Parameter analysis and identification of the Greitzer model by analogy with the Van der Pol equation

M.J. Nieuwenhuizen

DCT 2008.065

Coach(es): J. van Helvoirt
Supervisor: M. Steinbuch

Technische Universiteit Eindhoven
Department Mechanical Engineering
Dynamics and Control Technology Group

Eindhoven, May, 2008
Abstract

In this thesis the two-state lumped parameter Greitzer model for surge in compressors is discussed. A resemblance is seen with the Van der Pol system. To enhance this resemblance, a transformation of coordinates is applied to the Greitzer system. A parameter analysis of this transformed Greitzer system is discussed. It is seen that an approximately linear relation exists between the period time of the limit cycle of the transformed Greitzer system and its parameter $\mu$. This relation resembles the relation between the period time of the limit cycle of the Van der Pol system its parameter. An expression for this relation is available. Based on this expression an expression for the period time of the Greitzer system is developed. This expression can be used in the identification of the model parameters of the Greitzer system. However, it is also seen that for large values of the stability parameter $\mu$, when the system exhibits relaxation-type behavior, it is not possible to determine the combination of the parameters uniquely and unambiguously.
# Contents

Abstract i

Nomenclature v

1 Introduction 1
   1.1 Operating principles of a compressor 1
   1.2 Surge 3
   1.3 Surge modeling and control 3
   1.4 The Van der Pol equation 4
   1.5 Research objectives 5
   1.6 Outline of this thesis 6

2 The lumped parameter Greitzer model 7
   2.1 System representation and model assumptions 7
   2.2 Model equations 8
   2.3 Parameter identification 11
   2.4 Compressor characteristic 13
   2.5 Throttle characteristic 15
   2.6 Stability of the operating point 16
   2.7 Dynamic behavior of the two-state Greitzer model 18
   2.8 Summary 20

3 The similarity between the Van der Pol system and the Greitzer model 21
   3.1 The Van der Pol equation 21
   3.2 The transformed Greitzer model 23
   3.3 Comparing the Greitzer equations with the Van der Pol equations 24
      3.3.1 The shape and frequency of the limit cycle 24
      3.3.2 Stability 25
   3.4 Summary 26
## Parameter study

4.1 The transformed Greitzer model ........................................ 28
4.2 The parameter $\mu$ .................................................... 30
4.3 The parameter $F$ ...................................................... 31
4.4 The throttle characteristic ............................................. 33
4.4.1 Quadratic vs. linear throttle characteristic ..................... 33
4.4.2 Varying $x_{1,0}$ .................................................. 34
4.4.3 Varying $S^#$ .................................................. 37
4.5 The parameters of the dimensional Greitzer model ............... 37
4.6 Summary ............................................................ 40

## Parameter identification based on the period time of the limit cycle of the Greitzer model

5.1 The period time of the limit cycle of the Van der Pol equation .... 41
5.2 The period time of the limit cycle of the transformed Greitzer system .... 42
5.3 The period time of the limit cycle of the dimensional Greitzer system ... 44
5.3.1 $x_{1,0}$ and $S^#$ .................................................. 45
5.3.2 $P$ as a function of dimensional parameters ..................... 46
5.4 An example of parameter identification ................................ 47
5.5 Sensitivity .......................................................... 50
5.6 Domain of utility .................................................... 53
5.7 Discussion .......................................................... 54

## Conclusion and recommendations

6. Conclusion and recommendations .................................... 55

## Bibliography

Bibliography .......................................................... 58

## A Model parameters

A Model parameters .................................................... 59

## B Greitzer model with variable rotor speed

B Greitzer model with variable rotor speed .......................... 61
B.1 Greitzer model equations with variable rotor speed ............... 61
B.2 The dynamic behavior of the Greitzer model with variable rotor speed .......................... 62
B.3 Parameter analysis of the variable speed model ................... 67

## C Conference paper submitted to the IEEE Conference on Decision and Control, 2008

C Conference paper submitted to the IEEE Conference on Decision and Control, 2008 69
# Nomenclature

## Roman uppercase

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Dimensions</th>
<th>S.I. unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>area</td>
<td>L²</td>
<td>m⁻¹</td>
</tr>
<tr>
<td>B</td>
<td>Greitzer stability parameter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>slope of the compressor characteristic</td>
<td>L⁻¹T⁻¹</td>
<td>Pa kg⁻¹s⁻¹</td>
</tr>
<tr>
<td>C₀</td>
<td>valley point of compressor characteristic</td>
<td>M L⁻¹T⁻²</td>
<td>Pa</td>
</tr>
<tr>
<td>C₀</td>
<td>tangential component of the gas velocity</td>
<td>LT⁻¹</td>
<td>m s⁻¹</td>
</tr>
<tr>
<td>F</td>
<td>lumped model parameter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>lumped model parameter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>compressor characteristic semi height</td>
<td>M L⁻¹T⁻²</td>
<td>Pa</td>
</tr>
<tr>
<td>J</td>
<td>rotor inertia</td>
<td>M L²</td>
<td>kg m²</td>
</tr>
<tr>
<td>L</td>
<td>length</td>
<td>L</td>
<td>m</td>
</tr>
<tr>
<td>N</td>
<td>rotor speed</td>
<td>T⁻¹</td>
<td>rpm</td>
</tr>
<tr>
<td>P</td>
<td>period time of limit cycle</td>
<td>T</td>
<td>s</td>
</tr>
<tr>
<td>R</td>
<td>radius</td>
<td>L</td>
<td>m</td>
</tr>
<tr>
<td>S</td>
<td>inverse of slope of throttle characteristic</td>
<td>L T</td>
<td>Pa⁻¹ kg s</td>
</tr>
<tr>
<td>T</td>
<td>slope of the throttle characteristic</td>
<td>L⁻¹T⁻¹</td>
<td>Pa kg⁻¹s⁻¹</td>
</tr>
<tr>
<td>U₁</td>
<td>impeller tip speed</td>
<td>LT⁻¹</td>
<td>m s⁻¹</td>
</tr>
<tr>
<td>V</td>
<td>volume</td>
<td>L³</td>
<td>m³</td>
</tr>
<tr>
<td>W</td>
<td>compressor characteristic semi width</td>
<td>M T⁻¹</td>
<td>kg s⁻¹</td>
</tr>
<tr>
<td>Z</td>
<td>compressibility factor</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

## Roman lowercase

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Dimensions</th>
<th>S.I. unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>speed of sound constant</td>
<td>L T⁻¹</td>
<td>m s⁻¹</td>
</tr>
<tr>
<td>b</td>
<td>constant</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b₀</td>
<td>constant</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*M = mass, L = length, T = time, Θ = temperature.*
### Symbol Description Dimensions S.I. unit

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Dimensions</th>
<th>S.I. unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_p$</td>
<td>specific heat at constant pressure</td>
<td>$L^2T^{-2}\Theta^{-1}$</td>
<td>J kg$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$c_v$</td>
<td>specific heat at constant volume</td>
<td>$L^2T^{-2}\Theta^{-1}$</td>
<td>J kg$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>$c_t^*$</td>
<td>dimensionless throttle parameter</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$d$</td>
<td>diameter</td>
<td>$L$</td>
<td>m</td>
</tr>
<tr>
<td>$k$</td>
<td>constant</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$k_f$</td>
<td>viscous friction parameter</td>
<td>$L^2 M T^{-1}$</td>
<td>m$^2$ kg s$^{-1}$</td>
</tr>
<tr>
<td>$\dot{m}$</td>
<td>mass flow</td>
<td>$M T^{-1}$</td>
<td>kg s$^{-1}$</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure</td>
<td>$M L^{-1} T^{-2}$</td>
<td>Pa</td>
</tr>
<tr>
<td>$q_1, q_2$</td>
<td>state variables of two-state Van der Pol system</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
<td>$T$</td>
<td>s</td>
</tr>
<tr>
<td>$u_t$</td>
<td>throttle valve opening</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$x$</td>
<td>variable</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$x_1, x_2$</td>
<td>state variables of transformed Greitzer model</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$y$</td>
<td>variable of Van der Pol equation</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

### Greek

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Dimensions</th>
<th>S.I. unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>parameter</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>ratio of specific heats $c_p/c_v$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>eigenvalue</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\mu$</td>
<td>model parameter</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\xi$</td>
<td>dimensionless time</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
<td>$M L^{-3}$</td>
<td>kg m$^{-3}$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>slip factor</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\varsigma$</td>
<td>dimensionless time constant</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\tau$</td>
<td>time constant</td>
<td>$T$</td>
<td>s</td>
</tr>
<tr>
<td>$\tau_c$</td>
<td>compressor torque</td>
<td>$M L^2 T^{-2}$</td>
<td>N m</td>
</tr>
<tr>
<td>$\tau_d$</td>
<td>motor drive torque</td>
<td>$M L^2 T^{-2}$</td>
<td>N m</td>
</tr>
<tr>
<td>$\phi$</td>
<td>dimensionless mass flow</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\psi$</td>
<td>dimensionless pressure difference</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\omega$</td>
<td>angular frequency</td>
<td>$T^{-1}$</td>
<td>rad s$^{-1}$</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>dimensionless characteristic curve</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
### Subscript

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>atmospheric intersection point</td>
</tr>
<tr>
<td>c</td>
<td>compressor</td>
</tr>
<tr>
<td>H</td>
<td>Helmholtz</td>
</tr>
<tr>
<td>p</td>
<td>plenum</td>
</tr>
<tr>
<td>r</td>
<td>control valve</td>
</tr>
<tr>
<td>s</td>
<td>(return piping) system</td>
</tr>
<tr>
<td>ss</td>
<td>steady-state</td>
</tr>
<tr>
<td>t</td>
<td>throttle valve</td>
</tr>
<tr>
<td>v</td>
<td>Van der Pol</td>
</tr>
</tbody>
</table>

### Superscript

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>*</td>
<td>defined in the $(\phi_c, \psi)$-coordinate system</td>
</tr>
<tr>
<td>#</td>
<td>defined in the $(x_1, x_2)$-coordinate system</td>
</tr>
</tbody>
</table>

### Symbols and Operations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{a}$</td>
<td>time derivative</td>
</tr>
<tr>
<td>$\hat{a}$</td>
<td>estimate</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

The general topic of this thesis is surge in a centrifugal compressor. Surge is an unstable operating mode of a compressor that occurs at low mass flow rates. It goes together with large fluctuations in pressure and mass flow and can cause the temperature to rise considerably. It can therefore severely damage a compressor and thus it has to be avoided. Much research has been done on the modeling and control of compressors in surge, but identification of the model parameters proved difficult. In this thesis a parameter analysis of a widely used compressor surge model will be discussed. Furthermore an analogy with the Van der Pol equation is used to develop a new method for parameter identification.

This chapter begins with a short description of the operating principles of a compressor in Section 1.1. Subsequently, the characteristics of surge will be discussed in Section 1.2. Section 1.3 gives an overview of available models and surge control techniques. In Section 1.4 the Van der Pol equation is introduced. Next, in Section 1.5, the motivation for this research and the research objectives are stated. Section 1.6 gives an outline of this thesis.

1.1 Operating principles of a compressor

A compressor is a machine that increases the pressure of a compressible medium. There are many different types of compressors. Positive displacement compressors pressurize gas by decreasing its volume. Turbo compressors (axial and centrifugal) pressurize gas by increasing its velocity and subsequently converting the kinetic energy in potential energy by decelerating the gas.

In an axial compressor, see Figure 1.1, the gas is accelerated by a row of rotating blades and decelerated by a following row of stator blades. A centrifugal compressor, see Figure 1.2, consists of a shaft with rotating blades (impeller), where the gas is accelerated, and an annular diverging channel (diffuser) where the gas is decelerated with a consequent rise of the static pressure.

Compressor performance

The performance of a compressor is often specified in a compressor map; a map that relates the mass flow through the compressor with the pressure rise over the compressor. Figure 1.3 gives an example of a compressor map. The compressor characteristic represents the steady-state
pressure rise that will be achieved as a function of the mass flow and it is dependent on the rotational speed of the impeller. The load or throttle line represents the pressure requirement of the system. The intersection point of the compressor characteristic and the throttle line is the steady-state operating point of the compressor.

For high mass flows, the operating range is bounded by the limited capacity of the compressor due to choked flow. This is called the Stonewall line. In the low mass flow region, the operating range is bounded by the occurrence of surge and rotating stall. The transition from stable to unstable mode is indicated by the surge line. Surge and rotating stall are flow instabilities that occur at low mass flows in a compressor and that can influence each other. The focus in this thesis is on compression systems where only surge is seen and rotating stall is not important. Therefore, rotating stall will not be discussed in this thesis. For details on rotating stall see (de Jager, 1995).
1.2 Surge

When a compressor enters surge, the noise level increases and the piping around the compressor can often be seen vibrating. This demonstrates the violence of surge oscillations. The large mechanical and thermal load associated with surge can damage the system and disrupt the (chemical) process that is connected to the compression system.

Figure 1.4 shows an example of pressure oscillations of a compression system during surge. It is first operated at a stable operating point. After throttling the flow to bring the operating point left of the surge line, the compressor enters a state of high pressure oscillations. Hand in hand with these pressure oscillations go large mass flow oscillations. This is shown in Figure 1.5, where the surge cycle is plotted in a compressor map. The type of surge shown in this figure is referred to as deep surge. It is characterized with large pressure and flow oscillations and flow reversal over part of the cycle. Surge can also occur with small pressure and flow oscillations. The flow then remains positive. This is referred to as mild surge. For more details on surge classification of different types of surge, see (de Jager, 1995). In this thesis the focus is on deep surge.

1.3 Surge modeling and control

Surge is a damaging operating mode of a compressor that limits the range of mass flows for which the compressor can be used. It is important to understand when and why surge occurs, and to avoid it. Therefore much research has been done on the modeling of surge and efforts have been made to design surge controllers.

Surge modeling

Emmons and Pearson (1955) used the analogy with a Helmholtz resonator to derive a linear model for the dynamic behavior of the gas in a compressor. A Helmholtz resonator is a container of air with an open hole (or neck). A volume of air in and near the open hole vibrates because of the compressibility of the air in the cavity. Greitzer (1976a) used the same analogy to derive his lumped parameter Greitzer model. The lumped parameter Greitzer model is a nonlinear model.
capable of describing surge in an axial compressor with a low pressure ratio. Greitzer reduced
the complicated compression system to a plenum volume with an inlet duct accommodating the
compressor and an outlet duct accommodating a throttle valve. This implies that the characteristics
of the compressor are lumped on an actuator disk in the inlet duct and the characteristics of
the load are lumped on an actuator disk in the outlet duct. Due to the simplicity and usability of
the model, the Greitzer model is still the most widely used model in the field of control of surge
in a compressor and therefore this model forms the starting point of the analysis in this thesis.

In 1981, Hansen et al. (1981) showed that the lumped parameter Greitzer model is also valid for
small centrifugal compressors. Fink et al. (1992) introduced an equation for the conservation of
angular momentum in the turbocharger spool to account for variations in rotor speed. Gravdahl
and Egeland (1999) elaborated on the work of Fink et al. (1992) and derived an equation for the
resistive torque of the compressor for an impeller with radial vanes. They also introduced an an-
alytical expression for the compressor map. An extended overview of the development of models
of the dynamic behavior of compressors is given by (Van Helvoirt, 2007; Paduano et al., 2001).

**Surge control**

The techniques that are currently used to avoid surge are surge avoidance and detection. With
surge avoidance, a safety margin some distance from the surge line is used. Operating points
left of the safety line are not used. With surge detection, the system is monitored and when the
onset of surge is detected, the flow is increased or the pressure behind the compressor is lowered
to avoid surge. A new and promising technique proposed by Epstein et al. (1989) is active surge
control, where feedback control is used to suppress surge. With this active control approach it
would be possible to stabilize operating points in the unstable range (mass flow lower than the
surge line). Pinsley et al. (1991) showed that it is possible to stabilize a small turbocharger system
beyond the surge line by using appropriate feedback. More recently, Willems (2000) showed
that active suppression of surge is possible on a laboratory scale turbocharger. Van Helvoirt
(2007) continued the work on active control by investigating the application on an industrial scale
compressor.

1.4 The Van der Pol equation

The Van der Pol equation is a well known and well researched equation in the field of nonlinear
equations. It originated in the field of electrical circuits. The Van der Pol equation is often used as
an example for nonlinear dynamics. The period time of the limit cycle of the Van der pol equation
is also researched and an analytical relation between the period time and the model parameter
is known (Dorodnicyn, 1947; Davis and Alfriend, 1967; Bavink and Grasman, 1969; Padin et al.,
2005).

The Van der Pol equation can rewritten to a system of two equations. Figure 1.6 shows the limit
cycle of the two-state Van der Pol system. The limit cycle is located around a cubic nullcline. This
limit cycle bears a clear resemblance with the surge cycle depicted in Figure 1.5. The surge cycle
is located around the compressor characteristic. Both the compressor characteristic and the cubic
nullcline in the Van der Pol system can be described by a cubic function (see Moore and Greitzer
(1986) for the compressor characteristic).
1.5 Research objectives

The compressor model is a lumped parameter model and therefore a large simplification of the real compression system. And the compressor characteristic can only be measured in the stable region and also the load characteristic cannot be measured very well. This implies a lot of uncertainties in the model.

It is therefore important to understand the effect of the various lumped parameters and the shape of the compressor and load characteristic on the dynamic behavior. Although there is a lot of literature on the Greitzer model and its derivatives on the use for modeling the aerodynamics of surge in a compressor system and for the control of surge, not much is written to improve the understanding and implications of the equations themselves. Only Oliva and Nett (1991) published a general nonlinear dynamical analysis of the stability of the point equilibria and periodic solutions of the equations.

