Modelling friction near sharp edges using a Eulerian reference frame: application to aluminium extrusion

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SUMMARY

When using a Eulerian finite element approach to model the material deformation that occurs in e.g. forming processes, the accurate capturing of friction is of crucial importance to the quality of the computational results. For the algorithm that incorporates the frictional phenomena into the system of equations, the direction of the contact surface normal in a node is an essential parameter. However, this normal is not uniquely defined in the nodes of a curved, discretized surface. Therefore, a substitute normal has to be reconstructed. The commonly used (averaging) methods to determine the normal are either mesh or geometry dependent which renders poor results on coarse meshes. Therefore, a new method is presented that reconstructs the direction of the normal from the flow field near the node. Comparing the flow fields on a coarse mesh with those obtained on a very fine mesh reveals that a more accurate solution field is obtained using the method introduced here. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: flow; friction; contact; finite element; Lagrange multiplier; surface normal

1. INTRODUCTION

Aluminium extrusion is a process in which a hot billet is placed in a container and pressed through a die (see e.g. Figure 1). The objective of this process is to produce a straight profile with a cross-sectional shape that meets the specifications defined by the customer. In this process, the geometry of the die has a dominant influence on the balancing of the exit flow and thus on the eventual shape of the profile. However, in current engineering practice, die design is a process that, besides on experience, is based on trial and error. To avoid the drawbacks of the trial and error process the trial-pressings can be replaced by numerical computations which predict the profile shape for a given die design.

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Two main reference frames can be indicated to formulate the kinematics of aluminium extrusion, the Eulerian and the Lagrangian. In the Eulerian formulation, the deformation \(-\text{rate}\) of a material is defined in terms of a reference frame that is fixed in space, i.e. the frame is not connected to the material that is deformed. In the Lagrangian formulation, the deformation is defined in terms of a reference frame that is attached to the material points. Both formulations can be used to assess the deformation of the configuration in discrete points, after subdividing the reference frame into a limited number of subregions, the finite elements.

The Eulerian formulation is well suited to model systems where material flows through a domain with fixed boundaries and known boundary conditions at the entrance and exit. Aluminium extrusion is an example of such a system. Using a finite element Eulerian framework to model aluminium extrusion implies that the boundary conditions have to be implemented in that framework as well. This is straightforward for Dirichlet type (displacements or velocities prescribed) and Neumann type (boundary tractions prescribed) boundary conditions. However, friction is considerably more difficult to take into account. In particular, the normal (usually occurring in the friction law) to a discretized curved surface is generally not uniquely defined in the nodes. To overcome this problem a substitute normal can be applied in the nodes.

The substitute normal in a node plays an important role in friction modelling. The direction of the normal can be reconstructed from the normals of the adjacent element faces. If the substitute normal is not generated correctly, this introduces an error in the solution field. If the difference between the normals of the individual element faces is large, as will be the case near sharp edges, the range in which the normal in the node can vary is also large. Therefore, the accuracy of the computed solution field near a sharp edge is largely dependent on the selection of the method to define the normal in the node.

2. MATHEMATICAL MODELLING OF FRICTION

The Stokes formulation for forming processes where plastic, incompressible deformations predominate [1] is governed by the conservation of mass (1) and momentum (2):

\[
\nabla \cdot \mathbf{v} = 0 \quad (1)
\]

\[
\nabla \cdot (-p \mathbf{I} + 2\eta \sigma(D, T) \mathbf{D}) = 0 \quad (2)
\]
where $v$ represents the velocity, $p$ the pressure, $\nabla$ the gradient operator, $\eta$ the viscosity, and $D = \frac{1}{2}(\nabla v + (\nabla v)^T)$ the rate of deformation tensor. In order to solve (1) and (2), additional boundary constraints have to be supplied for every point on the boundary. These constraints can be of Dirichlet (velocity prescribed) or Neumann (external load prescribed) type, however, when modelling the friction between a flowing continuum and a rigid, stationary obstacle these constraints are more complicated.

2.1. Constitutive models

In aluminium extrusion, friction is a complex and hardly understood phenomenon. Due to the high pressures and temperatures, it is extremely difficult to perform in situ experiments. Consequently, most information on friction phenomena is obtained by studying the bearing surface after extrusion [2–4]. These studies render some insight into the mechanisms that govern friction but do not supply sufficient information to specify the constitutive relations that can be applied to model friction in an extrusion analysis. Yet, there are indications that some models are more suitable than others.

