $(u, p)$ approximation in incompressible flows, the discretised formulation of $\text{div} \, u = 0$ is

$$\int_{\Omega} q_h \, \text{div} \, u_h \, dx = 0 \quad \text{for all } q \in P_k(K),$$

where $u_h \in RT_k(K)$. From property i. of the $RT_k$ space, i.e. $\text{div} \, u_h \in P_k(K)$, and (D.13) we get

$$\text{div} \, u_h = 0,$$

i.e. the divergence is zero exactly and not only in a weak form. The subspace that fulfills (D.14) is denoted by $RT_0^0$:

$$RT_0^0(K) = \{ q \, | \, q \in RT_k(K), \, \text{div} \, q = 0 \}.$$
The $RT_0^k$ space is not a practical space to work with, because

$$\int_K \text{div} \, q \, dx = \int_{\partial K} q \cdot n \, ds = 0,$$

(D.16)

which means that there is constraint on the normal components on the element boundary. These are, in return, coupled with the normal components of neighbour elements because of the continuity requirement \textit{iii}. However, relaxing the constraint (D.16) leads to a space that can be used in practice:

$$RT_1^k(K) = \{ q | q \in RT_k(K), \text{div} \, q \in P_0(K) \},$$

(D.17)

i.e. the divergence is a constant per element. The number of variables that can be eliminated $n_{\text{elim}}$ is of course equal to the reduction of div $u_h$ from $P_k(K)$ to $P_0(K)$:

$$n_{\text{elim}} = \dim P_k(K) - 1 = \begin{cases} \frac{1}{2}(k + 1)(k + 2) - 1 & \text{for } d = 2, \\ \frac{1}{6}(k + 1)(k + 2)(k + 3) - 1 & \text{for } d = 3. \end{cases}$$

(D.18)

Reducing $RT_k$ to $RT_1^k$ can be achieved by elimination of internal degrees of freedom. This can be seen as follows. For $u_h \in RT_1^k(K)$ we have for the constant divergence

$$\text{div} \, q_h = \frac{\int_{\partial K} q_h \cdot n \, ds}{\int_K dx}.$$  

(D.19)

Using this and

$$\int_K \phi \, \text{div} \, q_h \, dx = \int_{\partial K} \phi q_h \cdot n \, ds - \int_K q_h \cdot \text{grad} \, \phi \, dx,$$

(D.20)

for any smooth $\phi$, we arrive at

$$\int_K q_h \cdot \text{grad} \, \phi \, dx = \int_{\partial K} (\phi - \frac{\int_K \phi \, dx}{\int_K dx}) q_h \cdot n \, ds.$$  

(D.21)

By substituting $n_{\text{elim}}$ independent function from $P_k(K) \setminus P_0(K)$ for $\phi$ into (D.21) we see that $n_{\text{elim}}$ variables from the internal variables given in (D.6) can be expressed in the normal velocities on the boundary and thus can be eliminated on element level.

The dimension of $RT_1^k$ can be calculated from (D.3) and (D.18)

$$\dim RT_1^k(K) = \dim RT_k(K) - n_{\text{elim}}$$

$$= \begin{cases} \frac{1}{2}(k + 2)(k + 3) & \text{for } d = 2, \\ \frac{1}{6}(k + 1)(k + 2)(2k + 9) + 1 & \text{for } d = 3. \end{cases}$$

(D.22)

and the number of internal degrees of freedom for $RT_1^k$ becomes

$$n^{1}_{\text{internal}} = \dim RT_1^k(K) - n_{\text{boun}} = \begin{cases} \frac{1}{2}k(k - 1) & \text{for } d = 2, \\ \frac{1}{6}(k + 1)(2k + 5) & \text{for } d = 3. \end{cases}$$

(D.23)

where the number of boundary degrees of freedom $n_{\text{boun}}$ are still given by (D.4). In figure D.2 we have depicted the triangular (reference) elements for the space $RT_1^k$ that are used in this report.
Internals:
\[ R(\gamma - \nu) d\Omega, \ R \gamma u d\Omega, \ R \gamma v d\Omega \]

\[ k = 0 \]

\[ k = 1 \]

\[ k = 2 \]

Internals: \( \int (y u - x v) d\Omega \)

\[ k = 3 \]

Internals: \( \int (y u - x v) d\Omega, \ \int x y u d\Omega, \ \int x y v d\Omega \)

Figure D.2: Triangular (reference) elements for the space \( RT_k^1 \) up to \( k = 3 \).
D.3 The $RT_{[k]}$, $RT_{[k]}^0$ and $RT_{[k]}^1$ spaces

The $RT_{[k]}$ space can be introduced as follows (Brezzi & Fortin 1991; Roberts & Thomas 1991):

$$RT_{[k]}(K) = P_{k+1}(x) \times P_k(y) e_x + P_k(x) \times P_{k+1}(y) e_y$$  \hspace{1cm} (D.24)

on a rectangle $K \equiv \Omega_x$ and $P_k(x)$ is given by

$$P_k(x) : \text{the space of polynomials of degree } \leq k \text{ in one space dimension.}$$  \hspace{1cm} (D.25)

Note that the space is defined per element $K$. A vector field $\mathbf{q}$ that is an element of the $RT_k$ space has the following properties

i. $\text{div } \mathbf{q} \in Q_k(K)$.

ii. the normal component $\mathbf{q} \cdot \mathbf{n}$ is a polynomial of degree $k$ on the boundaries of the element.

iii. the normal components $\mathbf{q} \cdot \mathbf{n}$ are continuous across element boundaries.

The properties ii. and iii. means that we need to define $4(k+1)$ normal components $\mathbf{q} \cdot \mathbf{n}$ on the boundaries of an element.