The first objective of this thesis is:

\textit{improve the understanding of the lumped parameter Greitzer model with a parameter analysis of the important parameters.}

With the complicated structure of an industrial compression system that is not designed to fit a simple lumped parameter model, as a laboratory compression system can be, one of the difficulties in the work of ? was to identify the relevant parameters of the system. However, the period time of the compression system can be easily measured. An analytical expression that links this period time with the parameters of the model could considerably simplify the identification process. Such an analytical expression is available for the period time of the limit cycle of the Van der Pol equation which, as is discussed in Section 1.3, clearly resembles the surge cycle of the lumped parameter Greitzer model.

The second objective of this thesis is:

\textit{develop a method for parameter identification of the lumped parameter Greitzer model based on the analogy with the Van der Pol equation, by means of using the analytical expression for the period time of the limit cycle of the Van der Pol equation.}

1.6 Outline of this thesis

In Chapter 2 the lumped parameter Greitzer model will be discussed and in Chapter 3 the resemblance between the lumped parameter Greitzer model and the Van der Pol equation will be investigated. The lumped parameter Greitzer model will be transformed to enhance the resemblance with the Van der Pol equation. Next, in Chapter 4, a parameter analysis of this transformed Greitzer model will be discussed. In Chapter 5 the known analytical expression for the period time of the limit cycle of the Van der Pol equation will be adapted to fit the relation between the period time of the surge cycle and the parameters of the lumped parameter Greitzer model, using the parameter analysis from Chapter 4. This expression is then used to identify the parameters of the lumped Greitzer model in a simulated example. Furthermore, the sensitivity of this identification method is investigated. Chapter 6 contains the conclusions and recommendations of this thesis.

The lumped parameter Greitzer model is a model for a compressor with constant rotor speed. In Appendix B an addition to the model for a variable rotor speed is discussed.

A conference paper dealing with the issues in Chapter 3 and 4 of this thesis is submitted to the IEEE Conference on Decision and Control. This paper can be found in Appendix C.
Chapter 2

The lumped parameter Greitzer model

In this chapter, the lumped parameter Greitzer model is discussed. The lumped parameter Greitzer model is a model for the behavior of the gas flow through and pressure rise over the compressor, a subsequent volume and a throttle (or the resistance of the system). In the remainder of this thesis, the lumped parameter Greitzer model will be referred to as the Greitzer model.

In Section 2.1, the structure of the lumped Greitzer model is described and the model assumptions are presented. Then, in Section 2.2, the equations of the model are introduced and the model is reduced and transformed to a dimensionless form. Section 2.3 deals with the model parameters and Sections 2.4 and 2.5 deal with the compressor and throttle characteristics, respectively. The stability of the operating point is discussed in Section 2.6 and a description of the dynamic behavior of the gas is given in Section 2.7. Finally, Section 2.8 gives a preview on the next chapters.

2.1 System representation and model assumptions

A schematic representation of the lumped parameter Greitzer model is shown in Figure 2.1. In his model, Greitzer (1976a) represents the compressor as an actuator disc in an inlet duct of constant area. This actuator disc represents a blade row over which the mass flow is continuous, but pressure changes are allowed to be discontinuous. The geometry of the duct ($L_c, A_c$) should be chosen in such a way that the changes in mass flow rate generate pressure differences that are equal to the actual pressure differences in the duct. In a similar manner the throttle is represented as an actuator disc in an outlet duct ($L_t, A_t$). The volume of the piping between the compressor and the throttle is represented by a plenum volume ($V_p$). Because the Greitzer model is a model for the gas flow, the boundary of the model is the interface between the gas and the surrounding material of the compressor, pipe and throttle. The model of the compression system can be coupled with models of other parts of the process system by applying the continuity laws to the mass flow and pressure in front of the inlet and behind the outlet duct.

Model assumptions

Greitzer designed his model analogous to a Helmholtz resonator. In the Helmholtz resonator, all the kinetic energy is associated with the movement of the gas in the duct and all the potential energy is associated with the compression of the gas in the plenum.
The modeled systems are confined to those having low inlet Mach numbers and relatively small pressure rises. There are, however, no restrictions on the amplitude of the oscillations of the mass flow and pressure rise, compared to the steady-state values of these quantities. Now assuming that these oscillations have a low frequency, it is reasonable to consider the flow in the ducts to be incompressible. Furthermore, Greitzer assumes an one-dimensional flow in the ducts and an isentropic compression process in the plenum. Also, the dimensions of the plenum are assumed to be smaller than the acoustic wavelength, which justifies the assumption that the pressure in the plenum uniform. The compressor and throttle behavior are assumed quasi-steady and the systems’ overall temperature ratio is near unity.

2.2 Model equations

Applying these assumptions to the model in Figure 2.1 yields the following mass and momentum balance (Greitzer, 1976a)

\[
\frac{d\dot{m}_c}{dt} = \frac{A_c}{L_c} (-\Delta p + \Delta p_c(\dot{m}_c, N)) \tag{2.1}
\]

\[
\frac{d\dot{m}_t}{dt} = \frac{A_t}{L_t} (\Delta p - \Delta p_t(\dot{m}_c, u)) \tag{2.2}
\]

\[
\frac{dp_p}{dt} = \frac{a^2}{V_p} (\dot{m}_c - \dot{m}_t) \tag{2.3}
\]

\[
\frac{d\Delta p_c}{dt} = \frac{1}{\tau} (\Delta p_{c, ss}(\dot{m}_c, N) - \Delta p_c(\dot{m}_c, N)) \tag{2.4}
\]

In this model \(\Delta p\) is the difference between the plenum pressure and the atmospheric pressure \((p_p - p_0)\). This will be referred to as pressure rise or pressure difference of the system. Equation (2.1) relates the change in the mass flow through the compressor \((\dot{m}_c)\) with the difference between the pressure rise of the system and the pressure rise of the gas as it goes through the compressor.
As mentioned in the previous chapter, the height of the compressor characteristic depends on the speed of the rotor \((N)\). Equation (2.2) describes the changes in mass flow through the throttle \((\dot{m}_t)\) as a function of the pressure rise of the system and the pressure drop over the throttle \((\Delta p_t)\). Equation (2.3) is a mass balance and relates the pressure rise of the system to the difference between the mass flow through the compressor and the mass flow through the throttle. Here, \(a\) is the speed of sound.

Greitzer (1976a) added a fourth equation (2.4) to the model to account for the lag in the compressor response during the onset of rotating stall. In this equation, \(\Delta p_{c,ss}\) is the measured or approximated steady-state compressor characteristic. Greitzer incorporated this equation, because the compressor does not respond quasi-steadily during the onset of stall. The time constant \(\tau\) is based on the time needed to develop a full rotating stall pattern.

The Greitzer model is often used in a nondimensional form where the mass flow is scaled as \(\phi = \dot{m}_c/\rho U_t A_c\), the pressure as \(\psi = \Delta p/\frac{1}{2}\rho U_t^2\) and the time variable is scaled with the Helmholtz frequency, \(\xi = t\omega_H = ta\sqrt{\frac{A_c}{V_p L_c}}\). This results in the equations

\[
\frac{d\phi_c}{d\xi} = B(\Psi_c(\phi_c, N) - \psi) \tag{2.5}
\]

\[
\frac{d\phi_t}{d\xi} = \frac{B}{G}(\psi - \Psi_t(\psi_t, u_t)) \tag{2.6}
\]

\[
\frac{d\psi}{d\xi} = \frac{1}{B}(\phi_c - \phi_t) \tag{2.7}
\]

\[
\frac{d\Psi_c}{d\xi} = \frac{1}{\varsigma}(\Psi_{c,ss}(\phi_c, N) - \Psi_c) \tag{2.8}
\]

with

\[
B = \frac{U_t}{2a\sqrt{\frac{V_p}{A_c L_c}}} = \frac{U_t}{2\omega_H L_c}
\]

\[
G = \frac{L_t A_c}{L_c A_t}
\]

The parameter \(B\) is also called the Greitzer stability parameter. A system with a large \(B\) will exhibit surge when the operating point is in the unstable area. When \(B\) is large, surge is dominant over rotating stall. When \(B\) is small, surge is not dominant and the system will exhibit rotating stall.

In this report, only compressors where surge is dominant over rotating stall will be considered. Therefore, equation (2.8) is omitted. Examples of such compressors can be seen in the works of, for example, Willems (2000), Meuleman (2002) and Van Helvoirt (2007).

The parameter \(G\) is the ratio of the equivalent length to area ratios for throttle and the compressor ducts, and can be interpreted as a measure of the inertia effects in the throttle, compared to those in the compressor duct. For most systems \(G\) will be small, since the duct length through the compressor will be much larger than the thickness of the throttle. For instance, in the work of Van Helvoirt (2007) the value of \(G\) for Rig A is 0.01. Therefore, equation (2.6) is often omitted.
from the model in the remainder of this thesis. The Greitzer model assumes an open compression system. In other words, the gas supplied to the compressor is sucked from the atmosphere and the gas is blown through the throttle into the atmosphere. Specifically, the gasflow in to and out of the compression system do not influence the pressure in front of the inlet duct and behind the throttle. This is, however, not the case in the work of Van Helvoirt (2007). In his work one of the test rigs was a closed system (see Figure 2.3). Van Helvoirt therefore adapted the model to include a system pressure instead of the atmospheric pressure. A second volume was introduced to account for the return piping, see Figure 2.2.

With the mentioned adjustments, the model is reduced to a two-state model

\[
\frac{d\phi_c}{d\xi} = B(\Psi_c(\phi_c, N) - \psi) \quad (2.9)
\]

\[
\frac{d\psi}{d\xi} = \frac{F}{B}(\phi_c - \phi_t(\psi, u_t)) \quad (2.10)
\]

with

\[
F = 1 + \frac{Z_s T_s V_p}{Z_p T_p V_s} \quad (2.11)
\]

In this model a new parameter \((F)\) is introduced. This parameter is a measure for the ratio of the plenum volume and the return piping volume. Here, \(T\) is the temperature and \(Z\) is the compressibility factor. Note that when \(V_s\) becomes infinitely large \(F\) goes to 1 and the original Greitzer model is obtained.

For completeness, equation (2.12) and (2.13) state the dimensional two-state Greitzer model.
\[
\frac{d\dot{m}_c}{dt} = \frac{A_c}{L_c} \left[ \Delta p_c (\dot{m}_c) - \Delta p \right] \tag{2.12}
\]
\[
\frac{d\Delta p}{dt} = \left( \frac{a^2}{V_p} + \frac{a^2}{V_s} \right) + \left[ \dot{m}_c - \dot{m}_t (\Delta p, u_t) \right] \tag{2.13}
\]

A detailed derivation of these equations can be found in Greitzer (1976a), Willems (2000), Meuleman (2002) or Van Helvoirt (2007).

As can be seen in equation 2.9, the compressor characteristic is a function of the speed of the rotor. This speed is assumed constant in the presented model, as is done in a large part of the literature (see e.g. Hansen et al., 1981; Willems, 2000; Van Helvoirt, 2007). However, Fink et al. (1992); Bøhagen (2007) and others incorporated spool dynamics in the model. This is discussed in Appendix B.

### 2.3 Parameter identification

In Section 2.2, the Greitzer model is presented. When this model is used for control, the parameters will have to be identified. In this section the identification of the geometrical parameters is discussed. In Section 2.4 and 2.5 the compressor and throttle characteristic will be discussed.

In the literature, the subject of geometric parameter identification is often not discussed. In papers and proceedings on experimental work on compressors often the geometric parameters are stated without an explanation on how these parameters are obtained (see for instance Fink et al., 1992; Hansen et al., 1981). In doctoral thesis’ the subject is discussed more often, and there it becomes evident that the lumped geometric parameters are not easy to determine and that \(L_c\) is often used as a tuning parameter to match the simulations with the measurements (see for instance (Van Helvoirt, 2007) and (Willems, 2000)).

One reason for this is that measuring in a process system is not simple. Pressure measurements often have considerable measurement noise. Also, the placement of pressure transducers is important. In a laboratory set-up the placement of the pressure transducers can be chosen freely. However, in an industrial set-up one is dependant on the position of the available pressure transducers. Furthermore, flow measurements are even more difficult. Reliable, transient mass flow measurements are often not available in industrial set-ups. The period time of the surge cycle can, however, easily be computed from the pressure measurements.

Another important difficulty is that the Greitzer model is a lumped model; the geometry of the system is lumped on two ducts of constant area and a volume. See for instance the system in Figure 2.3 to 2.5. This is an industrial test rig as it was used by Van Helvoirt (2007). In Figure 2.5, a cross-section of the compressor is shown. In this example, the gas flows through a narrow channel through the compressor, then through the much wider pipe between the compressor and the throttle and finally through a narrow throttle valve. The geometry of the ducting through the compressor has to be lumped on the parameters \(L_c\) and \(A_c\) and the geometry of the throttle has to be lumped on \(L_t\) and \(A_t\). The length of the inlet duct \((L_c)\) could be chosen as the length of the ducting through the compressor (indicated in Figure 2.5). It is common in literature to choose the frontal area of the impeller eye as the area of the inlet duct \((A_c)\) (e.g. Van Helvoirt, 2007; Gravdahl and Egeland, 1997). The length of the outlet duct \((L_t)\) is chosen as the thickness...
Figure 2.3: Scheme of compressor test rig from Van Helvoirt (2007).

Figure 2.4: Photograph of the centrifugal compressor with top casing removed, courtesy of Siemens.

Figure 2.5: Aerodynamic components of the centrifugal compressor, courtesy of Siemens.
of the valve and the area of the outlet duct \( (A_t) \) is usually chosen as the area of the valve. The plenum volume \( (V_p) \) is the volume of the piping between the compressor and the throttle. It begins directly after the compressor and incorporates all the ducting between the compressor and the throttle valve. These are, however, choices that the researcher makes. Meuleman et al. (1998), for instance, choose the area \( A_c \) equal to the compressor outlet duct area.

The choices of the geometrical parameters are ambiguous, because even though the geometry of the inlet duct in the model is quite a natural simplification for an axial compressor, a centrifugal compressor, has a much more complicated geometry, see Figure 2.5. This makes it more difficult to choose a \( L_c \) and an \( A_c \) correctly.

Greitzer (1976a) acknowledged this problem even in the axial compressor and stated the requirement that a given change in mass flow generates pressure differences that are equal to the actual pressure differences in the duct. This leads to the geometric requirement

\[
\frac{L}{A} = \int_{\text{actual ducting}} \frac{dl}{A(l)} \quad (2.14)
\]

In the example above, the boundary between the inlet duct and the plenum volume \( V_p \) lies directly behind the compressor. The plenum volume is the volume of the piping between the compressor and the throttle valve.

Van Helvoirt (2007) used various methods to calculated the geometric parameters. He used equation (2.14), estimated the parameters via approximate realizations from step response measurements and also tuned the model to match measurements with simulations of the pressure oscillations during surge. With these three methods, he determined \( B \) with an accuracy of around \( \pm 18\% \).

The ambiguity of identifying the geometric parameters becomes clear from comparing (Meuleman et al., 1998), (Willems, 2000) and (Gravdahl et al., 2004). The compression system used in all three works is the same, but the parameter values vary from \( A_c = 7.9 \cdot 10^{-3} \) to \( A_c = 9.56 \cdot 10^{-3} \), from \( V_p = 2.03 \cdot 10^{-2} \) to \( V_p = 3.75 \cdot 10^{-2} \) and from \( L_c = 1 \) to \( L_c = 2.45 \).

### 2.4 Compressor characteristic

The relationship between the dimensionless mass flow through the compressor at given rotor speed and the resulting pressure rise is given by a so-called compressor characteristic. The set of characteristics for all speeds is called a compressor map (see also Figure 2.6).

Operating points on the compressor characteristic for high mass flow, between the surge line and the stonewall line, are stable. Therefore, this part of the compressor characteristic can be measured with steady-state measurements.

The part of the compressor characteristic for negative mass flows is often approximated. Hansen et al. (1981), however, measured the negative flow branch by feeding air to the compressor exit to force a steady negative flow. Another approach was used by Fink et al. (1992). He used a system with a large plenum to measure the surge cycle in deep surge. Due to the large time scale
associated with a large plenum, the state of the compressor follows the negative branch of the compressor characteristic closely. Fink et al. (1992) used this measurement as an approximation for the negative branch of the steady-state compressor characteristic.

The unstable part of the compressor characteristic, between zero mass flow and the surge line, is usually assumed to be smooth and is approximated by a cubic polynomial. However, Fink et al. (1992) was able to measure this part in his experimental setup with steady-state measurements. By placing the throttle directly after the compressor outlet, he reduced the plenum volume and thereby the B parameter to a sufficiently low value that this branch became stable. Gravdahl and Egeland (1999) used another approach and calculated the entire compressor characteristic by the use of energy transfer and loss analysis. Most of the time, however, only steady-state measurements of the stable branch of the compressor characteristic are available.