To model friction, a variety of constitutive laws can be used to define the surface tractions that are exerted on a material when it is sliding along a surface. Usually, the tangential surface tractions, $(I - nn) \cdot \sigma \cdot n$, with $n$ the outward unit surface normal vector and $\sigma$ the stress tensor, are related to the size of the normal surface traction $n \cdot \sigma \cdot n$ and to the relative sliding velocity along the tooling $(v - v_{tool})$. As a point of departure a Norton–Hoff type law is considered which gives the following relation [5]:

$$(I - nn) \cdot \sigma \cdot n = -\mu |n \cdot \sigma \cdot n|^\alpha (v - v_{tool})^{\beta - 1} (v - v_{tool})$$ (3)

where $\mu$ represents the friction coefficient, and $\alpha$ and $\beta$ are additional dimensionless constitutive parameters. Notice that for this law the dimension of $\mu$ depends on the choice of $\alpha$ and $\beta$. For extrusion the velocity of the tooling is given by (see also Figure 1):

$$v_{tool} = \begin{cases} 0 & \forall x \in \Gamma_{die} \cup \Gamma_{container} \\ v_{ram} & \forall x \in \Gamma_{ram} \end{cases}$$ (4)

where $x \in \Gamma$ indicates a position on the contact surface. For the particular values of $\alpha = 0$, $\beta = 1$, (3) reduces to the Norton friction law. For $\alpha = 0$, $\beta = 0$, (3) represents the constant friction law, while for $\alpha = 1$, $\beta = 0$, the Coulomb friction law is recovered.

In extrusion analysis, all of these laws have been used to model friction (see e.g. Reference [4] for Norton, Reference [6] for constant friction, and Reference [7] for Coulomb). However, in extrusion practice, it has been observed that locally increasing the length of the bearings above a certain value does not further improve the balance of the extrudate. This suggests that towards the exit of the bearing the frictional traction decreases and even vanishes for large bearing lengths. This trend cannot be captured by adopting the constant friction law. Also, the Norton law is incapable of describing this behaviour since it predicts that a particle that is sliding along the bearing will experience a frictional traction proportional to the sliding velocity. Since the velocity of a particle will not change significantly within the bearing, the Norton law suggests a nearly constant friction. Furthermore, there are indications that the sliding velocity has very little influence on the friction forces in metal–metal contact [8, 9]. Additionally, experiments in extrusion with non-parallel bearings reveal that the contact...
pressure plays a crucial role [3, 4]. Therefore, in this research a Coulomb friction law has
been employed.

2.2. Coulomb friction

Coulomb’s law of friction may be expressed [10] by a friction criterion (5) which defines
the limits of the frictional traction in relation to the normal traction, a slip rule (6) which
prescribes the frictional tractions to be opposed to the sliding velocity, and a complementarity
condition (7) which ensures that stick and slip are mutually exclusive conditions:

\[
\|(I - nn) \cdot \sigma \cdot n\| \leq -\mu (n \cdot \sigma \cdot n)
\]

(5)

\[
\frac{v - v_{\text{tool}}}{\|v - v_{\text{tool}}\|} = -\frac{(I - nn) \cdot \sigma \cdot n}{\|(I - nn) \cdot \sigma \cdot n\|}
\]

(6)

\[
(\|v - v_{\text{tool}}\|)(\|(I - nn) \cdot \sigma \cdot n\| + \mu (n \cdot \sigma \cdot n)) = 0
\]

(7)

In every point on the aluminium surface that is in contact with the tooling, the stress state
and the velocity must satisfy these three equations. Also, in case of sliding the size of the
frictional surface tractions, \(\mu (n \cdot \sigma \cdot n)\) should not exceed the local yield stress of aluminium.
In extrusion, this requirement is implicitly met because numerical experiments show that, in
the areas where aluminium slides along the tooling, the contact pressures are considerably
lower than the actual yield stress of aluminium there.

Due to the difficulties of performing in situ experiments, it is impossible to measure the
Coulomb friction coefficient, \(\mu\). Therefore, the friction coefficient has been estimated by per-
forming a number of simulations with varying Coulomb coefficients for a die design that
rendered a well-balanced exit flow in extrusion practice. From these computations, the flow
field that was obtained with a Coulomb friction coefficient of 0.4 displayed the most uniform
exit flow. Therefore, in this work a Coulomb friction coefficient of 0.4 will be used.