Similar to the $RT_0^k$ and $RT_1^k$ spaces we can define $RT_0^k$ and $RT_1^k$ spaces. In the practical implementation we will use the $RT_1^k$ space.

D.4 Construction of the $RT_k$ space; curved elements

Introduction of the space $RT_k$ by means of (D.1) has advantages for theoretical considerations but is not practical. The space $RT_{[k]}$ given by (D.24) is only for rectangles. For a practical implementation of these elements and for constructing elements with curved boundaries we work with a transformation of a reference element (Raviart & Thomas 1977; Raviart 1981).

The transformation of a reference element to the real element is similar to the deformation of bodies in continuum mechanics:

$$(\xi_1, \xi_2, \xi_3) \rightarrow \mathbf{x}(\xi_1, \xi_2, \xi_3),$$  \hspace{1cm} (D.26)

where $(\xi_1, \xi_2, \xi_3)$ are co-ordinates in the reference space, also denoted by $\hat{\mathbf{x}}$. The deformation of line elements is determined by the deformation gradient tensor $\mathbf{F}(\hat{\mathbf{x}})$:

$$d\mathbf{x} = \mathbf{F}(\hat{\mathbf{x}}) \cdot d\hat{\mathbf{x}} \quad \text{with } F_{ij} = \frac{\partial x_i}{\partial \xi_j},$$  \hspace{1cm} (D.27)

where $d\hat{\mathbf{x}}$ and $d\mathbf{x}$ are infinitesimal line elements in the reference space and real space respectively. The transformation of a reference triangle has been depicted in figure D.3. The transformation of the reference rectangle is similar.

The deformation of volume and area elements is given by

$$dV = J d\hat{V},$$  \hspace{1cm} (D.28)

$$da = J F^{-T} \cdot d\hat{a},$$  \hspace{1cm} (D.29)

where $J = \text{det } \mathbf{F}$. The vector quantities for $RT$ elements transform such that the component normal to a surface is preserved by the transformation except for the stretching of the surface:

$$\hat{\mathbf{q}} \cdot d\hat{\mathbf{a}} = \mathbf{q} \cdot da$$  \hspace{1cm} (D.30)
With (D.29) it follows that

\[ q = J^{-1} F \cdot \tilde{q}. \]  

(D.31)

This is called the Piola transformation. By defining the usual operators in both the reference and the real space, we find that

\[ \int_{\partial \Omega} \phi q \cdot d\alpha = \int_{\partial \hat{\Omega}} \hat{\phi} \hat{q} \cdot d\hat{\alpha}, \]  

(D.32)

\[ \int_{\Omega} \phi \text{div} q \, dx = \int_{\hat{\Omega}} \hat{\phi} \hat{\text{div}}\hat{q} \, d\hat{x}, \]  

(D.33)

\[ \int_{\Omega} \phi q \cdot \text{grad}(\cdot) \, dx = \int_{\hat{\Omega}} \hat{\phi} \hat{q} \cdot \hat{\text{grad}}(\cdot) \, d\hat{x}, \]  

(D.34)

where \( \phi \) is a scalar quantity. For curved elements the transformation \( F \) is not constant and computing differential operations other than the ones above become rather complicated. For example computing the velocity gradient \( L = (\text{grad} v)^T \) is, although straightforward, rather cumbersome. Contrary to standard finite elements the transformation \( F \) is not isoparametric. We use the blending function method (see, for example, Szabó & Babuška 1991) to generate curved elements.
Appendix E

Forces and torques on a rigid body submerged in a fluid

We consider a rigid body submerged in a fluid as depicted in figure E.1. The force $F$ and torque $M_P$ with respect to a point $x_P$ acting on the body $\mathcal{B}$ due to the flow of the fluid is given by

$$F = \int_{\partial \mathcal{B}} (-p n + 2\eta_s n \cdot d + n \cdot \tau) \, dS; \quad (E.1a)$$

$$M_P = \int_{\partial \mathcal{B}} (x - x_P) \times (-p n + 2\eta_s n \cdot d + n \cdot \tau) \, dS. \quad (E.1b)$$

The contribution of the viscous terms can be simplified as follows. Assume that the body is rotating with a rate $w_0$. Consider a frame that is fixed to the body and thus has the same rotation rate of $w_0$. In this frame we have on the wall

$$2n \cdot d = 2 \frac{\partial u'_0}{\partial n} n + \left( \frac{\partial u'_1}{\partial t_1} \right) t_1 + \left( \frac{\partial u'_2}{\partial t_2} \right) t_2$$

$$= \frac{\partial u'_1}{\partial n} t_1 + \frac{\partial u'_2}{\partial n} t_2$$

$$= w' \times n$$

$$= (w - w_0) \times n,$$  \quad (E.2)

where $(n, t_1, t_2)$ is a local Cartesian (not curved) system, $w'$ is the vorticity in the moving frame\(^1\). We have used that $u' = 0$ on the wall and div $u' = 0$. The relation

\(^1\)Only quantities that are frame dependent are denoted with a prime.
between the vorticity in the stationary frame $w$ and the moving frame is $w'$ is

$$w' = w - w_0.$$  \hfill (E.3)

Substitution of (E.2) into (E.1) and using

$$\int_{\partial \mathcal{B}} n \, dS = 0 \tag{E.4}$$

we obtain

$$F = \int_{\partial \mathcal{B}} (-pn + \eta_s w \times n + n \cdot \tau) \, dS, \tag{E.5a}$$

$$M_P = \int_{\partial \mathcal{B}} (x - x_P) \times (-pn + \eta_s (w - w_0) \times n + n \cdot \tau) \, dS. \tag{E.5b}$$

Note that $w_0$ cannot be eliminated for $M_P$. 


Bibliography