Moore and Greitzer (1986) introduced an axisymmetric cubic polynomial to approximate the compressor characteristic. This Moore-Greitzer polynomial in dimensionless form is

$$\Psi_c(\phi_c, N) = C_0^*(N) + H^*(N) \left[ 1 + \frac{3}{2} \left( \frac{\phi_c}{W^*(N)} - 1 \right) - \frac{1}{2} \left( \frac{\phi_c}{W^*(N)} - 1 \right)^3 \right]$$ (2.15)

The definitions of the semi-width $W^*$, the semi-height $H^*$, and the valley point $C_0^*$ are given in Figure 2.7. These parameters are determined by fitting equation (2.15) on the steady-state measurements of the compressor characteristic. This does not always result in a plausible compressor characteristic. For instance, in the work of Willems (2000), the model with the approximated compressor characteristic predicts a higher minimal plenum pressure rise than the measurements showed. Willems solved this problem by shifting the approximated $C_0^*$ downwards to improve the fit between surge measurements and surge simulation. The original approximation was kept for $\phi_c \geq 2W^*$ and the approximation with the shifted $C_0^*$ was used for $\phi_c \geq 2W^*$. Meuleman (2002) elaborated on this method and calculated the valley point $C_0^*(N)$ with
Figure 2.7: Definitions of parameters in the cubic polynomial, equation (2.18).

\[
\frac{p_p}{p_s}(0, N) = \left(1 + \frac{\pi^2 (N/60)^2 (d_2^2 - d_1^2)}{2 c_p T_s} \right)^{\gamma - 1} \tag{2.16}
\]

\[
C_0^*(N) = \left(\frac{p_p}{p_s}(0, N) - 1\right) p_s \frac{1}{2 \rho U_t^2} \tag{2.17}
\]

with \(d_2\) the impeller tip diameter, \(d_1\) the impeller eye diameter and \(\gamma\) the ratio of specific heats \(c_p/c_v\). For completeness, the Moore-Greitzer polynomial in dimensional form is stated in equation (2.18).

\[
\Delta p_c(\dot{m}_c, N) = C_0(N) + H(N) \left[1 + \frac{3}{2} \left(\frac{\dot{m}_c}{W(N)} - 1\right) - \frac{1}{2} \left(\frac{\dot{m}_c}{W(N)} - 1\right)^3\right] \tag{2.18}
\]

Here, \(C_0\), \(W\) and \(H\) are the valley point, the semi-width and semi-height, respectively, of the compressor characteristic in the \((\dot{m}_c, \Delta p_c)\)-plane.

### 2.5 Throttle characteristic

The throttle or load characteristic is often described as a simple quadratic characteristic. Willems (2000) and Meuleman (2002) used equation (2.19) as the throttle characteristic.

\[
\phi_t = c_t^* u_t \sqrt{\psi_t} \quad \text{if} \quad \psi_t > 0 \tag{2.19}
\]

Here, \(u_t\) is the dimensionless throttle position (between 0 and 1) and \(c_t\) is a dimensionless throttle parameter, which is a measure for the capacity of the fully opened throttle.
In reality of course, the relation between the mass flow and pressure difference over the throttle or load can be much more complicated. Manufactures provide these expressions for the throttle flow characteristics and industrial standards are available. Van Helvoirt (2007) discusses the difference between the throttle characteristic via equation (2.19) and the industrial standard IEC 60534-2-1(1998). He also investigates the effect of using a variable density instead of a constant density. The resulting effects on the throttle characteristic are large and result in a significantly different slope of the throttle characteristic in the intersection point.

Furthermore, the slope of the throttle characteristic is much steeper for a choked throttle or almost horizontal, for instance for gas re-injection in gas fields, where the discharge volume is very large.

In this report, often the throttle characteristic will be assumed to be linear: \( \phi_t = S^* \psi_t / + k \), with \( S = 1/T^* \) and \( T^* \) the slope of the characteristic in the compressor map and \( k \) a constant. Although this is physically correct only in some systems, it can help to understand the influence of the throttle characteristic on the shape and frequency of the surge cycle.

2.6 Stability of the operating point

The stability of the two-state greitzer model can be studied by analyzing the linearized model. Linearizing equation (2.9) and (2.10) around the intersection point \((\phi_c,0,\psi_0)\) gives the linearized model

\[
\begin{bmatrix}
\dot{\phi}_c \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
BC^* & -B \\
\frac{F}{B} & -\frac{F}{BT^*}
\end{bmatrix}
\begin{bmatrix}
\phi_c \\
\psi
\end{bmatrix}
\]

(2.20)

Where \( C^* \) and \( T^* \) are the slopes of the compressor characteristic and the throttle characteristic in the intersection point, respectively.

Calculating the eigenvalues gives the expression

\[
\lambda_{1,2} = \frac{-\left(\frac{F}{BT^*} - BC^*\right) \pm \sqrt{\left(\frac{F}{BT^*} - BC^*\right)^2 - 4F(1 - C^*/T^*)}}{2}
\]

(2.21)

The stability of the operating point is influenced by the Greitzer stability parameter \( B \), the parameter \( F \) and the slope of the compressor characteristic, and the slope of the throttle line. The two expressions between brackets in equation (2.21) determine the stability boundary. When both expressions are positive, the system has two negative eigenvalues and is thus stable. When \((1 - C^*/T^*) < 0\), the slope of the compressor characteristic is steeper than the slope of the throttle line and the system is statically unstable. The system then has one positive and one negative eigenvalue. The equilibrium point is a saddle point. This is illustrated in Figure 2.8. When \((\frac{F}{BT^*} - BC^*) < 0\), the system is dynamically unstable. Both eigenvalues are positive and the equilibrium point is an unstable focus or unstable node, see Figure 2.9. Dynamic instability usually occurs at a mass flow directly to the left of the top of the compressor characteristic. The parameter \( F \) introduced by Van Helvoirt (2007) influences only the dynamic instability boundary.
Figure 2.8: Phase portrait around statically unstable operating point, with compressor characteristic (black), throttle characteristic (gray) and simulation (---).

Figure 2.9: Phase portrait around dynamically unstable operating point, with compressor characteristic (black), throttle characteristic (gray) and simulation (---).

Figure 2.10: The mechanical analogue of the compression system (Cumpsty, 1989).
2.7 Dynamic behavior of the two-state Greitzer model

The Greitzer model has two states, $\phi_c$ and $\psi$. The rate of change of these states are both driven by a difference term:

- $\Psi_c(\phi_c, N) - \psi$
  This is the difference between the steady-state pressure rise that the compressor can achieve ($\Psi_c$) and the actual pressure rise ($\psi$) in the system. This difference influences the mass flow through the compressor. For instance, when the pressure rise over the compressor is lower than the achievable pressure rise, the mass flow through the compressor will increase to increase the pressure rise. When the pressure rise over the compressor is higher than the achievable pressure rise, less mass will flow through the compressor as a result of the high pressure in the plenum.

- $\phi_c - \phi_t(\psi, u_t)$
  This is the difference between the mass flow into $\phi_c$ and out of the plenum $\phi_t(\psi, u_t)$. It influences the pressure in the plenum. When the mass flow through the compressor into the plenum is larger than the mass flow through the throttle, the pressure in the plenum will increase. Similarly, when the mass flow into the plenum is smaller than the mass flow out of the plenum, the pressure in the plenum will decrease. This influence on the pressure is larger when the plenum is small.

With these two difference terms, the behavior of the compression system can be understood.

**Stable operating point**

The operating point is stable when the point of intersection of the compressor and throttle characteristic is on the stable branch of the compressor characteristic. When from this point the throttle is closed a bit further, but still remains in the stable area (see Figure 2.11, (1)), less mass is flowing
out of the plenum (due to the new throttle characteristic) than the mass flow $\psi_c$ into the plenum. Consequently, the pressure in the plenum increases. Because now the pressure rise over the compressor is higher than the achievable pressure rise by the compressor, the compressor is not able to force the same mass flow into the plenum and the mass flow will reduce. Because the slope of the compressor characteristic is negative, the state of the system will follow the compressor characteristic until the new operating point is reached.

**Unstable operating point**
When from this point the throttle is closed further and the new operating point lies in the unstable area (Figure 2.11, (2)), the system will go into surge. This is illustrated in Figure 2.12. First the plenum fills, because the throttle reduces the mass flowing out of the system. At the same time, the mass flow over the compressor will reduce (see above). However, when the state reaches mass flows where the compressor is unstable (1), a further reduction of the mass flow does not result in a higher achievable pressure rise over the compressor, but a lower achievable pressure rise (indicated by the compressor characteristic). As a consequence, the mass flow will reduce even further and faster, because the compressor is not able to sustain the pressure difference. For a system with a $B$ sufficiently high to achieve deep surge, the mass flow will even reverse. At some point, the state of the system reaches point (2) and the compressor will work in effect as a throttle for the reversed flow. The mass flows out of the plenum to the ducting in front of the compressor and thus the pressure difference over the compressor decreases ($2 \rightarrow 3$). When the pressure difference has decreased to a certain point, the compressor will be able to produce a pressure rise and the mass flow will reverse again ($3 \rightarrow 4$). The plenum will fill again until point (1) is reached. The mass flow will reduce again and a second surge cycle will begin.

**The Greitzer stability parameter $B$**
The Greitzer stability parameter $B$ not only determines the stability of the operating point, but also the size and shape of the surge cycle. Figure 2.13 gives an impression of the behavior of the Greitzer model for different values of $B$. For very low values of $B$ the system is stable. For high values of $B$ the oscillation are of the relaxation type in which two differen time scales can be distinguished.
2.8 Summary

In this chapter the Greitzer model is discussed. It is reduced to a dimensionless two-state model. The limit cycle of this two state model resembles the limit cycle of the Van der Pol system, as presented in Chapter 1. In Chapter 3 this similarity is discussed.

In Section 2.3, 2.4 and 2.5 the parameters of the Greitzer model are discussed. These parameters are difficult to determine, but have a large impact on the stability of the operating point and the shape and period time of the surge cycle. An insight in this influence is therefore important. In Chapter 4 a parametric analysis of the parameters is presented. In Chapter 5 the similarity between the Greitzer model and the Van der Pol system will be used to derive a method for parameter identification.
Chapter 3

The similarity between the Van der Pol system and the Greitzer model

It has been shown in Chapter 1 that there are similarities between the limit cycle of the Greitzer model and the limit cycle of the Van der Pol system. The Van der Pol equation is a well known equation in the field of nonlinear dynamics. It has received extensive attention in the literature and the equation is used as an example for nonlinear analysis techniques in many textbooks on nonlinear systems, e.g. (Khalil, 2000; Strogatz, 2000). The period time of the limit cycle of the Van der Pol equations is also a much researched topic (Dorodnicyn, 1947; Bavink and Grasman, 1974; Davis and Alfriend, 1967; Padin et al., 2005). Analytical relations for the period time of the Van der Pol system are available. For the identification of the parameters of the Greitzer model, it would be very helpful to have an analytical relation between the model parameters and the period time of the surge cycle, because the period time is easily measured.

In this chapter the similarity between the Greitzer model and the Van der Pol system is explored. First, in Section 3.3.1 the Van der Pol system is introduced. In Section 3.2 a coordinate transformation is applied to the Greitzer model to improve the similarity with the Van der Pol system. Subsequently, in Section 3.3 the similarity is discussed on the basis of the shape and period time of the limit cycle. Finally it is argued that the behavior is similar when the stability parameter is large. This will be useful in later chapters, where a method of parameter identification for the Greitzer model is designed based on the similarity with the Van der Pol system.

3.1 The Van der Pol equation

The Van der Pol equation is a well known equation that originated from in the study of circuits containing vacuum tubes. It was first studied by a Dutch physicist, Balthasar van der Pol, in the 1920’s, who described an electrical circuit used in an early radio receiver (Pol, 1926). It is an equation that describes self-sustaining oscillations in which energy is fed into small oscillations and removed from large oscillations.
The unforced Van der Pol equation in its most know form is

\[ \ddot{y} - \mu(1 - y^2)\dot{y} + y = 0 \]  

with \( \mu > 0 \) and a \( \dot{y} \) indicates the time derivative.

Equation 3.1 can be rewritten with the Liénard transformation \( q_2 = y - \frac{1}{3}y^3 - \frac{1}{\mu}\dot{y} \) and \( q_1 = y \) to the system of equations (3.2) and (3.3) (see Strogatz, 2000).

\[
\begin{align*}
\dot{q}_1 &= \mu \left[ q_1 - \frac{1}{3}q_1^3 \right] - q_2 \\
\dot{q}_2 &= \frac{1}{\mu} q_1
\end{align*}
\]

Figure 3.1 gives an impression of the behavior of the Van der Pol system for different values of \( \mu \). In the left subfigures, the cubic nullcline \( q_2 = q_1 - \frac{1}{3}q_1^3 \) is drawn. From this figure, the influence of \( \mu \) on the limit cycle becomes clear. A small \( \mu \) gives a large, rounded limit cycle. When \( \mu \) is large, the limit cycle lies close around the cubic nullcline in a similar way as the surge cycle lies around the compressor characteristic (see Figure 2.13). From Figure 3.1 it can also be seen that for large \( \mu \), \( q_1 \) has two time scales. It changes slowly when the state is close to the nullcline and abrupt when it jumps from one branch to the other branch. This is very similar to the behavior of the Greitzer model for large \( B \) parameters. The shape of the nullcline shows considerable similarity with the compressor characteristic when the latter is described by the cubic polynomial from Moore and Greitzer (1986), equation (2.15).
3.2 The transformed Greitzer model

Using the cubic Moore-Greitzer polynomial, equation (2.15) to describe $\Psi_c(\phi_c, N)$ and a simple quadratic throttle characteristic, equation (2.19) in the two-state Greitzer model, equation (2.9) and (2.10, yields

$$\frac{d\phi_c}{d\xi} = B \left[ C^*_0 + H^* \left( 1 + \frac{3}{2} \left( \frac{\phi_c}{W^*} - 1 \right) - \frac{1}{2} \left( \frac{\phi_c}{W^*} - 1 \right)^3 \right) - \psi \right]$$  \hspace{1cm} (3.4)

$$\frac{d\psi}{d\xi} = \frac{F}{B} \left[ \phi_c - c_{t1} u_t \sqrt{\psi} \right]$$  \hspace{1cm} (3.5)

The similarity between the Greitzer model and the Van der Pol system becomes more apparent when a different coordinate system is used for the Greitzer model. The coordinate transformation that will be used is

$$x_1 = \frac{\phi_c - W^*}{W^*}$$  \hspace{1cm} (3.6)

$$x_2 = \frac{2}{3H^*} (\psi - C^*_0 - H^*)$$  \hspace{1cm} (3.7)

This transformation of coordinates is illustrated in Figure 3.2. In effect, the coordinate system $(\phi_c, \psi)$ is shifted to put the point of inflection of the compressor characteristic at $(0,0)$ in the $(x_1, x_2)$ coordinate system. Furthermore, the system is scaled to put the minimum and maximum of the compressor characteristic at $(-1, -\frac{2}{3})$ and $(1, \frac{2}{3})$, like they are in the nullcline of the Van der Pol system.
Applying the transformation of coordinates to the equations (3.4) and (3.5) gives

\[
\frac{dx_1}{d\xi} = \mu \left[ \left( x_1 - \frac{1}{3} x_1^3 \right) - x_2 \right] \tag{3.8}
\]
\[
\frac{dx_2}{d\xi} = \frac{F}{\mu} \left[ x_1 - \left( \frac{c_t u_t}{W^*} \sqrt{\frac{3H^*}{2}} x_2 + C_0^* + H^* - 1 \right) \right] \tag{3.9}
\]

with

\[
\mu = \frac{3H^* B}{2W^*} \tag{3.10}
\]

The Greitzer system in \((x_1, x_2)\)-coordinates will be referred to in this thesis as the transformed Greitzer system. Note that this transformation of coordinates does not affect the period time of the limit cycle.

3.3 Comparing the Greitzer equations with the Van der Pol equations

By comparing equation (3.8) and (3.9) with equation (3.2) and (3.3) it can be seen that the Greitzer system and the Van der Pol system have much similarities. The two systems differ only in the throttle characteristic term \(\frac{c_t u_t}{W^*} \sqrt{\frac{3H^*}{2}} x_2 + C_0^* + H^* - 1\) and the multiplication factor \(F\) in equation (3.9).

3.3.1 The shape and frequency of the limit cycle

The effect of the presence of a throttle characteristic on the shape and period time of the limit cycle is shown in Figure 3.3. This figure shows a set of limit cycles and oscillations with various values of \(\mu\) of the Van der Pol system and of the transformed Greitzer system. The throttle valve characteristic in the Greitzer system is quadratic and intersects the compressor valve at 80 percent of the top mass flow, i.e. the intersection point \(x_{1,0}\) lies at \(x_1 = 0.6\).

Figure 3.3 reveals similarities and differences between the two systems. In Figure 3.3a-d, with \(\mu = 10\), the limit cycles are very similar, but the (dimensionless) period time of the Greitzer model is significantly higher. In the Van der Pol system (see Figure 3.3b) the state of the system \((q_1, q_2)\) follows the positive branch of the nullcline at the same speed as it follows the negative branch. In the Greitzer model (see Figure 3.3d) the state \((x_1, x_2)\) follows the positive branch of the compressor characteristic faster than it follows the negative branch. This is caused by the extra term in equation (3.5).

In Figure 3.3(e-p) it can be seen that the differences between the shape and size of the limit cycle increase with decreasing \(\mu\). Decreasing \(\mu\) implies that the equation that contains the difference between the Van der Pol system and the transformed Greitzer system becomes relatively more important (equation (3.3) and (3.9)). The operating point of the Greitzer model even becomes
stable for sufficiently small $\mu$ (Figure 3.3o), while the limit cycle of the Van der Pol system only increases. Another difference is that the limit cycle of the Greitzer model is non-symmetric, while the limit cycle of the Van der Pol system is always symmetric around the $q_2$-axis.