3. FINITE ELEMENT IMPLEMENTATION

The Stokes problem associated with extrusion, governed by (1) and (2), can be approximated
using the classical Galerkin finite element method. To do so first the spaces of trial solutions
\(\mathcal{V}\) (for the velocity \(v\)) and \(\mathcal{P}\) (for the pressure \(p\)) are defined on the aluminium domain \(\Omega\)
with boundary \(\Gamma\):

\[
\mathcal{V} = \{v \in (H^1(\Omega))^n\} \quad \mathcal{P} = \{p \in L^2(\Omega)\}
\]

(8)

where \(n \in \{1, 2, 3\}\) represents the dimension of the problem, and \(H^1(\Omega)\) and \(L^2(\Omega)\) are the
standard Hilbert and Lebesgue spaces, respectively. The variational form of the Stokes problem
can then be written as:

\[
\begin{align*}
\text{find } (v, p) \in \mathcal{V} \times \mathcal{P} \text{ such that } \\
\int_{\Omega} 2\eta \mathbf{D}_w : \mathbf{D}_v \, d\Omega - \int_{\Omega} (\nabla \cdot \mathbf{w}) p \, d\Omega = \int_{\Gamma} \mathbf{w} \cdot \mathbf{s} \, d\Gamma & \quad \forall w \in \mathcal{V} \\
\int_{\Omega} q(\nabla \cdot v) \, d\Omega = 0 & \quad \forall q \in \mathcal{P} \\
BC(v, p) = 0 & \quad \text{on } \Gamma
\end{align*}
\]

(9)

where \( \mathbf{D}_w \) and \( \mathbf{D}_v \) represent the rate of deformation tensor related to the fields \( \mathbf{w} \) and \( \mathbf{v} \), respectively. The variable \( \mathbf{s} = (-p \mathbf{I} + 2\eta \mathbf{D}_v) \cdot \mathbf{n} \) denotes the stress vector on \( \Gamma \). The function \( BC(v, p) \) is used to introduce the appropriate boundary conditions on the boundary of the aluminium domain, \( \Gamma \), and will be discussed further on in this chapter. Next, the finite dimensional subspaces \( \mathcal{V}^h \subset \mathcal{V} \) and \( \mathcal{P}^h \subset \mathcal{P} \) are defined on a family of discretizations \( \mathcal{F}^k \) of the domain, where \( \mathcal{F}^k \) is supposed to contain elements, \( K \), of triangular (2D) or tetrahedral (3D) shape. Now the following subspaces are chosen:

\[
\mathcal{V}^h = \{ v^h \in \mathcal{V}^h; v^h|_K \in (P_1(K))^n, \forall K \in \mathcal{F}^k \} \\
\mathcal{P}^h = \{ p^h \in \mathcal{P}; p^h|_K \in P_1(K), \forall K \in \mathcal{F}^k \}
\]

(10) (11)

where \( P_1(K) \) denotes the space of linear polynomial functions on the element \( K \). Now the problem formulation (9) can be written in terms of \( v^h \in \mathcal{V}^h \) and \( p^h \in \mathcal{P}^h \) by:

\[
\begin{align*}
\text{find } (v^h, p^h) \in \mathcal{V}^h \times \mathcal{P}^h \text{ such that } \\
\int_{\Omega} 2\eta \mathbf{D}_{w^h} : \mathbf{D}_{v^h} \, d\Omega - \int_{\Omega} (\nabla \cdot \mathbf{w}^h) p^h \, d\Omega = \int_{\Gamma} \mathbf{w}^h \cdot \mathbf{s} \, d\Gamma & \quad \forall w^h \in \mathcal{V}^h \\
\int_{\Omega} q^h(\nabla \cdot \mathbf{v}^h) \, d\Omega = 0 & \quad \forall q^h \in \mathcal{P}^h \\
BC(v^h, p^h) = 0 & \quad \text{on } \Gamma
\end{align*}
\]

(12)

where the weighting functions are chosen identically, in conjunction with the (Bubnov-)Galerkin method. It is well-known that, for the given choices of \( \mathcal{V}^h \) and \( \mathcal{P}^h \), the problem definition (12) is ill-posed because it does not satisfy the inf–sup or Babuška–Brezzi (LBB) stability condition (see e.g. References [11, 12] and references therein). As a result, spurious pressure modes will cause severe oscillations in the pressure solution and the velocity solution will lock in many cases, irrespective of the mesh size.