3.3.2 Stability

The stability of the operating point of the Greitzer model is discussed in Section 2.6 on the basis of the linearized model. Linearizing the transformed Greitzer model in equations (3.8) and (3.9) gives
\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt}
\end{bmatrix} = \begin{bmatrix}
\mu C^\# - \frac{\mu}{E} \\
\frac{\mu}{E} - \frac{\mu}{\mu T^\#}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

Here, \(C^\#\) and \(T^\#\) are respectively the slope of the transformed compressor characteristic and the slope of the transformed throttle characteristic in the intersection point.

The eigenvalues are

\[
\lambda_{1,2} = -\left(\frac{\mu}{\mu T^\#} - \mu C^\#\right) \pm \frac{\sqrt{\left(\frac{\mu}{\mu T^\#} - \mu C^\#\right)^2 - 4F(1 - \frac{C^\#}{T^\#})}}{2}
\]

(3.11)

Linearizing the Van der Pol system of equations (3.2) and (3.3) around the equilibrium point \((0,0)\) results in

\[
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix} = \begin{bmatrix}
\mu - \frac{\mu}{\mu} \\
\frac{\mu}{\mu} - 0
\end{bmatrix} \begin{bmatrix}
q_1 \\
q_2
\end{bmatrix}
\]

(3.12)

with the eigenvalues

\[
\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}
\]

(3.13)

Figure 3.4 shows the eigenvalues of the Van der Pol system and the transformed Greitzer system with an intersection point in \((0,0)\). The real part of the eigenvalues of the Van der Pol system is always positive and the real part of the eigenvalues of the Greitzer system is negative for small \(\mu\). When the intersection point is in \((0,0)\), \(C^\#\) is 1 and the difference between the eigenvalues of the transformed Greitzer system and the Van der Pol system becomes very small for large \(\mu\). When the intersection point is not in \((0,0)\) (in Figure 3.4 shown with \(C^\# = 0.75\)), the fast eigenvalue of the transformed Greitzer system for large \(\mu\) is considerably lower.

### 3.4 Summary

In this chapter the similarity between the Greitzer model and the Van der Pol system was explored. To enhance the resemblance a coordinate transformation was applied to the Greitzer model. It was then shown that the difference between the transformed Greitzer model and the Van der Pol system lies in the presence of the throttle characteristic and the parameter \(F\). For small \(\mu\), the operating point of the transformed Greitzer model becomes stable, in contrast to the Van der Pol system that has no stable equilibrium point. For large values of \(\mu\), the behavior of both systems is very similar. The shape of the limit cycle is almost the same. However, the period time differs.

Note that the parameters that can be defined in the \((\dot{m}_c, \Delta p)\), the \((\phi_c, \psi)\) and the \((x_1, x_2)\) coordinate system have no superscript in the \((\dot{m}_c, \Delta p)\) coordinate system, a * in the \((\phi_c, \psi)\) coordinate system and a # in the \((x_1, x_2)\) coordinate system.
Figure 3.4: Eigenvalues of the Van der Pol system (−) and the transformed Greitzer system in (0,0) for $T^\# = 2$ and $F = 1$, with $C^\# = 1$ (−−), $C^\# = 0.75$ (··).

In the next chapter, the effect of the different parameters and the throttle characteristic on the limit cycle will be discussed. In Chapter 5, the similarity between the two systems will be exploited to find a relation between the parameters and the period time of the Greitzer system, based on the known relation between $\mu$ and the period time of the Van der Pol system.
Chapter 4

Parameter study

In this chapter, a parameter study of the transformed Greitzer model will be discussed. This is done for two reasons.

Firstly, the parameters of the Greitzer model are difficult to determine, as was discussed in Chapter 2. The throttle characteristic differs according to the method that was used to model the throttle. Furthermore, $L_c$ is difficult to determine and it is therefore often used as a tuning parameter. The parameter study in this chapter provides insight in the effect of the different parameters on the shape and period time of the surge cycle. This insight can aid in dealing with model uncertainties and support the effective tuning of the lumped parameters.

Secondly, the effect of the parameters on the relation between the parameter $\mu$ and the period time of the transformed Greitzer model has to be determined. In Chapter 3 the similarity was shown between the Greitzer model and the Van der Pol system. In this chapter it will be shown that a linear relationship exists between $\mu$ and the period time of the limit cycle of the Greitzer model. In Chapter 5 the known relation between $\mu$ and the period time of the Van der Pol system will be used to derive an equation for the relation between $\mu$ and the period time of Greitzer model. Because this relation differs for different parameters of the Greitzer model, it is important to know the effect of the parameters on this linear relation.

This chapter gives an overview of the effects of various parameters and the throttle characteristic on the shape and period time of the limit cycle and on the stability of the intersection point. In Section 4.1 the equations on which the parameter study is performed will be discussed. In Section 4.2 the influence of $\mu$ on the limit cycle will be shown. Then, in Section 4.3, the influence of $F$ will be addressed. In the next section, the influence of the parameters of the throttle characteristic will be discussed. In Section 4.5 the parameter study will be linked to the parameters of the original Greitzer model. The chapter will be concluded with a summary of the effects of the parameters.

4.1 The transformed Greitzer model

The parameter study presented in this chapter is performed on the transformed Greitzer model. For the purpose of simplicity, the transformed cubic throttle characteristic, see equation (3.9), is replaced with a linear approximation. In this chapter, this linear approximation will be referred to as the transformed throttle characteristic. The slope of the transformed throttle characteristic will
be denoted with $T^\#$ (and the inverse of the slope $S^\#$) and the intersection point of the compressor and throttle characteristic will be denoted as $x_{1,0}$. The validity of the linear approximation is discussed in Section 4.4.

The equations on which the parameter study is performed are

\[
\frac{dx_1}{d\xi} = \mu \left( x_1 - \frac{1}{3} x_1^3 - x_2 \right) \tag{4.1}
\]

\[
\frac{dx_2}{d\xi} = \frac{F}{\mu} \left[ x_1 - g(x_2) \right] \tag{4.2}
\]

In (4.2), $g(x_2)$ is the linearized throttle characteristic according to

\[
g(x_2) = S^\# x_2 + k \tag{4.3}
\]

with

\[
k = x_{1,0} - S^\# (x_{1,0} - \frac{1}{3} x_{1,0}^3) \tag{4.4}
\]

For any time instance $\xi$, the term between brackets in equation (4.1) represents the vertical difference between the transformed compressor characteristic and the momentary state $x_2$ in the phase plane. Similarly, the term between brackets in equation (4.2) represents the horizontal distance between the transformed throttle characteristic and the momentary state $x_1$. Consequently, the direction of change of the state in an arbitrary point $p$ depends on the horizontal distance of $p$ to transformed throttle characteristic and the vertical distance to the transformed compressor characteristic, see also Figure 4.1.

![Figure 4.1](image_url)

Figure 4.1: An illustration of equations (4.1) and (4.2) for a certain point $p$ on the surge cycle. The figure shows the vertical distance between point $p$ and the compressor characteristic and the vertical distance between point $p$ and the throttle characteristic.
The transformed Greitzer model with a linear throttle characteristic, (4.1) and (4.2), is very suitable for a parameter analysis, because the equations are scaled such that the compressor characteristic has a standard width and height and there are only four parameters ($\mu$, $F$, $x_{1,0}$ and $S^\#$) to be discussed. Unlike in the dimensionless Greitzer model, the semi-width and the semi-height of the compressor curve are no parameters in the transformed Greitzer model, because they are taken into account in $\mu$.

In the next paragraphs, the influence of the parameters on the shape and period time of the limit cycle are discussed. The amplitude of $x_1$ and $x_2$, defined as $\frac{1}{2}\left[\max(x_i) - \min(x_i)\right]$, will be used to characterize the shape of the limit cycle. Furthermore, the effect of the parameters on the stability of the intersection point will be considered.

### 4.2 The parameter $\mu$

In Figure 4.2 the effect of varying $\mu$ on the surge cycle is depicted. Figures 4.2a and b show the effect of $\mu$ on the amplitude of $x_1$ and $x_2$. When $\mu$ is large, oscillations are of the relaxation type and have an amplitude of $x_1$ and $x_2$ of approx. 2 and $\frac{2}{3}$, respectively. When $\mu$ is decreased to a value lower than 10, the amplitude of $x_2$ increases and the amplitude of $x_1$ decreases; the oscillations lose their relaxation type behavior and the limit cycle does not fit closely around the compressor characteristic any more. For very small $\mu$, the amplitude decreases very fast with decreasing $\mu$; the system goes from deep surge to mild surge and then to stability. This is similar to the effect of $B$ on the dimensionless Greitzer model, see Section 2.7.

The effect of varying $\mu$ on the stability of the intersection point is depicted in Figure 4.2d. This figure shows the eigenvalues of the transformed Greitzer model (equation (3.11)) as a function of $\mu$. Here it can be seen that for low $\mu$ the intersection point is stable for low $\mu$. The critical value of $\mu$ indicates the transition of stable to unstable operation and depends on the parameters of the model. The critical value of $\mu$ is determined by the dynamic instability criterium. In Section 2.6 this criterium was defined for the dimensionless Greitzer system. For the transformed Greitzer system, the dynamic instability criterium is

$$\frac{F}{\mu P^\#} - \mu C^\# < 0$$

The effect of varying $\mu$ on the period time of the surge cycle is depicted in Figure 4.2c. From this figure it can be seen that for $\mu > 5$, the period time of the limit cycle (from here on denoted as $P^\#$) increases approximately linearly with $\mu$. The approximately linear relation between $\mu$ and $P^\#$ corresponds to the earlier observation that the limit cycle oscillations are of the relaxation type for large $\mu$. In terms of an actual compression system this implies that, for high values of $\mu$, the period time is determined by the time needed to fill and empty the plenum. The time needed for reversing the flow is comparatively short. This can also be seen from equation (4.1) and (4.2) by noting that the rate of change in mass flow, described by $\frac{dx_1}{d\xi}$ depends on $\mu$, while the change in pressure rise, described by $\frac{dx_2}{d\xi}$ depends on the reciprocal of $\mu$. The effect of $\mu$ is in agreement with findings reported in literature, e.g. Van Helvoirt (2007) and Greitzer (1976b).
Figure 4.2: Simulation of transformed Greitzer system with varying $\mu$ and $S^\#: 0.5$, $x_{1,0} = 0.5$ and $F = 1$; (a) amplitude of $x_1$, (b) amplitude of $x_2$, (c) dimensionless period time, (d) real part of eigenvalues.

4.3 The parameter $F$

The parameter $F$ in equation (2.11) is a measure for the volume ratio between the plenum and the system volume. In the original Greitzer model, the system volume is the environment. It is infinitely large and thus $F$ is 1. However, when the compressor operates in a closed system it does not have an infinitely large volume, like the environment, to extract its gas or air from and this affects the operation of the compressor. An $F$ of approximately 1 implies that the plenum volume $V_p$ is much smaller than the system volume $V_s$. When both the plenum and system volume are of the same size, $F$ is approximately 2. Note that a value of 2 is very extreme. The value of $F$, for most compressor systems, is close to 1. Van Helvoirt (2007), for instance, uses a closed system with $F = 1.1$.

Unlike $\mu$, $F$ only affects the mass balance of the model, equation (4.2). Since $x_2$ is derived from the pressure difference between the plenum and the system volume, a large $F$ (i.e. a small system volume) implies that adding more gas to the plenum volume, results in decreasing the
The effect of varying $F$ can be seen in Figure 4.3. For increasing $F$ the amplitude of the limit cycle increases. The period time of the limit cycle is shorter, because with a large $F$ the system volume is relatively small and increasing pressure in the plenum results in a decreasing pressure in the system. Therefore the maximum pressure difference is reached faster. This can be seen in Figure 4.3c, where the period of the surge cycle is plotted as a function of $\mu$ and with varying values of $F$.

The effect of varying $F$ on the eigenvalues of the linearized system is depicted in Figure 4.3d. Increasing $F$ increases the critical value of $\mu$. This agrees with the criterium for dynamic instability of the transformed Greitzer system, equation (4.5).
4.4 The throttle characteristic

In this section the effect of the transformed throttle characteristic on the limit cycle will be discussed. In order to simplify this analysis, the nonlinear transformed throttle characteristic is replaced with a linear approximation $S^#x_2 + k$, as was discussed in Section 4.1. In Section 4.4.1, the effect of this linearization will be discussed and the effect of varying of $x_{1,0}$ and $S^#$ will be discussed in Section 4.4.2 and 4.4.3.

4.4.1 Quadratic vs. linear throttle characteristic

Replacing the quadratic throttle characteristic with a linear characteristic, while retaining the intersection point and the slope of the characteristic in the intersection point, results in an error in the predicted throttle flow. This error is positive for all $x_2$, as can be seen in Figure 4.4.

As was mentioned in Section 4.1, $\frac{dx_2}{d\xi}$ is determined by the difference between the momentary state and $g(x_2)$. This difference is affected by the linearization of the throttle characteristic. Because the error of the linearization compared to the quadratic model (difference in distance) is always positive, $\frac{dx_2}{d\xi} \big|_{lin} \leq \frac{dx_2}{d\xi} \big|_{quad}$ for all $x_2$.

This can be deduced by writing equation (4.2) with a quadratic or linear throttle characteristic, respectively, as

$$\frac{dx_2}{d\xi} \big|_{quad} = \frac{F}{\mu} \left[ x_1 - g(x_2) \right]_{quad} (4.6)$$

$$\frac{dx_2}{d\xi} \big|_{lin} = \frac{F}{\mu} \left[ x_1 - g(x_2) \right]_{lin} (4.7)$$

It can then be seen that

$$\frac{dx_2}{d\xi} \big|_{lin} = \frac{dx_2}{d\xi} \big|_{quad} - \frac{F}{\mu} \left[ g(x_2) \big|_{lin} - g(x_2) \right]_{quad} (4.8)$$
Because
\[ g(x_2)|_{\text{lin}} - g(x_2)|_{\text{quadr}} \geq 0 \quad \forall x_2 \quad (4.9) \]
it holds that
\[ \frac{dx_2}{d\xi}|_{\text{lin}} \leq \frac{dx_2}{d\xi}|_{\text{quadr}} \quad \forall x_2 \quad (4.10) \]

For an actual compression system, this means that the net flow into the plenum is affected by the linearization of the throttle characteristic. Because the mass flow error of the linearized throttle characteristic compared to the quadratic characteristic is always positive, the linearized model will always predict a larger net mass flow into the plenum. The result is that the linearized model predicts that the pressure rises faster and decreases slower compared to the model with a quadratic throttle characteristic.

The magnitude of the throttle flow error depends on the slope of the throttle characteristic and the position of the intersection point. In terms of an actual compression system, the largest flow errors are made for systems with a low \( C_0/H \) ratio. With a specific throttle characteristic, the largest error is made for low \( \Delta p \).

The effect of linearizing the throttle characteristic on the period time of the limit cycle when it is of the relaxation type is small. The largest errors seen in the predicted period time for large \( \mu \) are approximatively 0.5\% and they are present when \( x_{1.0} \) is high. It also does not effect the linearity of the relationship between \( \mu \) and the period time of the limit cycle. This is shown in Figure 4.5.

### 4.4.2 Varying \( x_{1.0} \)

The linear throttle characteristic has two parameters, namely the inverse of the slope of the throttle characteristic \( S^\# \) and the intersection point \( x_{1.0} \). In this section the effect of varying \( x_{1.0} \) is studied.

Figure 4.6 shows the effect of varying \( x_{1.0} \) on the shape and the period time of the limit cycle. In Figure 4.6c the relation between \( \mu \) and the simulated period time of the limit cycle is plotted for various values of \( x_{1.0} \). It can be seen that an \( x_{1.0} \) closer to the top of the compressor curve results in a longer period time. This can be explained by considering equation (4.2). This equation states that \( \frac{dx_2}{d\xi} \) (related to the time needed to fill and empty the plenum) depends on the horizontal difference between the momentary state \( (x_1, x_2) \) and \( g(x_2) \). For increasing \( x_1 \) this difference decreases for \( x_1 > g(x_2) \) and it increases for \( x_1 < g(x_2) \). In terms of percentage, the decrease of the horizontal difference is larger than the increase and therefore the period time of the surge cycle increases for increasing values of \( x_{1.0} \).

For an actual compression system, changing the point of intersection of the compressor and throttle characteristics corresponds to changing the operating point of the system. It is known from literature that moving the (unstable) operating point towards a lower mass flow results in a slightly higher surge frequency (Van Helvoirt, 2007). This is due to the fact that the time
Figure 4.5: Period time of limit cycle of transformed Greitzer model with quadratic (*) and linear (o) throttle characteristic; $S^\# = 0.3$, $x_{1,0} = 0.5$.

Figure 4.6: Simulation of transformed Greitzer system with varying $\mu$ and $x_{1,0}$, $S^\# = 0.5$, $F = 1$; (a) amplitude of $x_1$, (b) amplitude of $x_2$, (c) dimensionless period time, (d) real part of eigenvalues.
Figure 4.7: Simulation of transformed Greitzer system with varying $\mu$ and $S^\#$, $x_{1,0} = 0.5$, $F = 1$; (a) amplitude of $x_1$, (b) amplitude of $x_2$, (c) dimensionless period time, (d) real part of eigenvalues and, with $x_{1,0} = 0.9$; (e) dimensionless period time.
needed to fill the plenum increases while the time needed to empty it decreases. This effect is also predicted from the simulation results and the discussion above.