Two approaches exist that are generally employed to remedy the ill-posedness of (12) [13]. The first method is to choose the polynomials that span the spaces of trial solutions \( \mathbf{v}^h \) and \( p^h \) of different order. In that case, the combination of these function spaces has to be selected such that the LBB condition will be satisfied. An example is the triangular or tetrahedral \( P_2P_1 \) element implemented in standard FE packages, where the velocity is interpolated quadratically.
and the pressure linearly. However, these elements introduce a large number of degrees of freedom for each element which results in (actually, for extrusion simulation unacceptably) large systems of equations to be solved.

The second approach is to stabilize the linear elements by modifying the Stokes equation through the inclusion of a certain amount of diffusivity to the formulation, without affecting the consistency of the solution (see Reference [14] for an overview). An example of such an approach is the method indicated as Galerkin least squares (GLS) [15]. This allows to circumvent the compatibility requirements imposed on \( \psi^h \) and \( \phi^h \) by the LBB condition and thus enables the use of equal order trial spaces. A disadvantage of these methods is that they usually require the selection of a stabilizing parameter that is a function of the mesh size and the viscosity, which makes the method less suitable in case of strong variations in the element sizes and viscosities within \( \Omega \). However, in the formulation of the MINI element [16], which is also a stabilized element, the stabilization parameter follows directly from the formulation. Therefore, the MINI element has been selected to be applied in this research.

### 3.1. MINI element

In the MINI element, an enriched space of trial functions \( \mathcal{V}^+_h \) is constructed out of the linear trial space \( \mathcal{V}^h \), and the space of bubbles \( \mathcal{V}_b^h \):

\[
\mathcal{V}^+_h = \mathcal{V}^h \oplus \mathcal{V}_b^h
\]

Bubbles are functions which are equal to zero at element boundaries and positive within the element to which they are associated. The bubble space is generally taken to be

\[
\mathcal{V}_b^h = \{ v_b^h \in (H^1_0(K))^n; v_b^h |_{K} \in (B_k(K))^n, \forall K \in \mathcal{K} \}
\]

where \( B_k(K) \) represents the space of bubbles constructed out of polynomials of degree \( k \) or less and all basis functions \( \phi \in B_k(K) \) satisfy:

\[
\begin{align*}
\phi(x) > 0 & \quad \forall x \in K \setminus \bar{K} \\
\phi(x) = 0 & \quad \forall x \in \partial K \\
\phi(x) = 1 & \quad \text{for } x \text{ in the geometrical centre of } K
\end{align*}
\]

where \( \partial K \) indicates the boundary of the element \( K \). The trial velocity solution \( v^+_h \in \mathcal{V}^+_h \) can now be split into a linear and a bubble contribution in a unique way:

\[
v^+_h = v^h + v_b^h \quad \forall (v^h, v_b^h) \in \mathcal{V}^h \times \mathcal{V}_b^h
\]
Using this split of variables, the discretized form of (12) at the element level, while omitting the boundary conditions, can be written as:

\[
\begin{align*}
\text{find } (v^h_b, v^b_b, p^b) & \in \mathbb{V}^h \times \mathbb{V}_b^h \times \mathbb{P}^h \\
& \text{such that}
\int_K 2\eta D_{v+} : D_{v+} dK - \int_K (\nabla \cdot (w^h + w^h_b)) p^b dK
\end{align*}
\]

\[
\int_{\partial K} (w^h + w^h_b) \cdot s_k d\partial K \quad \forall (w^h, w^h_b) \in \mathbb{V}^h \times \mathbb{V}_b^h
\]

\[
\int_K q^h (\nabla \cdot (v^h + v^h_b)) dK = 0 \quad \forall q^h \in \mathbb{P}
\]

(17)

where \( s_k \) represents the surface traction acting on \( \partial K \).

The operations that will be performed to simplify (17) are based on the property of the bubble space that for any constant tensor \( A \) on element \( K \) the following holds:

\[
\int_K A : (\nabla v^h_b) dK = 0
\]

which is a direct result of the fact that, by construction, \( v^h_b = 0 \) on \( \partial K \). This property, together with the observation that, as a result from the linear interpolation \( \nabla v^h_b \) is constant within an element, leads to the orthogonality of the subspaces:

\[
\int_K \nabla v^h : \nabla v^h_b dK = 0 \quad \forall (v^h, v^h_b) \in \mathbb{V}^h \times \mathbb{V}_b^h
\]

(19)

The orthogonality property (19) can be applied to simplify (17) if the viscosity (at the element level) is assumed constant for the moment:

\[
\begin{align*}
\text{find } (v^h_b, v^b_b, p^b) & \in \mathbb{V}^h \times \mathbb{V}_b^h \times \mathbb{P}^h \\
& \text{such that}
\int_K 2\eta D_{v+} : D_{v+} dK - \int_K (\nabla \cdot (w^h + w^h_b)) p^b dK
\end{align*}
\]

\[
\int_{\partial K} (w^h + w^h_b) \cdot s_k d\partial K \quad \forall (w^h, w^h_b) \in \mathbb{V}^h \times \mathbb{V}_b^h
\]

\[
\int_K q^h (\nabla \cdot (v^h + v^h_b)) dK = 0 \quad \forall q^h \in \mathbb{P}
\]

(20)

Elaboration of (20) for one element leads to a linear system of equations with the following structure:

\[
\begin{bmatrix}
K & 0 & L \\
0 & K_b & L_b \\
-L^t & -L_b^t & 0
\end{bmatrix}
\begin{bmatrix}
v \\
v_b \\
p
\end{bmatrix}
= \begin{bmatrix}
f \\
f_b \\
0
\end{bmatrix}
\]

(21)

where \( v, v_b, \) and \( p \) are the columns containing the nodal velocities and pressures and \( f, f_b \) are columns with nodal loads. The matrices in (21) are related to the integrals over \( K \).
Since the bubble velocities, $v_b$, are only defined inside the element considered, the system in (21) can be reduced by static condensation of the bubble degrees of freedom to render:

$$
\begin{bmatrix}
K & L_b & L_b^T K_b^{-1} L_b \\
-L_b^T & L_b L_b^T & L_b K_b^{-1} L_b \\
L_b^T K_b^{-1} & L_b K_b^{-1} & f_b \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{v} \\
\mathbf{p} \\
f_b \\
\end{bmatrix}
= 
\begin{bmatrix}
f \\
f_b \\
\end{bmatrix}
$$

The structure of this element matrix is similar to that obtained with other stabilization techniques. An additional feature of this structure is that the zero diagonal matrix, which is still present in (21), has disappeared (the zero matrix tends to cause problems with iterative solvers).

It is clear that the stabilization solely stems from the bubble. Therefore, it is obvious that the shape of the bubble will have some influence on the performance of the element. Numerical experiments revealed that the quadratic bubble, as introduced in Reference [16] rendered the best results.

### 3.2. Friction

To incorporate the frictional conditions to be imposed on the surface traction, denoted by $\sigma \cdot \mathbf{n}$ in (5)–(7), into the discretized system (20), this traction has to be defined in terms of discretized entities. To this end, one of three approaches is commonly pursued: a Lagrange multiplier method [17], an augmented Lagrange method [18], or a penalty method [19]. Here, the Lagrange multiplier method is adopted to avoid the introduction of a penalty parameter [20]. In this method, a Lagrange multiplier, $\lambda$, which can be identified as the traction $\sigma \cdot \mathbf{n}$, is used to enforce the friction equations on the boundary $\Gamma_{\text{tool}}$. Furthermore, the inequalities in (5)–(7) have to be reformulated into a set of equalities. This is done by the introduction of three distinct frictional states that allow the formulation of a set of equalities associated with each state.

#### 3.2.1. Formulation of friction using three distinct states.

To rewrite the inequalities in (5)–(7) into equalities, three distinct states are introduced, being stick, slip and release, comparable to the approach suggested in Reference [21]. The friction states 'stick' and 'slip' follow directly from the complementarity condition. The friction state 'release', on the other hand, is introduced to cope with negative contact pressures. An unrealistic negative contact pressure implies that aluminium would lose contact with the die and create a free surface that might regain contact again. To accurately capture this behaviour, a contact analysis would be required which introduces numerous numerical instabilities. Moreover, in extrusion practice it has been observed that, even if aluminium releases from the die, the gap between the die and aluminium is minimal. Therefore, in the separate state ‘release’, negative contact pressures are coped with by prescribing that the frictional tractions are zero and by neglecting the gap between aluminium and the die. Neglecting this gap implies that the velocities of the aluminium remain tangential to the tooling surface.