The effect of varying \( x_{1,0} \) on the eigenvalues of the linearized system is depicted in 4.6d. Increasing \( x_{1,0} \) increases the critical value of \( \mu \). This agrees with the criterium for dynamic instability of the transformed Greitzer system, equation (4.5).

### 4.4.3 Varying \( S^\# \)

The second parameter that affects the limit cycle of the transformed Greitzer system is the slope \( 1/S^\# \) of the transformed throttle characteristic. The effect of varying of \( S^\# \) between 0 and 1 on the limit cycle is shown in Figure 4.7. Note that a higher value of \( S^\# \) implies a less steep throttle characteristic. Figure 4.7c shows that in the case of \( x_{1,0} = 0.5 \) the period time of the limit cycle increases when \( S^\# \) is increased. The same trend can be observed for \( x_{1,0} < 0.5 \). However, for the case \( x_{1,0} = 0.9 \) (Figure 4.7e) the period time decreases with increasing \( S^\# \). Again, these observations can be explained by noting the effect of increasing \( S^\# \) on \( \frac{dc_2}{d\xi} \) and the related filling and emptying times of the plenum.

In terms of actual compression system, the effect of \( S^\# \) on \( P^\# \) can be explained by noting the effect of decreasing the slope of the throttle characteristic (increasing \( S^\# \)) on the net inflow into the plenum. For \( x_2 < x_{2,0} \) the net inflow increases while it decreases for \( x_2 > x_{2,0} \). For intersection points at low mass flows (\( x_{1,0} < 0.5 \)) the decreasing net inflow has a larger effect and hence the period time of the surge cycle increases. However, for intersection points near the top of the compressor characteristic, the effect of a decreasing net inflow is much smaller, yielding a decrease in the period time.

The effect of varying \( S^\# \) on the eigenvalues of the linearized system is depicted in Figure 4.7d. Increasing \( S^\# \) increases the critical value of \( \mu \). This agrees with the criterium for dynamic instability of the transformed Greitzer system, equation (4.5). This effect can also be seen in Figure 4.7a and b. These figures show that for low \( \mu \), the amplitude of the limit cycle decreases for increasing \( x_{1,0} \). This illustrates the transition from deep surge to mild surge and stability.

### 4.5 The parameters of the dimensional Greitzer model

In the previous sections the effect of the nondimensional parameters on the nondimensional period time is discussed. In this section the parameter study is related to the parameters of the dimensional Greitzer model.

The two-state Greitzer model in dimensional form with an infinite system volume \( (F = 1) \) is

\[
\frac{d\dot{m}_c}{dt} = \frac{A_c}{L_c}\left[\Delta p_c(\dot{m}_c) - \Delta p\right] \tag{4.11}
\]

\[
\frac{d\Delta p}{dt} = \frac{a^2}{V_p}\left[\dot{m}_c - \dot{m}_t(\Delta p)\right] \tag{4.12}
\]
with the cubic Moore-Greitzer polynomial in dimensional form

\[
\Delta p_c(\dot{m}_c) = C_0 + H \left[ 1 + \frac{3}{2} \left( \frac{\dot{m}_c}{\dot{m}_c W} - 1 \right) - \frac{1}{2} \left( \frac{\dot{m}_c}{W} - 1 \right)^3 \right]
\] (4.13)

The relation between the period time of the dimensional Greitzer model and the period time of the transformed Greitzer model is

\[
P = \frac{P^\#}{\omega_H}
\] (4.14)

This implies that the results of the parameters studies for parameters that do not influence \( \omega_H \) can be applied to the dimensional model: when the parameter study predicts an increasing period time of the limit cycle of the transformed Greitzer model, the period time of the limit cycle of the dimensional Greitzer model also increases. This is the case for the parameters of the throttle characteristic and \( F \) (assuming the reason for changing \( F \) is not a change in \( V_p \)).

However, varying \( \mu \) can effect \( \omega_H \), because with \( W^* = \frac{W}{\rho A_c U_t} \) and \( H^* = \frac{H}{\frac{H}{2 \rho U_t^2}} \), \( \mu \) can be written as

\[
\mu = \frac{3}{2} H \frac{V_p}{W} \sqrt{\frac{A_c}{L_c}} \sqrt{\frac{a^2}{V_p}}
\] (4.15)

and

\[
\omega_H = \sqrt{\frac{A_c}{L_c}} \sqrt{\frac{a^2}{V_p}}
\] (4.16)

Now the effect of increasing \( \frac{A_c}{L_c} \) on the period time can be deduced by the following reasoning. Suppose \( \frac{A_c}{L_c} \) of the dimensional system is increased with a multiplication factor \( x \) to \( \frac{A_c}{L_c} = x \frac{A_c}{L_c} \). The parameter \( \mu_1 \) is then multiplied with \( \sqrt{x} \), yielding \( \mu_2 = \sqrt{x} \mu_1 \). Now from Section 4.2 it is known that the dimensionless period time increases when \( \mu \) increases. Suppose that the relation between \( P^\# \) and \( \mu \) is approximated with

\[
P_1^\# = b \mu_1 + c
\] (4.17)

then

\[
P_2^\# = b \sqrt{x} \mu_1 + c
\] (4.18)
and

\[ P_2^\# = \sqrt{x}P_1^\# + c(1 - \sqrt{x}) \]  

(4.19)

Noting that \( \omega_{H2} = \sqrt{x}\omega_{H1} \) and substituting equation (4.19) into equation (5.14) gives the new dimensional period time

\[
P_2 = \frac{P_2^\#}{\omega_{H2}} = \frac{\sqrt{x}P_1^\# + c(1 - \sqrt{x})}{\sqrt{x}\omega_{H1}} \]

(4.20)

\[
= P_1 + \frac{c(1 - \sqrt{x})}{\sqrt{x}\omega_{H1}} \quad \text{(4.21)}
\]

\[
< P_1 \quad \text{(4.22)}
\]

Thus the dimensional period time decreases slightly when \( \frac{Ac}{Lc} \) is increased.

This way of reasoning can also be followed for \( \frac{a^2}{V_p} \) and the results are stated in Table 4.1. Note that this is only valid for large \( \mu \), because for \( \mu < 4 \) the relation between \( P^\# \) and \( \mu \) is not linear. From Table 4.1, it can be concluded that \( \frac{a^2}{V_p} \) has a much larger influence on the period time than \( \frac{Ac}{Lc} \).

Table 4.1: The effect of multiplying \( \frac{Ac}{Lc} \) or \( \frac{a^2}{V_p} \) with \( x \) on \( \mu \), \( \omega_H \) and the period time.

<table>
<thead>
<tr>
<th>multiplication factor</th>
<th>( \mu_2 )</th>
<th>( P_2^# )</th>
<th>( \omega_{H2} )</th>
<th>( P_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{Ac}{Lc} )</td>
<td>( x )</td>
<td>( \sqrt{x}\mu_1 )</td>
<td>( \sqrt{x}P_1^# + c(1 - \sqrt{x}) )</td>
<td>( \sqrt{x}\omega_{H1} )</td>
</tr>
<tr>
<td>( \frac{a^2}{V_p} )</td>
<td>( x )</td>
<td>( \sqrt{\frac{x}{z}}\mu_1 )</td>
<td>( \sqrt{\frac{x}{z}}P_1^# + c(1 - \sqrt{\frac{x}{z}}) )</td>
<td>( \sqrt{\frac{x}{z}}\omega_{H1} )</td>
</tr>
</tbody>
</table>

39
4.6 Summary

In this chapter, a parameter study was performed on the transformed Greitzer system with a linear throttle characteristic. The effect of the stability parameter $\mu$, $F$, the intersection point $(x_{1,0}, x_{2,0})$ and the inverse of the slope of the throttle characteristic $S^\#$ on limit cycle were discussed.

In short, the effects of the parameters on the dimensionless period time $P^\#$ are:

- Increasing $\mu$ increases $P^\#$.
- Increasing $F$ decreases $P^\#$.
- Increasing $x_{1,0}$ increases $P^\#$.
- Increasing $S^\#$ increases $P^\#$ for low $x_{1,0}$ and decreases $P^\#$ for high $x_{1,0}$
- Increasing $F$, $x_{1,0}$ or $S^\#$ increases the critical value of $\mu$.

It was seen that an approximately linear relation exists between the stability parameter and the period time of the limit cycle. The approximate linearity of this relation is preserved when $F$, the intersection point or the slope of the throttle characteristic is varied. The fact that the nature of the relationship between $\mu$ and $P^\#$ does not change for different values of $S^\#$ and $x_{1,0}$ will be used in the next chapter to derive an equation for the relation between $\mu$ and $P^\#$, based on the known equation for the relation between $\mu$ and the period time of the Van der Pol system.
Chapter 5

Parameter identification based on the period time of the limit cycle of the Greitzer model

In chapter 2 it was discussed that the identification of the model parameters of the Greitzer system is difficult. However, they have a large influence on the stability of the system. The stability parameter $B$ is very important in the Greitzer model. It determines whether the system will exhibit stall or surge or is stable. In Van Helvoirt (2007) the parameter $B$ is determined in three different ways with an accuracy of $\pm 18\%$. Improving the accuracy in determining the model parameters will be very helpful in the active control design of a compressor.

The aim of this chapter is to find a relation between the parameters of the Greitzer system and the period of the limit cycle ($P$) that takes into account the slope of the throttle characteristic ($1/S$) and the position of the intersection point ($m_{c,0}$). Because the period time of the surge cycle can be easily measured, this can help to identify the parameters of the Greitzer model.

First an expression will be presented for the period time of the limit cycle of the Van der Pol system. Then, in Section 5.2, this expression is modified to represent the period time of the transformed Greitzer system. Next, in Section 5.3, the equation for the period time of the dimensional Greitzer system is developed. In Section 5.4 an example is given of a parameter identification. Subsequently, the sensitivity of this method is discussed and some remarks are made on the domain of utility. The chapter is concluded with a discussion.

5.1 The period time of the limit cycle of the Van der Pol equation

The period of the limit cycle of the Van der Pol equation ($P_v$) was calculated by Bavink and Grasman (1969) for $\mu \gg 1$ as

$$P_v = (3 - 2 \ln 2)\mu + 3a\mu^{-1/3} - \frac{2}{3} \mu^{-1} \ln \mu +$$
$$+ (\ln 2 - \ln 3 + 3b_0 - 1 - \ln \pi - 2 \ln Ai'(-\alpha))\mu^{-1} + O(\mu^{-1}) \quad (5.1)$$
with numerical values $a = 2.338...$, $b_0 = 0.1723...$ and $Ai'(-\alpha)$ is the derivative of the Airy function at $-a$. For the Airy function, see Abramowitz and Stegun (1964). In short, equation (5.1) is obtained by dividing the limit cycle of equation 3.1 in 8 parts and calculating the solution for each part. The constants of the solutions are chosen such that the solutions of the different parts match. The solutions for each part are then integrated over time and added to obtain equation (5.1), see Bavink and Grasman (1969).

With the constants, equation (5.1) becomes (Bavink and Grasman, 1969)

$$P_v = 1.613706\mu + 7.014321\mu^{-1/3} - \frac{2}{3}\mu^{-1}\ln\mu - 1.3234\mu^{-1} + O(\mu^{-1})$$  \hspace{1cm} (5.2)

Figure 5.1 shows the calculated period time of the Van der Pol equation via equation 5.2 and the simulated period time of the Van der Pol equation. For large $\mu$ ($\mu > 4$), the difference between the simulated and the calculated period is smaller than 1%. For $\mu > 10$, the deviation is less than 0.1%.

### 5.2 The period time of the limit cycle of the transformed Greitzer system

In this section equation (5.2) will be used as a basis for developing an equation that describes the period time of the limit cycle of the transformed Greitzer system as a function of $\mu$ and the transformed linear throttle characteristic, $P^\# = f(\mu, S^\#, x_{1,0})$.

Equation (5.2) represents the period time of the Van der Pol system. The Van der Pol system is identical to the transformed Greitzer system with $S^\# = 0$ and $x_{1,0} = 0$. In Chapter 4 it was seen that the values of $S^\#$ and $x_{1,0}$ influence the steepness and the height of the function $P^\# = f(\mu, S^\#, x_{1,0})$. The steepness and height of $P_v(\mu)$ are determined by the constants of the
Figure 5.2: Contribution of the terms of equation (5.2) to the calculated period time of the limit cycle of the Van der Pol system.

first two terms of equation 5.2. This can be seen in Figure 5.2 that shows the contribution of the four terms of equation (5.2) to $P_p(\mu)$. The first term determines the steepness, the second term the height. The third and fourth term comparatively small for high $\mu$.

Therefore, an appropriate approximation of $P^\#$ as a function of $\mu$ is

$$P^\# = \gamma \mu + \alpha \mu^{-1/3} - \frac{2}{3} \mu^{-1} \ln \mu - 1.3234 \mu^{-1}$$  \hspace{1cm} (5.3)

with $\gamma(S^\#, x_{1,0})$ and $\alpha(S^\#, x_{1,0})$ because they determine the steepness and the height of $P^\#(\mu)$.

To find values for $\gamma(S^\#, x_{1,0})$ and $\alpha(S^\#, x_{1,0})$, the period times of the limit cycle for each combination of $S^\# = [0 : 0.1 : 0.9]^1$ and $x_{1,0} = [0 : 0.1 : 0.9]$ and $\mu$ are calculated from simulations of the transformed Greitzer model for $\mu_{1...n} = [4 : 1 : 30]$. With the resulting period times values of $\alpha$ and $\gamma$ are estimated for each combination of $S^\#$ and $x_{1,0}$ by solving

$$b = Ax$$  \hspace{1cm} (5.4)

in a least squares sense by minimizing the Euclidian norm $||Ax - b||_2$.

The calculated $\alpha$ and $\gamma$ at each combination of $S^\#$ and $x_{1,0}$ are plotted in Figure 5.3(a) and 5.3(b). In Figure 5.4, $\frac{1}{n} \sum \left( P^\#_{\text{sim}} - P^\#_{\text{eq}} \right)^2$, with $P^\#_{\text{eq}}$ via equation (5.3), is plotted as a measure for the accuracy of the estimates for $\alpha$ and $\gamma$. In this figure it can be seen that the approximation is least

---

The used MATLAB-like notation $[a : b : c]$ is short for a row from $a$ to $b$ with spacing $c$. 

43
accurate for high \(x_{1,0}\) and low \(S^\#\) values. These values represent steep throttle characteristics near the top of the compressor curve.

When values of \(\alpha\) and \(\gamma\) are needed for intermediate values of \(x_{1,0}\) and \(S^\#\), they are calculated with a 2D second order spline-fit on the grid of calculated \(\alpha\) and \(\gamma\) values, respectively.

### 5.3 The period time of the limit cycle of the dimensional Greitzer system

In the previous section a function was developed for the dimensionless period time \(P^\# = f(\mu, S^\#, x_{1,0})\). However, this function cannot be used for parameter identification, because the parameters that are needed to make the equations nondimensional, are also parameters
that need to be identified. Therefore, in this section the function \( P^# = f(\mu, S^#, x_{1,0}) \) from equation (5.3), will be transformed into a function that is based on the dimensional parameters: \( P = f\left(\frac{a^2}{\mu^2}, \frac{\Delta_0}{L_0}, H, W, S, m_{c,0}\right) \).

First it will be shown that \( x_{1,0} \) and \( S^# \) can be written in terms of the dimensional parameters \( H, W, S \) and \( m_{c,0} \). Then with the definition of \( P^# \) and \( \mu \), equation (5.3) will be transformed to a relation between the dimensional period time of the surge cycle and the model parameters.

5.3.1 \( x_{1,0} \) and \( S^# \)

The parameters \( \alpha \) and \( \gamma \) are functions of the inverse of the slope of the transformed throttle characteristic and the intersection point of the transformed compressor and throttle characteristic.