Associated with each state, a set of equations is defined which enforce that the material either sticks to the tooling, slides along the tooling or ‘releases’ (i.e. slips along the tooling with zero frictional traction). With $\mathbf{n}$ the outward unit normal to aluminium surface, a function
\( F(v, \lambda) \) is defined according to:

\[
F(v, \lambda) = \begin{cases} 
\text{stick} & \{(v - v_{\text{tool}}) = 0 \\
\text{slip} & \{(v - v_{\text{tool}}) \cdot \mathbf{n} = 0 \\
\lambda \cdot (I - \mathbf{n}) - \mu(\lambda \cdot \mathbf{n}) \frac{v_{\text{tool}}}{\|v_{\text{tool}}\|} = 0 \\
\text{release} & \{(v - v_{\text{tool}}) \cdot \mathbf{n} + \lambda \cdot (I - \mathbf{n}) = 0 \}
\end{cases}
\]

(23)

The function \( F(v, \lambda) \) is applied in an iterative procedure that proceeds as follows. At the start of the first iteration each point on the boundary is estimated to be in either the stick, the slip or the release state. For this distribution of states on the friction boundary, the solution field can be obtained by combining \( F(v, \lambda) \) with the equations that govern the flow of aluminium.

For the solution field computed using this set of equations, it is assessed whether for every point on the friction boundary the inequalities in (5)–(7) are satisfied. For each state, these inequalities can be simplified according to:

\[
\begin{cases} 
\text{stick} & \|\lambda \cdot (I - \mathbf{n})\| + \mu\lambda \cdot \mathbf{n} \leq 0 \\
\text{slip} & (v - v_{\text{tool}}) \cdot (\lambda \cdot (I - \mathbf{n})) \leq 0 \\
\text{release} & -\lambda \cdot \mathbf{n} \leq 0
\end{cases}
\]

(24)

If the inequalities are not satisfied for a point on the boundary, the state of that point has to be adjusted. If a stick state has been assumed for a given iteration and the frictional traction, which is computed using this assumption, exceeds the Coulomb friction limit, slip is assumed for the next iteration. Similarly, if slip is assumed and the inner-product of the relative sliding velocity and the frictional traction is not negative, stick is assumed for the subsequent iteration. Additionally, if the contact pressure becomes negative after an iteration with a stick or a slip assumption, release is assumed for the following iteration. Moreover, if slip was assumed in a previous iteration and the contact pressure has become positive, slip is assumed for the next iteration. This process is continued until (5)–(7) are satisfied on the entire friction boundary.

It should be noted that the function \( F(v, \lambda) \) becomes non-linear for the slip state since it then contains a multiplication between \( v \) and \( \lambda \). In that case, \( F_i \) is linearized for an iteration \( i \) into \( F_i(v_{i-1}, \lambda_i) \). In other words, the velocity field of the previous iteration is used to construct the linearized form of the function \( F_i \). If \( v_{i-1} \) is zero, which is the case if the state in iteration \( i - 1 \) was stick, an estimate for \( v_{i-1} \) is used that is based on the fact that for a converged solution the sliding direction has to be opposite to the frictional traction. Consequently, the estimate is constructed according to \( v_{i-1} = -\lambda \cdot (I - \mathbf{n}) \).

Due to the non-linearity of \( F \) for the slip state, the velocity field can still change between iterations even though the states of the individual points no longer change. It is therefore possible that additional iterations are required to obtain a converged solution after the correct frictional state has been obtained in every point.

3.2.2. Discretization of the Lagrange multipliers. Before \( F(v, \lambda) \) can be incorporated into the finite element formulation, the Lagrange multipliers that operate on the friction boundary \( F^{\kappa} \), have to be discretized. The discretization has to be such that it does not introduce spurious modes or locking in the other degrees of freedom of the velocity field [22].

It is therefore subject to similar stability considerations as the discretization of the pressure degrees of freedom in the mixed formulation. In Reference [23], it has been suggested that, for piecewise linear integration of the Lagrange multiplier space, the distance between two adjacent points of \( T^k \) should be larger than the edge length of the elements in \( T^k \) to avoid locking of the solution field. As a result, it would be complicated to generate a discretization of the surface on which the Lagrange multiplier is applied. However, if each component of \( \lambda \) is approximated with a piecewise constant the distance between two points in \( T^k \) is less constrained and can be taken equal to the edge length in \( T^k \) [23]. This piecewise constant approximation is referred to as the virtual finite element method. In this method, \( N \) control points, \( x_{ci}, i = 1, 2, \ldots, N \) are introduced on the boundary. At these control points, the friction equations are enforced locally (comparable to the collocation method). This leads to the following subspace of trial solutions for \( \lambda^h \):

\[
\Lambda^h = \{ \lambda^h \in (L^2(\Gamma_{\text{tool}}))^d; \lambda^h|_{x_{ci}} \in P_d(x_{ci}), \forall i \in 1, \ldots, N \} \tag{25}
\]

Now the appropriate weighting function that expresses the local operation of the Lagrange multiplier is a Dirac function \( \delta(x) \) which defines the subspace:

\[
\mathcal{Z}^h = \left\{ \chi^h = \sum_{i=1}^N z_i \delta(x - x_{ci}) \right\} \tag{26}
\]

In principle, friction should be modelled on the entire tooling surface \( \Gamma_{\text{tool}} \). However, both numerical experiments and observations in extrusion practice indicate that aluminium sticks to the ram and the container surface. Therefore, at these surfaces stick is assumed a priori and friction is only modelled on those parts of the tooling surface associated with the die. Then, using the enriched solution space for the velocities \( v^h \) (13) (rather than the linear space, \( v^h \)), the linear space for the pressures \( p^h \) and the piecewise constant space for the Lagrange multipliers \( \Lambda^h \) along with their spaces of weighting functions, the flow problem in (12) can be written as:

\[
\begin{align*}
\text{find } (v^h, p^h, \lambda^h) & \in v^h \times p^h \times \Lambda^h \text{ such that } \\
& \int_{\Omega} 2\eta D_{v^h} : D_{v^h} \, d\Omega - \int_{\Omega} (\nabla \cdot w^h_+) p^h \, d\Omega \\
& - \int_{\Gamma_{\text{die}}} w^h_+ \cdot \lambda^h \, d\Gamma = \int_{\Gamma_{\text{tool}}} w^h_+ \cdot s \, d\Gamma \quad \forall w^h_+ \in v^h_+ \\
& \int_{\Omega} q^h (\nabla \cdot v^h) \, d\Omega = 0 \quad \forall q^h \in p^h \\
& \int_{\Gamma_{\text{die}}} \chi^h \cdot \left( \mathcal{S}(v^h_+, \lambda^h) \right) \, d\Gamma = 0 \quad \forall \chi^h \in \mathcal{Z}^h \\
& v^h = v_{\text{tool}} \quad \text{on } \Gamma_{\text{tool}} \setminus \Gamma_{\text{die}}
\end{align*}
\tag{27}
\]

where the surface integral over the surface traction \( s \) is not taken over the die area or the outflow area. This is allowable because on the die surface the surface tractions are introduced by the friction Lagrange multipliers and on the outflow surface the surface tractions are zero, according to a stress-free exit. Since the friction boundary coincides with a domain boundary, it is most convenient to locate the control points on the nodes of the boundary. This implies
that the friction is enforced at the nodal level and therefore (23) and (24) have to be written in terms of nodal entities. The velocities in these relations are readily available but the surface normal $\mathbf{n}^k$ poses a serious problem (see also Figure 2). In node $k$, the direction of the normal is not uniquely defined for the discretized boundary. In fact, if there are $m$ element faces (for a 2D flow $m = 2$) connected to node $k$, there are $m$ normals defined in node $k$: $\mathbf{n}^k_1, \ldots, \mathbf{n}^k_m$, where $\mathbf{n}^k_i$ represents the face normal of the $i$th face connected to $k$. Only if $\mathbf{n}^k_i = \mathbf{n}^k_j, \forall i, j \in [1, m]$ the normal in $k$ is uniquely defined. Since this is not the case for curved boundaries, as exemplified in Figure 2, the normal $\mathbf{n}^k$ has to be determined in an appropriate manner.