In Section 2.2, \( \phi_c \) is defined as \( \phi_c = \frac{\dot{m}_{c,0}}{\rho A_{c} U_t} \). It then follows that \( W^* = \frac{W}{\rho A_c U_t} \). Substituting the dimensional intersection point \( \dot{m}_{c,0} \) in the transformation \( x_1 = \frac{\phi_c - W^*}{W^*} \) (see section 3.2) leads to

\[
x_{1,0} = \frac{\dot{m}_{c,0} - W}{W}
\]  

Similarly, the inverse of the slope of the transformed throttle characteristic at the intersection point \((m_{c,0}, \Delta p_0)\) can be calculated with

\[
S^# = \left. \frac{dx_1}{dx_2}\right|_{(x_{1,0}, x_{2,0})}
\]

Noting that \( x_2 = \frac{2}{3H^*}(\psi - C_0^* - H^*) \) (Section 3.2), \( S^# \) can be written as

\[
S^# = \left. \frac{d(\phi_c - W^*)}{d(\frac{2}{3H^*}(\psi - C_0^* - H^*))}\right|_{(\phi_{c,0}, \psi_0)} = \left. \frac{d(\frac{\phi_c}{W})}{d(\frac{2}{3H^*}\psi)}\right|_{(\phi_{c,0}, \psi_0)}
\]

and with \( \psi = \frac{\Delta p}{\frac{2}{3H^*}U_t^2} \) (see Section 2.2), \( H^* \) can be written as, \( H^* = \frac{H}{\frac{2}{3}U_t^2} \), so

\[
S^# = \left. \frac{d(\frac{\dot{m}_{c,0}}{W})}{d(\frac{2}{3H^*}\Delta p)}\right|_{(m_{c,0}, \Delta p_0)}
\]

\[
= \frac{3H}{2W} \left. \frac{d\dot{m}_{c}}{d\Delta p}\right|_{(m_{c,0}, \Delta p_0)}
\]

\[
= \frac{3H}{2W} S
\]

Now \( \alpha \) and \( \gamma \) can be determined as a function of \( m_{c,0} \) and \( S \).
5.3.2 \( P \) as a function of dimensional parameters

The dimensionless period time \( P^\# \) can be written in terms of the dimensional period \( P \) and the parameters of the dimensional model as

\[
P^\# = P \omega_H \\
= P \sqrt{\frac{a^2}{V_p}} \sqrt{\frac{A_c}{L_c}}
\]

(5.13)

(5.14)

The parameter \( \mu \), defined as

\[
\mu = \frac{3}{2} \frac{H^*}{W^*} B
\]

(5.15)

can be written in terms of the parameters of the dimensional model as

\[
\mu = \frac{3}{2} \frac{H}{W} \frac{\rho A_c U_t}{\frac{1}{2} \rho U_t^2} \frac{U_t}{2 \omega H L_c}
\]

(5.16)

\[
\mu = \frac{3}{2} \frac{H}{W} \frac{A_c}{L_c} \frac{1}{\omega_H}
\]

(5.17)

\[
\mu = \frac{3}{2} \frac{H}{W} \sqrt{\frac{V_p}{a^2}} \sqrt{\frac{A_c}{L_c}}
\]

(5.18)

With equation (5.14) and (5.18), equation (5.3) can be written in terms of dimensional parameters

\[
P = \gamma(S, m_{c,0}) \frac{\mu}{\omega_H} + \alpha(S, m_{c,0}) \mu^{-1/3} - \frac{2}{3} \ln \mu \omega_H - \frac{1.3234}{3 \omega_H} \frac{1}{\mu \omega_H}
\]

(5.19)

\[
= \gamma(S, m_{c,0}) \frac{3H}{2W} \sqrt{\frac{V_p}{a^2}} \sqrt{\frac{A_c}{L_c}} + \alpha(S, m_{c,0}) \frac{3H}{2W} \frac{1}{\frac{1}{2} \rho U_t^2} \frac{U_t}{\sqrt{a^2 V_p} \sqrt{A_c}} - \frac{2}{3} \ln \left( \frac{3H}{2W} \sqrt{\frac{V_p}{a^2}} \sqrt{\frac{A_c}{L_c}} \right) - \frac{1.3234}{3 \omega_H}
\]

(5.20)

\[
= \gamma(S, m_{c,0}) \frac{3H}{2W} \frac{V_p}{a^2} + \alpha(S, m_{c,0}) \left( \frac{3H}{2W} \right)^{1/3} \left( \frac{V_p}{a^2} \right)^{1/3} \left( \frac{A_c}{L_c} \right)^{-2/3} - \frac{4W}{9H} \ln \left( \frac{3H}{2W} \sqrt{\frac{V_p}{a^2}} \sqrt{\frac{A_c}{L_c}} \right) \left( \frac{A_c}{L_c} \right)^{-1} - \frac{1.3234}{3 \omega_H}
\]

(5.21)

With this result the period time of the limit cycle of the dimensional Greitzer system \( P \) is now expressed as a function of the dimensional model parameters.
5.4 An example of parameter identification

In the previous section an equation was developed to relate the period time of the surge cycle to the parameters of the dimensional Greitzer system. In this section, this equation will be used to identify the parameters of a simulated compressor system.

A two-state Greitzer model in the form of equations (4.11) and (4.12) is simulated. The compressor characteristic is modeled as a cubic Moore-Greitzer polynomial, equation (4.13), and the throttle characteristic is modeled as a linear characteristic. The model parameters are stated in Table 5.1. The parameters of the compressor characteristic, $H$ and $W$, and the throttle characteristic, $S$ and $m_{c,0}$, are assumed to be known. The simulated model is similar to Test rig A from the work of Van Helvoirt (2007). Figure 5.5 and 5.6 show the surge cycle and the pressure oscillations. The period time of the surge cycle $P^*$ can be determined from the pressure oscillations, here 0.8310 seconds.

In order to find the relation between $P$ and $\frac{V_p}{a^2}$ and $\frac{A_c}{L_c}$, the parameters $\alpha$ and $\gamma$ in equation (5.21) have to be determined. To determine these parameters, first $S^#$ and $x_{1.0}$ are determined from equation (5.6) and (5.12) and then the method explained in Section 5.2 is followed. The values for $\alpha$ and $\gamma$ are determined from the obtained spline function. The resulting values of $\alpha$ and $\gamma$ are stated in Table 5.1. However, $\alpha$ and $\gamma$ do not have to be determined for the entire range of $S^# = [0 : 0.1 : 0.9]$ and $x_{1.0} = [0 : 0.1 : 0.9]$. A set of simulations of the transformed Greitzer system with the $S^#$ and $x_{1.0}$ from this example, calculated via equation (5.6) and (5.12), would suffice and the $\alpha$ and $\gamma$ could be calculated from this set of period times via equation (5.5).

Now, three scenario’s will be discussed: (1) $a^2 \frac{V_p}{V_p}$ is known and $\frac{A_c}{L_c}$ has to be identified, (2) $\frac{A_c}{L_c}$ and $a$ are known and $V_p$ has to be identified and (3) both $a^2 \frac{V_p}{V_p}$ and $\frac{A_c}{L_c}$ are unknown.

In scenario (1), where $a^2 \frac{V_p}{V_p}$ is known, the relationship between $P$ and $\frac{A_c}{L_c}$, as defined in equation (5.21) can be determined, see Figure 5.8. Note that $P$ is on the horizontal axis and $\frac{A_c}{L_c}$ is on the vertical axis. From this figure, the estimated $\frac{A_c}{L_c}$ can be determined as 0.0119 (at $P = 0.8310$). The actual $\frac{A_c}{L_c}$ is 0.0113 and the deviation is 5.2%. In Figure 5.8 it can also be seen that a deviation $\frac{A_c}{L_c}$ around the actual value of 0.0113 only leads to a small deviation in the period time.

<table>
<thead>
<tr>
<th>model parameter</th>
<th>value</th>
<th>calculated parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$3.2613 \cdot 10^4$</td>
<td>$B$</td>
<td>3.75</td>
</tr>
<tr>
<td>$W$</td>
<td>0.3689</td>
<td>$\mu$</td>
<td>23.5</td>
</tr>
<tr>
<td>$C_0$</td>
<td>$1.9209 \cdot 10^5$</td>
<td>$S^#$</td>
<td>0.25</td>
</tr>
<tr>
<td>$A_c$</td>
<td>0.0034</td>
<td>$x_{1.0}$</td>
<td>0.85</td>
</tr>
<tr>
<td>$L_c$</td>
<td>0.3</td>
<td>$\alpha$</td>
<td>7.9474</td>
</tr>
<tr>
<td>$V_p$</td>
<td>0.32</td>
<td>$\gamma$</td>
<td>2.1537</td>
</tr>
<tr>
<td>$a$</td>
<td>340</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>$5.3044 \cdot 10^5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m_{c,0}$</td>
<td>0.6824</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U_t$</td>
<td>144</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1: Parameters for example.
Figure 5.5: Compressor characteristic ( - - ), throttle characteristic ( - - ) and the surge cycle ( - ) for example.

Figure 5.6: Pressure oscillations of example.

Figure 5.7: Relation between $\mu$ and $P^\#$.

Figure 5.8: Relation between $P$ and $\frac{A_c}{L_c}$ with known $V_p$.

Figure 5.9: Relation between $V_p$ and $P$ with known $\frac{A_c}{L_c}$.

Figure 5.10: Relation between $V_p$ and $\frac{A_c}{L_c}$. 

\[ \mu = 107.3 \]

\[ \mu = 13.9 \]
Figure 5.11: Surge cycles with $\mu = 13.9$ (black) and $\mu = 107.3$ (gray).

Figure 5.12: Pressure oscillations with $\mu = 13.9$ (–) and $\mu = 107.3$ (··).

In scenario (2), where only $V_p$ is unknown and $\frac{A_c}{L_c}$ and $a$ are known, the relationship between $P$ and $V_p$ can be plotted, see Figure 5.9. This results in an estimated $\hat{V}_p$ of 0.3193. The deviation to the actual $V_p$ is $-0.22\%$. From Figure 5.9 it can also be seen that a deviation in $V_p$ leads to a deviation in $P$ of approximately the same percentage. Because the system is clearly demonstrating relaxation behavior, the period time is mainly determined by the filling and emptying of the plenum, which is regulated by equation (4.12) and $V_p$ is the most important parameter of the system. Note that the error percentages of $\hat{V}_p$ can vary, depending on the quality of the simulation of the surge cycle and the approximation of the relation between the period time and $\mu$. This will be discussed in Section 5.5.

In scenario (3), where both $V_p$ and $\frac{A_c}{L_c}$ are unknown, a plot can be constructed that shows all the combinations of $\hat{V}_p$ and $\frac{A_c}{L_c}$ that result in the simulated $P^\star$. This is shown in Figure 5.10. The various combinations lead to various values of $\mu$. The combination in the steep part imply a value of $\mu$ much higher than the combinations in the flat part of the plot. For instance, in the indicated point in the bottom left of the plot, $\mu = 13.9$, and in the indicated point in the in the top right, $\mu = 107.3$.

Because there is no unique combination of $\hat{V}_p$ and $\frac{A_c}{L_c}$ that corresponds to $P^\star$, combination of $\hat{V}_p$ and $\frac{A_c}{L_c}$ has to be chosen based on simulations of the pressure oscillations. To show the difficulty in selecting a combination of $\hat{V}_p$ and $\frac{A_c}{L_c}$, the system is simulated with the parameters in the indicated points in Figure 5.10 with $\mu = 13.9$ and $\mu = 107.3$. Figure 5.11 and 5.12 show the surge cycle and the pressure oscillations, respectively, of the Greitzer system with these parameters. The difference between the time traces of the pressure oscillations is small. It is therefore difficult to select the correct set of parameters from Figure 5.10 when $\mu$ is large. It is then impossible to determine the parameters $\hat{V}_p$ and $\frac{A_c}{L_c}$ separately and unambiguously.

Note that a $\mu$ of 107.3 is not within the range of $\mu$ that was used to calculate $\alpha$ and $\gamma$. It is however used here to illustrate that a wide range of $\mu$ can lead to pressure oscillations with only small deviations that will be difficult distinguish in the measured, noisy pressure signal.
5.5 Sensitivity

In the previous section an example was given of how equation (5.21) can be used for the identification of the model parameters of the Greitzer system. In this section the sensitivity of the estimated parameters with respect to the period time will be discussed.

In order to investigate the errors and the sensitivity of the parameter identification method, simulations were done with four models with various values of $\mu$. The parameters that were used can be found in Appendix A. These parameters are for a large part taken from experimental set-ups in the literature, however, in contrast to the actual systems, each of the compressor characteristics is modeled with the Moore-Greitzer polynomial and a linearized throttle characteristic is used.

For each model the values of $\hat{A}_cL_c$ and $\hat{V}_p$ are estimated in the same way as in scenario (1) and (2) in Section 5.4. This is done for a range of $S^\#$ and $x_{1,0}$ from 0.1 to 0.9. The error percentage is plotted in Figure 5.13 for $\hat{A}_cL_c$ and in Figure 5.14 for $\hat{V}_p$. From Figure 5.13 it can be seen that the error percentage of $\hat{A}_cL_c$ varies between approximately 1% and 7%. The error percentage of $V_p$ varies between approximately 0.3% and -1% and is dependant on $\mu$, as can be seen in Figure 5.14.

Figure 5.13 and 5.14 also show the sensitivity of the error in $\hat{A}_cL_c$ and $\hat{V}_p$, respectively, to a deviation of $P$ calculated with equation (5.21). The sensitivity of $\hat{A}_cL_c$ to these errors is calculated with

$$
\frac{\%_{\text{err}}(\hat{A}_c)}{\%_{\text{err}}(P)} = \frac{\partial(\hat{A}_c)}{\partial(P)} \frac{P}{\hat{A}_c} \tag{5.22}
$$

Here, $\frac{\partial(\hat{A}_c)}{\partial(P)}$ is calculated by taking the inverse of the derivative with respect to $\hat{A}_c$ of equation (5.21). Equation (5.22) calculates the percentage that $\hat{A}_cL_c$ will deviate as a result of a deviation of $P$ of 1%. In Figure 5.13 it can be seen that $\hat{A}_cL_c$ is very sensitive to errors in the period time, especially for large $\mu$ where an error of 1% in $P$ leads to an error of 30% for $\hat{A}_cL_c$.

In the actual compression system $\frac{\hat{A}_c}{L_c}$ determines the change in mass flow. Because the oscillations are of the relaxation type for large $\mu$, the change in mass flow is very fast and it contributes little to the period time of the surge cycle. A small change in $\frac{\hat{A}_c}{L_c}$ therefore has only little effect on the period time of the surge cycle. On the other hand, to obtain a small change in the period time, a large change is needed in $\frac{\hat{A}_c}{L_c}$. Thus a small error in $P$ will lead to a large error in $\hat{A}_cL_c$.

Similarly, the sensitivity of $\hat{V}_p$ can be calculated with

$$
\frac{\%_{\text{err}}(\hat{V}_p)}{\%_{\text{err}}(P)} = \frac{\partial V_p}{\partial P} \frac{P}{\hat{V}_p} \tag{5.23}
$$

The sensitivity of $V_p$ is plotted for the four models in Figure 5.14. The sensitivity of $V_p$ for large $\mu$ is approximately 1. Because the period time is almost completely determined by the time to fill and empty the plenum (that is determined by $V_p$), an error of 1% in the period time will lead to an error of approximately 1% in $\hat{V}_p$.

The deviation of $P^*$ to the calculated $P$ can have two causes. First, the simulation of $P^*$ or $P^\#$ can contain errors or, secondly, the the calculated relation between $P^\#$ and $\mu$ can contain errors.
Figure 5.13: Errors and sensitivity of $\frac{\hat{A}_c}{L_c}$; (a),(b): $\mu = 23.5$, (c),(d): $\mu = 17.2$, (e),(f): $\mu = 10.7$, (g),(h): $\mu = 4.1$. For parameters, see Appendix A.
Figure 5.14: Errors and sensitivity of $\hat{V}_p$; (a),(b): $\mu = 23.5$, (c),(d): $\mu = 17.2$, (e),(f): $\mu = 10.7$, (g),(h): $\mu = 4.1$. For parameters, see Appendix A.
The error in the simulated period time can be caused by the ODE-solver that is used. In this thesis, the ODE-solver ode45 from MATLAB was used for both the simulation of the period times of the transformed Greitzer system (to calculate $\alpha$ and $\gamma$) and the simulation of the period time of the Greitzer model for which the parameters have to be determined, $P^\star$. To illustrate the effect of the ODE-solver, the simulation of $P^\star$ of the Greitzer model in Section 5.4 is also done with the ODE-solver ode15s from MATLAB. The deviation between the simulated period time with ode45 and ode15s in the calculated period time is plotted in Figure 5.15b. Because sensitivity of $\frac{\Delta_\alpha}{L_c}$ to deviations in the period time is approximately 30 (see Figure 5.13b), the resulting error of $\frac{\Delta_\alpha}{L_c}$ is higher than the error when all simulations are done with ode45. This can also be seen by comparing Figure 5.13a with Figure 5.15a).

The second source of errors is the relation of $P^\#(\mu)$ to $\mu$, equation (5.3). If this equation does not accurately describe the relation $P^\#(\mu)$, then the estimated parameters will contain errors. There can be two reasons for the inaccuracy in $P^\#(\mu)$; first, the estimation of $\alpha$ and $\gamma$ is not optimal or, secondly, $P^\#(\mu)$ cannot be accurately described by equation (5.3). Equation (5.3) was developed for the Van der Pol system. The transformed Greitzer model differs from the Van der Pol system, therefore equation (5.3) is not as accurate in describing the period time of the Greitzer model as it is in describing the period time of the Van der Pol system.

### 5.6 Domain of utility

In the example in Section 5.4, a range values of $\mu$ from 4 to 30 was used to determine the parameters of the relation between $\mu$ and $P$. In this section the effect of selecting a value of $\mu$ outside this range is discussed.

$\mu > 30$

When $\mu$ is very large, the equations become increasingly stiff and the system becomes more difficult to simulate accurately. Therefore it is more difficult to determine $\alpha$ and $\gamma$ accurately.
Also, $\frac{\Delta}{L_c}$ becomes very sensitive to errors in the period time. However, because the period time is larger for high $\mu$, the relative error in the period time is small.

Furthermore, $\frac{\Delta}{L_c}$ becomes more sensitive for errors in $\frac{\alpha_2}{V_p}$. When $\frac{\Delta}{L_c}$ is relatively large (large $\mu$), the time needed to reverse the mass flow becomes very short. The period time then depends on the time needed to fill and empty the plenum, which is only dependant on $\frac{V_p}{\alpha_2}$. When $V_p$ is estimated only 1% to high, this causes a large error in $\frac{\Delta}{L_c}$ and $\mu$.

$\mu < 4$

When the system has a small $\mu$, the relation between $P^#$ and $\mu$ can no longer be assumed to be linear. It is then not guaranteed that the relation between $\mu$ and $P$ can be described by equation (5.3). Especially for high $x_{1,0}$ it can differ much and the relation is not valid.

5.7 Discussion

In this chapter a method is developed to identify the parameters of the Greitzer model. As was seen in Section 5.5, when $V_p$ is known, $\frac{\Delta}{L_c}$ (in which $L_c$ is often a tuning parameter) can be determined with an accuracy of approximately 3 to 5%. This is more accurate than the accuracy of 18%, to which $B$ was determined in (Van Helvoirt, 2007). However, it was seen that the proposed method is very sensitive to errors in the period time. The achieved accuracy depends heavily on the quality of the simulation of the limit cycle and the accuracy of the determination of $\alpha$ and $\gamma$.

It was also seen that it is impossible to determine the $\hat{V}_p$ and $\frac{\Delta}{L_c}$ separately and unambiguously when $\mu$ is large. Only a set of possible combinations can be determined. Each combination implies a different $\mu$ (and $B$). This can have large implications for the controller design. The zero and the poles of the closed loop transfer function directly depend on $\frac{\Delta}{L_c}$ and $V_p$ in the dimensional system (Van Helvoirt, 2007) and on $B$ in the nondimensional Greitzer system (Willems, 2000). However, this problem is also present in the current parameter identification methods.

In this chapter, a linear approximation of the throttle characteristic is used. In Section 4.4.1 it is shown that the linearization of the throttle characteristic causes a slightly lower period time for large $\mu$. The error for high $x_{1,0}$ can be up to 0.2. For the example in Section 5.4 this would mean an error percentage of $P$ of approx. 0.4%. With a sensitivity of 30, this leads to an error of 12% in $\frac{\Delta}{L_c}$ when $V_p$ is known.

The compressor characteristic in this thesis was modeled with a Moore-Greitzer polynomial and the slope of the throttle characteristic and the position of the intersection point are assumed known in this chapter. However, as was discussed in Section 2.5, the slope of the throttle characteristic in the intersection point can differ very much as a result of the method that is used to model the throttle. Furthermore, the compressor characteristic can not always be described by the Moore-Greitzer polynomial. This method could, however, also be applied with different compressor characteristics and throttle characteristics, as long as the relation between $P^#$ and $\mu$ can be described by equation (5.3).
Chapter 6

Conclusion and recommendations

In this thesis the two-state Greitzer model for surge in compressors was discussed. The first objective was to improve the understanding of the Greitzer model by way of a parameter analysis. The second objective was to develop a method for parameter identification based on an analogy with the Van der Pol equations, by means of the available analytical expression for the period time of the limit cycle of the Van der Pol equation.

Conclusions

Parameter identification of the Greitzer model is difficult because the Greitzer model is a lumped parameter model and not all the parameters have an explicit physical meaning. Furthermore, measurements on an industrial compressors are difficult. Only pressure oscillations and the period time of the surge cycle are easily measured.

The Van der Pol system has a limit cycle that resembles the limit cycle of the Greitzer model. To enhance this resemblance, a coordinate transformation was applied to the Greitzer model to equalize the scaling and position of the compressor characteristic and the cubic nullcline of the Van der Pol system. The resulting transformed Greitzer model has no dependance on the height and width of the compressor characteristic. These width and height are taken into account in the new stability parameter, $\mu$. The period time of the limit cycle of the transformed Greitzer model is equal to the period time of the limit cycle of the well-known dimensionless Greitzer model. The only differences between the equations of the Van der Pol system and the transformed Greitzer model are the term that represents the throttle characteristic and a multiplication factor in the mass balance (introduced by Van Helvoirt (2007)). The effect of the term that represents the throttle characteristic is that (with operation points in on the unstable branche of the compressor characteristic) the system becomes stable for low $\mu$, as opposed to the Van der Pol system that has no stable equilibrium points. For high values of $\mu$, the shape of the limit cycle almost the same for both systems, but the period time differs.

The effect of the throttle characteristic on the period time of the transformed Greitzer model was further investigated. For this purpose, the throttle characteristic was linearized and the slope of the throttle characteristic and the intersection point of the throttle and compressor characteristic were varied. Firstly, it was seen that for high values of $\mu$ the relation between the dimensionless period time of the limit cycle and $\mu$ is approximately linear. This approximate linearity is preserved when the slope and the intersection point are varied. However, the slope and intersection point do effect the slope and height of this approximately linear relation.
Because the nature of the relation between the dimensionless period time and $\mu$ does not change when varying the slope of the throttle characteristic and the intersection point, the expression for the period time of the limit cycle of the Van der Pol equation can be adapted to fit the period time of a compressor surge cycle. The result is an equation that relates the period time of the surge cycle to the parameters of the Greitzer model.

With this equation the parameters of the Greitzer model can be identified. The relevant parameters to be identified are $\frac{A_c}{L_c}$ and $\frac{a^2}{V_p}$. When $\frac{a^2}{V_p}$ is assumed to be known a priori, as is usually assumed, $\frac{A_c}{L_c}$ can be estimated with an accuracy of below 7%. The other way around, when $\frac{A_c}{L_c}$ is assumed known a priori, $\frac{a^2}{V_p}$ can be determined with an accuracy of below 0.3% in the studied cases. However, this method does require very good a priori knowledge of $\frac{a^2}{V_p}$ when $\frac{A_c}{L_c}$ is to be determined. When $\frac{a^2}{V_p}$ is only known approximately, the possible combinations of $\frac{a^2}{V_p}$ and $\frac{A_c}{L_c}$ that lead to the measured period time of the limit cycle can be determined. The correct combination has to be chosen based upon comparisons of the measured and simulated pressure oscillations. However, because for high values of $\mu$ the differences in the pressure oscillations are very small, it is often impossible to determine $\frac{a^2}{V_p}$ and $\frac{A_c}{L_c}$ separately and unambiguously.

A different combination of $\frac{A_c}{L_c}$ and $\frac{a^2}{V_p}$ implies a different value of the stability parameter. This parameter uncertainty can be a problem for the design of a controller. However, this problem is also present in the methods currently used for parameter identification. Finally, the developed parameter identification method is very sensitive for deviations in the simulated and measured period time.

**Recommendations**

The next step for this research is to test the proposed method on a real compressor. Here a new challenge will occur, namely the compressor characteristic is usually not a purely cubic function and throttle characteristic is not linear in reality. Although it is expected that the method will also work for these conditions, this has not been tested yet.

Furthermore, because the selected combination of $\frac{A_c}{L_c}$ and $\frac{a^2}{V_p}$ influences the design of the controller, it would be helpful to have guidelines on how to choose the combination of $\frac{A_c}{L_c}$ and $\frac{a^2}{V_p}$ that approaches the physical reality the from the possible combinations.

Lastly, in this thesis the parameter identification is investigated for values of $\mu$ between 4 and 30. It is recommended to extend the range that is investigated. For values of $\mu$ larger than 30, not much will change. But in the range of $\mu$ below 4 the pressure oscillations are not of the relaxation-type and the relation between the period time and $\mu$ is not approximately linear. For an intersection point near the point of inflection of the compressor characteristic, the proposed expression can probably be used. However, for intersection points near the top of the compressor curve, the relation between the period time of the limit cycle of the transformed Greitzer model and $\mu$ does not resemble the relation between the period time of the limit cycle of the Van der Pol system and $\mu$. For this range a different expression will have to be found. For this purpose the FitzHugh-Nagumo model (also called the Bonhoeffer-Van der Pol model) (Rocşoreanu et al., 2000) should be mentioned. The FitzHugh-Nagumo model is a generalized Van der Pol model and resembles the Greitzer model even closer than the Van der Pol system, because it can describe the transformed Greitzer system with the linear throttle characteristic.
Bibliography


Appendix A

Model parameters

Table A.1: Model parameters with $\mu = 23.5$ and $\mu = 17.2$, mainly taken from (Van Helvoirt, 2007), Test rig A.

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_c$</td>
<td>0.0034</td>
</tr>
<tr>
<td>$L_c$</td>
<td>0.3</td>
</tr>
<tr>
<td>$V_p$</td>
<td>0.32</td>
</tr>
<tr>
<td>$a$</td>
<td>340</td>
</tr>
<tr>
<td>$H(N = 15038)$</td>
<td>3.261e4</td>
</tr>
<tr>
<td>$W(N = 15038)$</td>
<td>0.369</td>
</tr>
<tr>
<td>$H(N = 9730)$</td>
<td>1.638e4</td>
</tr>
<tr>
<td>$W(N = 9730)$</td>
<td>0.253</td>
</tr>
<tr>
<td>$\mu(N = 15038)$</td>
<td>23.5</td>
</tr>
<tr>
<td>$\mu(N = 9730)$</td>
<td>17.2</td>
</tr>
</tbody>
</table>
Table A.2: Model parameters with $\mu = 10.7$, mainly taken from (Theotokatos and Kyrtatos, 2004).

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_c$</td>
<td>0.0033</td>
</tr>
<tr>
<td>$L_c$</td>
<td>1.22</td>
</tr>
<tr>
<td>$V_p$</td>
<td>0.025</td>
</tr>
<tr>
<td>$a$</td>
<td>340</td>
</tr>
<tr>
<td>$W$</td>
<td>0.1</td>
</tr>
<tr>
<td>$H$</td>
<td>$0.09 \cdot 10^5$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>10.7</td>
</tr>
</tbody>
</table>

Table A.3: Model parameters with $\mu = 4.1$, mainly taken from (Willems, 2000).

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>25000</td>
</tr>
<tr>
<td>$A_c$</td>
<td>0.0079</td>
</tr>
<tr>
<td>$L_c$</td>
<td>1.8</td>
</tr>
<tr>
<td>$V_p$</td>
<td>0.0203</td>
</tr>
<tr>
<td>$a$</td>
<td>340</td>
</tr>
<tr>
<td>$H$</td>
<td>$1.55 \cdot 10^4$</td>
</tr>
<tr>
<td>$W$</td>
<td>0.1587</td>
</tr>
<tr>
<td>$\mu$</td>
<td>4.1</td>
</tr>
</tbody>
</table>
Appendix B

Greitzer model with variable rotor speed

In this thesis the angular velocity of the impeller of the compressor was assumed to be constant. However, a compressor is a variable-speed machine and the spool dynamics can therefore influence the performance. Especially when a gas turbine is used to power the compressor (e.g. Meuleman, 2002; Fink et al., 1992; Gravdahl et al., 2004; Willems, 2000), the speed variations of the impeller can be high (up to 10% in Fink et al., 1992), because a gas turbine does not deliver a constant power. When an electrical power source is used as in (Van Helvoirt, 2007), the speed variations of the rotor are smaller, but still can be present. The speed variations influence the amplitude of the pressure oscillations and the mass flow variations. In this appendix, a variable rotor speed model is discussed, compared to measurements in literature and a short parameter study is performed.

B.1 Greitzer model equations with variable rotor speed

Fink et al. (1992) proposed to include the spool dynamics in the lumped Greitzer model, which then becomes

\[
\frac{d\dot{m}_c}{dt} = \frac{A_c}{L_c}(\Delta p_c(\dot{m}_c, \omega) - \Delta p) \tag{B.1}
\]

\[
\frac{d(\Delta p)}{dt} = \frac{a^2}{V_p}(\dot{m}_c - \dot{m}_t(\delta p, u_t)) \tag{B.2}
\]

\[
\frac{d\dot{\omega}}{dt} = \frac{1}{J}(\tau_d - \tau_c) \tag{B.3}
\]

where \(\Delta p_c(\dot{m}_c, \omega)\) is the variable speed compressor characteristic. Equation (B.3) is added to the model. This equation is widely used for including spool dynamics in the compressor mode (e.g. Gravdahl et al., 2004; Gravdahl and Egeland, 1999; Fink et al., 1992). The angular acceleration is determined by the difference between the drive torque \(\tau_d\) and the compressor torque \(\tau_c\) multiplied with the reciprocal of the rotor inertia \(J\).
The compressor torque is defined as change in angular momentum of the gas (Gravdahl and Egeland, 1999). This is the momentum transfers to the gas by the rotor

$$\tau_c = \dot{m}(R_2C_\theta - R_1C_\theta)$$  \hspace{1cm} (B.4)

Here, $R_1$ and $R_2$ are the impeller eye radius and impeller tip radius, respectively, and $C_\theta$ is the tangential component of the gas velocity at the impeller eye or tip. With the assumption of radial vanes in the impeller and no prewhirl, the compressor torque for forward and backward mass flow becomes (Gravdahl and Egeland, 1999)

$$\tau_c = \sigma R_2^2|\dot{m}_c|\omega$$  \hspace{1cm} (B.5)

with the slip factor $\sigma$. Next to the compressor torque, other resistive forces can exist. Bøhagen (2007), for example, included a simple form of viscous friction $k_f$ yielding

$$\frac{d\omega}{dt} = \frac{1}{J}(\tau_d - (k_c|\dot{m}_c| + k_f)\omega)$$  \hspace{1cm} (B.6)

### B.2 The dynamic behavior of the Greitzer model with variable rotor speed

The compressor characteristic is now defined as a function of the rotor speed. Figure B.1 shows an example of the fitted polynomials of a compressor characteristic for various rotor speeds and some simulations of time dependant compressor curves with variable rotor speed. The data corresponds to a compressor discussed in the works of Bøhagen (2007). The speed variations are not very large (about 2%).

#### The surge cycle with variable rotor speed

In Figure B.2 and B.3, the surge cycle with a motor drive torque of 16.7 Nm is depicted. Starting at the top of the compressor curve (1) in Figure B.2 and following the compressor curve, the pressure increases to the pressure maximum (2). The mass flow decreases and thus the speed of the impeller increases. Following the cycle to the point of mass flow reversal (3), the speed of the impeller increases even more. The angular acceleration depends on the motor drive torque and the resistive torque of the compressor. Since a low (absolute) mass flow implies a low compressor torque, the angular speed of the impeller increases. After the flow reversal, when the mass flow is negative and the flow goes backwards through the compressor, the absolute mass flow increases and the angular acceleration decreases. The angular speed, however, still increases. It is important to note that the model used for the acceleration of the impeller contains only the absolute speed. So according to the model, a backward flow does not create more resistance than a forward flow. Now, following the surge cycle down the negative part of the compressor curve, the pressure drops and the speed of the impeller increases. This increase in impeller speed lasts until the resistance to the impeller is larger than the drive torque, which is the case when the surge cycle passes the mass flow at the top of the compressor curve (7). From this point on the rotor speed increases again. Note that the rotor accelerates in the left half (low mass flow, point 1 to 7...
Figure B.1: Compressor map with fitted polynomials for various rotor speeds (⋯) (Bøhagen, 2007) and simulation of the time dependant compressor curve with variable rotor speed.

Figure B.2: Simulation of a surge cycle with speed dependant compressor characteristic (from Bøhagen, 2007) and a motor drive torque 16.7 Nm. The numbers correspond with the line numbers in Figure B.3.
Figure B.3: Simulation of surge cycle with motor drive torque of 16.7 Nm. The lines indicate:
1: Top of compressor curve, 2: Top of surge cycle, 3: Point of flow reversal for high pressure, 4: Minimum mass flow, 5: Point of flow reversal for low pressure, 6: valley point of surge cycle, 7: top of compressor curve, 8: maximum flow. The numbers correspond with the numbers in Figure B.2.
As can be seen in Figure B.3, the rotor speed is high when the pressure difference is low and vice versa. So in the top half of the surge cycle, the rotor speed is lower than in the down half, as is indicated in Figure B.2. This is illustrated in Figure B.4, where the rotor speed differences are enlarged in the simulation by using a lower rotor inertia ($J$) and the surge cycle is placed closer around the compressor curve by enlarging $A_c$. Here it can be seen clearly that the upper part of the surge cycle follows the lower part of the variable speed compressor curve and the lower part of the surge cycle follows the top part of the compressor curve.

**Rotor speed measurements**

Results of measurements of the rotor speed in surge can be found in (Bøhagen, 2007; Fink et al., 1992; Gravdahl et al., 2004). Fink et al. (1992) shows results of compressor surge measurements for mild surge and deep surge. The deep surge measurements clearly show that the rotor speed increases when surge sets in (fast decrease of mass flow). In both mild and deep surge measurements, it is seen that the rotor speed has a maximum where the pressure has a minimum and vice versa. Gravdahl et al. (2004) measured the rotor speed at a setpoint change. In this work it can be seen that the rotor speed increases when the mass flow increases, just like was seen in Figure B.3. Both Fink et al. (1992) and Gravdahl et al. (2004) dit their measurements with an impeller with radial blades (or radial at exit). Gravdahl et al. (2004) explicitly states that there is no prewhirl in the compressor. So the compressor satisfies the assumptions made in the model. Measurements on the rotor speed were also done on Test rig B discussed by Van Helvoirt (2007). Along with the pressure measurements, the speed of the impeller was monitored with a keyphaser device, which gives a pulse at every rotation of the impeller. In contrast to the measurements in the previous section, the impeller did not have radial blades. This means it does not meet the assumptions made for equation (B.3).
Figure B.5: Pressure rise and impeller speed during surge of Test rig B from Van Helvoirt (2007); $U_{\text{tip},0} = 310 \text{ m/s}$

In Figure B.5 the pressure rise and the rotor speed of a measurement in surge are plotted. Because the measurement method is very crude, it is difficult to determine the amplitude of the rotor speed variations. Figure B.5b shows the mean rotor speed over 1 second. The speed variations are approximately 4% in this measurement. The compressor curves for varying rotor speeds are plotted in Figure B.6. A speed variation of 4% at the top of the compressor curve equals a pressure rise difference of circa 8%.

In line with the theoretical predictions, the maximum rotor speed occurs when the pressure rise is at its minimum. When the pressure increases, the rotor speed decreases quickly to reach the minimum at about halfway the increase in pressure, after which is slowly rises again. This is clearly different than the theoretical predictions for impeller with radial blades. An explanation might be found in the inertia of the drive. The pressure decrease is very fast in this compression system and the drive could possibly not be able to react that fast. However, this is not verified by actual measurements.
B.3 Parameter analysis of the variable speed model

The parameters in the rotor speed model are: \( J \), \( \tau_d \), \( \sigma R_t \) and \( k_c \). Figure B.7 shows the effect of the various parameters on the variable speed compressor curve and surge cycle. As was also seen in Figure B.4, \( J \) effects the magnitude of the speed variations. The parameters \( \tau_d \), \( b_1 \) and \( b_0 \) mainly effect the mean rotor speed. In Figure B.7a-b it is seen that a higher \( b_1 \) or \( b_0 \) gives a higher rotor speed. Both \( b_1 \) and \( b_0 \) effect the resistance of the compressor. Because the equilibrium speed of the rotor is determined by the compressor torque minus the resistance, a higher resistance results in a lower speed. Likewise, a higher compressor torque \( \tau_d \) gives a lower rotor speed, as can be seen in Figure B.7c.
Figure B.7: Variation of parameters of rotor speed model; (a) variation of $\sigma R_t$: $\sigma R_t = 0.02$ (··), $\sigma R_t = 0.0355$ (- -), $\sigma R_t = 0.04$ (--) (b) variation of $b_0$: $b_0 = 0.001$ (··), $b_0 = 0.0022$ (- -), $b_0 = 0.004$ (.), (c) variation of $\tau_d$: $\tau_d = 18$ (··), $\tau_d = 14.2$ (---), $\tau_d = 10$ (---), (d) variation of $J$: $J = 0.0148$ (---), $J = 0.001$ (--).
Appendix C

Conference paper submitted to the IEEE Conference on Decision and Control, 2008
**Abstract**— This paper presents the idea to exploit the similarity between the Van der Pol equation and the Greitzer lumped parameter model. The Greitzer model is a widely used nonlinear model to describe surge transients in turbocompressors. One of the difficulties in applying this model is the identification of the model parameters. Usually, a priori knowledge is combined with a tuning procedure for the model parameters to match simulation results with experimental data. In contrast to the Greitzer model, there are various analytical approximations available for the period time of the Van der Pol oscillator. We propose to use the similarity with the Van der Pol equation and apply the available approximations for the identification of the model parameters in the Greitzer model.

In this paper we will focus on demonstrating the similarity between both models. For this purpose we will introduce a coordinate transformation for the Greitzer model. In the subsequent parameter study we show that the transformed model exhibits the same qualitative behavior as the original compression system model. Furthermore, the parameter study reveals a linear relation between the most important model parameter and the limit cycle period time. These results form a solid basis for further research into exploiting the similarity with the Van der Pol system.

**I. INTRODUCTION**

Surge is an unstable operating mode of a compression system that occurs at low mass flows. Over the past decades, progress has been made with the analysis, modeling, and suppression of this compressor instability and promising experimental results on laboratory setups were reported [1–4]. One of the most relevant contributions in the analysis and modeling of surge is the nonlinear lumped parameter model for compressor transients [5], the so-called Greitzer model.

Despite the progress over the years, the modeling and identification of surge dynamics in industrial scale installations remains a challenging task [6]. One of the problems when using the Greitzer model to describe surge is the adequate selection of all model parameters. Usually, a priori knowledge is combined with a tuning procedure to match the simulation results with actual surge measurements. However, using surge data alone offers limited possibilities to fit the various model and scaling parameters individually.

There are some applications of formal identification methods for compressor dynamics known in literature [7–10]. However, these identification methods are based on linear system theory and application is mostly limited to small and dedicated laboratory setups.

In this paper we will explore an apparent similarity between the widely used Greitzer model and the well-known Van der Pol equation. The Van der Pol equation has received extensive attention in the literature [11–13]. In particular we mention the work on approximating the period time of the Van der Pol oscillator [14–17].

The idea is to use the similarity between the Van der Pol equation and the Greitzer model and apply the results of the approximation techniques for the Van der Pol limit cycle to the Greitzer model. Our final aim is to replace the tuning approach to determine the Greitzer model and scaling parameters with an analytical approximation based on the measurable period time of the surge cycle. In this paper we present a first step towards achieving this goal by demonstrating the similarity between the Van der Pol equation and the Greitzer model.

More specifically, after introducing the Van der Pol equation and Greitzer model, we discuss the similarity in the periodic solutions of both models. Subsequently, we introduce a coordinate transformation to improve the similarity of the Greitzer model with the Van der Pol system. We will then carry out a parameter study with the transformed Greitzer model. The presented results illustrate that the transformed Greitzer model still exhibits the same qualitative behavior as the original compression system model. More importantly, the parameter study reveals an interesting linear relationship between the stability parameter and the limit cycle period time. We will discuss our findings and argue that they are a promising first step in achieving our final goal.

**II. THE DYNAMIC SYSTEMS**

In this section we will briefly introduce the Van der Pol system and the Greitzer lumped parameter model for compressor transients. Furthermore, we will introduce the model equations for both systems and we will briefly discuss the similarity between the periodic solutions of both systems. In order to exploit this similarity we will introduce a transformation of variables for the Greitzer model. In the remainder of this paper we will use the transformed Greitzer model to further investigate its similarity with the Van der Pol system.

A. The Van der Pol system

In this paper we will consider a Van der Pol system of the following form

\[
\dot{q}_1 = \mu \left[ \left( q_1 - \frac{1}{3} q_1^3 \right) - q_2 \right] \quad (1)
\]

\[
\dot{q}_2 = -q_1 \quad (2)
\]
that can be obtained from the well known autonomous Van der Pol equation $\ddot{y} - \mu(1 - y^2) \dot{y} + y = 0$ by using the Liénard transformation $q_1 = y$ and $q_2 = y - \frac{1}{3} y^3 - \frac{1}{5} \ddot{y}$. Figure 1 gives an impression of the behavior of the Van der Pol system for different values of $\mu$. Note that we have drawn the cubic nullcline $q_2 = q_1 - \frac{1}{3} q_1^3$ in the presented phase plots.

### B. The Greitzer model

The Greitzer lumped parameter model describes the transient behavior of turbocompressors [5]. We will consider the two-state Greitzer model for the schematic compression system in Fig. 2. The corresponding dimensionless equations are of the following form

\[
\begin{align*}
\frac{d\phi_c}{d\xi} &= B [\Psi_c(\phi_c) - \psi] \quad (3) \\
\frac{d\psi}{d\xi} &= \frac{1}{B} [\phi_c - \Phi_t(\psi)] \quad (4)
\end{align*}
\]

with $\phi_c$ the compressor mass flow, $\psi$ the pressure difference, $\xi$ the dimensionless time variable, and $B$ the so-called stability parameter.

The function $\Psi_c(\phi)$ represents the nonlinear compressor characteristic. This characteristic is often described by the cubic Moore-Greitzer polynomial [18] that, for a given compressor speed, is given by

\[
\Psi_c(\psi) = C_0 + H \left[ 1 + \frac{3}{2} \left( \frac{\phi_c}{W} - 1 \right) - \frac{1}{2} \left( \frac{\phi_c}{W} - 1 \right)^3 \right] \quad (5)
\]

with $H$, $W$, and $C_0$ according to Figure 4. The function $\Phi_t(\psi) = c_t u_t \sqrt{\psi}$ describes the nonlinear throttle characteristic, with valve coefficient $c_t$ and valve opening $u_t$. Figure 3 gives an impression of the behavior of the Greitzer model for different values of $B$.

### C. Similarity in limit cycle oscillations

From Fig. 1 we observe that the Van der Pol system exhibits a limit cycle that has a circular shape when $\mu$ is small. On the other hand, for large values of $\mu$ the limit cycle oscillations are of the relaxation type in which two different time scales can be distinguished. The relaxation type behavior is similar to the behavior of the Greitzer model at high $B$ values, see also Fig. 3. Furthermore, we point out that the shape of the nullcline in the Van der Pol system shows a similarity with the cubic polynomial (5) used to describe the compressor characteristic in the Greitzer model.

For low values of $\mu$ and $B$ the difference between the Van der Pol system and Greitzer model become more profound. This is mainly due to the second term in (4) that represents the mass flow through the throttle. Furthermore, we point out that the intersection point between the compressor and throttle characteristics is usually not located at the point $(W, C_0 + H)$ so the limit cycle of the Greitzer model is not symmetrical, in contrast to that of the Van der Pol system. Finally, we point out that for extremely low values of $B$, the Greitzer model does no longer exhibit a periodic solution anymore. In contrast, the Van der Pol system remains unstable, regardless the value of $\mu$.

### D. Coordinate transformation

It is worthwhile to explore the similarity between the Van der Pol system and the Greitzer model in more detail.

---

**Fig. 1.** Periodic solution of the Van der Pol system for $\mu = 0.5$ (top), $\mu = 4$ (middle), and $\mu = 10$ (bottom).

**Fig. 2.** Schematic representation of the Greitzer model.

**Fig. 3.** Periodic solution of the Greitzer lumped parameter model for $B = 0.1$ (top), $B = 0.4$ (middle), and $B = 2$ (bottom).
However, the comparison is hampered by the fact that the position and scaling of the limit cycles are different for the two models. To solve this, we propose the following change of variables for the Greitzer model

\[ x_1 = \frac{\phi_c - W}{W} \]
\[ x_2 = \frac{2}{3H} (\psi - C_0 - H) \]

that effectively scales and shifts the \((\phi_c, \psi)\) coordinate system such that the point of inflection of the compressor characteristic coincides with the point \((0,0)\) in the \((x_1, x_2)\) coordinate system. Substitution of \(x_1\) and \(x_2\) in (3)–(4) yields

\[ \frac{dx_1}{d\xi} = \frac{3HB}{2W} \left[ x_1 - \frac{1}{3} x_1^3 - x_2 \right] \]
\[ \frac{dx_2}{d\xi} = \frac{2W}{3HB} \left[ x_1 - \left( \frac{c_1 u - \sqrt{\frac{3H}{2}} x_2 + C_0 + H - 1} {W} \right) \right] \]

Finally, after comparing (8)–(9) with the Van der Pol system in (1)–(2) we write

\[ \mu = \frac{3HB}{2W} \]

so the transformed Greitzer model can be written as

\[ \frac{dx_1}{d\xi} = \mu \left[ x_1 - \frac{1}{3} x_1^3 - x_2 \right] \]
\[ \frac{dx_2}{d\xi} = \frac{1}{\mu} \left[ x_1 - g(x_2) \right] \]

with \(g(x_2) = \frac{c_1 u - \sqrt{\frac{3H}{2}} x_2 + C_0 + H - 1} {W}\) the only remaining difference with the Van der Pol system, see also Fig. 5. From this figure we see that the shapes of both limit cycles are almost identical. However, the period times of both limit cycles differ by approximately 10%.

We point out that for any time instance \(\xi\), the \([\ldots]\) term in (11) represents the vertical distance between the transformed compressor characteristic and the momentary state \(x_2\) in the phase plane. Similarly, the \([\ldots]\) term in (12) represents the horizontal distance between the transformed throttle characteristic and the momentary state \(x_1\). In terms of an actual compression system \(\frac{dx_1}{d\xi}\) describes the speed at which the flow changes, while \(\frac{dx_2}{d\xi}\) describes the speed at which the pressure difference over the compressor changes. The latter is directly related to the time needed to fill or empty the plenum volume.

### III. Parameter Study

In the previous section we derived a transformed Greitzer model that is similar to the Van der Pol system. We will now study the effect of the various parameters in (11)–(12) on the period time of the resulting limit cycle. In order to simplify this analysis, we will replace the nonlinear transformed throttle characteristic with a linear approximation \(g(x_2) \approx S x_2 + a\). Here, \(S\) denotes the inverse of the slope of \(\Phi_t(\psi)\) in \((x_1, x_2)\) coordinates at the intersection point \((x_{10}, x_{20})\) of the compressor and throttle characteristics, see also Fig. 6. In the remainder of this paper we will only use the linear approximation of \(g(x_2)\).

It can be shown that linearizing the throttle characteristic yields a slightly higher prediction for the period time of the limit cycle but this effect is significant for values of \(\mu < 2\) only. Furthermore, at relatively large values for \(C_0\)
that are typical for actual compression systems the difference between the quadratic and linear throttle characteristic is small as can be seen from Fig. 6. Given these arguments, the effects of linearizing the throttle characteristic are considered to be negligible for our current analysis.

We will now study the effect of varying $\mu$, $x_{10}$, and $S$ on the limit cycle period time of the transformed Greitzer model. We will also briefly discuss the physical interpretation of the presented results to illustrate that the transformed Greitzer model still describes the behavior of an actual compression system in a qualitative manner.

A. Varying $\mu$

With a given compressor and throttle characteristic, the period time of a surge limit cycle is determined by the stability parameter $B$. In the transformed Greitzer model this parameter is represented by $\mu$ according to (10). In Fig. 7 the effect of varying $\mu$ on the period time $T$ of the limit cycle is depicted. From this figure we see that for $\mu > 5$ the period time linearly increases with $\mu$.

The linear relation between $\mu$ and $T$ corresponds to our earlier observation that the limit cycle oscillations are of the relaxation type for large $\mu$. In terms of an actual compression system this implies that for high values of $B$ the period time is determined by the time needed to fill and empty the plenum and that the time needed for reversing the flow is negligible. This can also be seen from (11)–(12) by noting that the rate of change in mass flow, described by $\frac{dx_1}{d\xi}$, depends on $\mu$, while the change in pressure rise, described by $\frac{dx_2}{d\xi}$, depends on the reciprocal of $\mu$. The effect of $\mu$ on the period time is in agreement with findings reported in literature [6], [19].

B. Varying $x_{10}$

The second parameter that will be investigated is the intersection point of the compressor and throttle characteristics. From (12) we see that $\frac{dx_2}{d\xi}$—related to the time needed to fill and empty the plenum—depends on the horizontal difference between the momentary state $(x_1, x_2)$ and $g(x_2)$. For increasing $x_{10}$ this difference decreases for $x_1 > g(x_2)$ and it increases for $x_1 < g(x_2)$. In terms of percentage the decrease of the horizontal difference is larger than the increase and therefore the period time of the surge cycle increases for increasing values of $x_{10}$.

For an actual compression system changing the intersection point of the compressor and throttle characteristics corresponds to changing the operating point of the system. It is know from literature that moving the (unstable) operating point towards a lower mass flow results in a slightly higher surge frequency [6]. This is due to the fact that the time needed to fill the plenum increases while the time needed to empty it decreases. This effect is also predicted by the transformed Greitzer model as can be seen from the simulation results and the discussion above.

C. Varying $S$

Another parameter that affects the period time of the limit cycle is the slope $1/S$ of the (linearized) throttle characteristic. The results for different values of $S$ between 0 and 1 are shown in Fig. 9. Note that a higher value of $S$ implies a less steep throttle characteristic. Again we see that the linear relation between $\mu$ and $T$ is not affected by varying $S$. Furthermore, the results show that in the case of $x_{10} = 0.5$ the period time of the limit cycle increases when $S$ is increased. The same trend can be observed for $x_{10} < 0.5$. However, for the case $x_{10} = 0.9$ the period time decreases with increasing $S$, see Fig. 10. Again, these observations can be explained by looking at the effect of increasing $S$ on $\frac{dx_2}{d\xi}$ and the related filling and emptying times of the plenum.

In terms of an actual compression system, the effect of $S$ on $T$ can be explained by noting the effect of decreasing the slope of the throttle characteristic (increasing $S$) on the net inflow into the plenum. For $x_2 < x_{20}$ the net inflow increases while it decreases for $x_2 > x_{20}$. For operating points at low mass flows ($x_1 < 0.5$) the decreasing net inflow has a larger effect and hence the period time of the surge cycle increases. However, for operating points near the top of the compressor characteristic the effect of a decreasing net inflow is much smaller, yielding a decrease in the period time. Again, a similar effect of the slope of the throttle characteristic on the surge period time was reported in literature [6].
The results presented in this paper are a promising first step towards exploiting the similarity with the Van der Pol system to develop a parameter identification method for the Greitzer compression system model. Future work will now focus on investigating how the various approximation methods for the period time of the Van der Pol oscillator can be applied to the transformed Greitzer model. Available simulation and experimental data will be used to validate the resulting approximation methods for the period time of surge oscillations in various compressor test rigs.

REFERENCES


IV. DISCUSSION AND CONCLUSIONS

In this paper we have explored the similarity between the Van der Pol system and the Greitzer lumped parameter model for surge transients in turbocompressors. By comparing the limit cycle oscillations of both models, we showed that they both exhibit a relaxation type oscillation for large values of the main model parameter, i.e. $\mu$ and $B$.

In order to compensate for the different positioning and scaling of the cubic compressor characteristic with respect to the nullcline in the Van der Pol system, we introduced a coordinate transformation. The resulting transformed Greitzer model has almost the same structure as the Van der Pol system, except for an additional term that represents the effect of the throttle in the original compression system model.

Subsequently, we investigated the effect of the stability parameter, operating point, and slope of the throttle characteristic on the period time of the limit cycle oscillations. By comparing the results of this study with known results from literature, we showed that the transformed Greitzer model exhibits the same qualitative behavior as the original Greitzer compression system model. Most importantly, the results revealed a linear dependency between the stability parameter and the period time for large values of $\mu$.