3.2.3. Defining the nodal normal on curved surfaces. To define the normal, the new reconstruction method first suggested in Reference [24] will be employed since it is basically independent of the discretization and renders accurate results on coarse grids. The method is based on using the local flow field to determine the direction of the local normal and can be explained as follows. The normal $\mathbf{n}^k$ in node $k$ is assumed to be a weighted average of the normals on the element faces connected to the node under consideration:

$$\mathbf{n}^k = \frac{\sum_{i=1}^{m} w_i \mathbf{n}^k_i}{\| \sum_{i=1}^{m} w_i \mathbf{n}^k_i \|}$$

(28)

with $w_i$ the weight of each face. Usually these weights are determined by, for example, the surface area of the associated element faces [21], or local smoothing techniques [25]. However, in the approach proposed in Reference [24] the weight factors, $w_i$ are determined such that $\mathbf{n}^k$ becomes positioned in the plane that is perpendicular to the direction of the average velocity $\mathbf{v}$ in a thin boundary layer near $k$. This approach is based on the observation that when material is sliding along a surface, the velocity in a thin boundary layer is mainly perpendicular to the surface normal.

To determine $w_i$ such that $\mathbf{n}^k$ is perpendicular to $\mathbf{v}$ in the vicinity of $k$ an iterative method is used. Initially the weights, $w_i$ are chosen equal for all faces connected to $k$. Then the velocity field is computed. Next the velocity field is used to update $w_i$ in the following manner. For each element face $e$ connected to node $k$ with position vector $\mathbf{x}^e$, the velocity $\mathbf{v}^e$ is calculated in a material point $p_i$ located at

$$\mathbf{x}^e = \mathbf{x}^k - \theta \mathbf{n}^k_i$$

(29)
where $\theta > 0$ represents the thickness of the boundary layer, which should be small compared to the local element size. From $\mathbf{v}^p$ the component orthogonal to $\mathbf{n}^k$ and thus tangential to the element face can be computed:

$$\mathbf{v}_t^p = (I - n^k n^k) \cdot \mathbf{v}^p$$

The weight $w_i$ of each element normal is now defined to be the length of the tangential velocity in the sampling point:

$$w_i = \| \mathbf{v}_t^p \|$$

It can be verified that in a 2D flow ($m=2$) this definition of $w_i$ is adequate since it leads to a normal in $k$ that is perpendicular to the average velocity in the two sampling points.

The benefits of the proposed definition of $\mathbf{n}^k$ can be illustrated by considering the flow field that can be computed for a 2D extrusion problem. Four velocity fields are computed on a relatively coarse mesh (750 elements, see Figure 3(a)) for the following definitions of the normal $\mathbf{n}^k$ in the node $k$ on the edge at the entrance of the outflow channel:

(a) normal of upstream element face,
(b) normal of downstream element face,
(c) average direction of (a) and (b),
(d) the flow-dependent normal discussed above.

The velocity fields are compared to a velocity field that is obtained on a very fine mesh where several elements were positioned along the arc at the entrance of the outflow channel (2500 elements, see Figure 3(b)). For the velocity fields on the fine and the coarse mesh particle tracks are calculated. The starting positions of the particles are chosen at the inflow as indicated with the dots in Figure 4. The particle tracks near the re-entrant corner of the fine mesh (Figure 4) are compared to the corresponding tracks in the coarse mesh in Figure 5.

The results in Figure 5 indicate that the new method of defining the normal on a curved surface renders more accurate results for a coarse mesh. For the substitute normal that is based on the flow field, the position and shape of the particle track on the coarse mesh practically coincides with that of the fine mesh, while the other methods show significant deviations.

It is remarked that the extra computational costs that are introduced by the method are small. This is due to the fact that the iterative procedure, necessary to determine the direction of the normals, can be executed simultaneously within the iterative procedure that is necessary to solve the friction equations. In the computations, the number of iterations that are necessary to solve the friction equations are only increased by 20 per cent as a result of the concurrent determining of the direction of the normals.
4. RESULTS

In the following, the performance of the friction modelling is investigated. First, a rather simple 3D profile is used to assess the influence of the Coulomb friction factor used. Second, a complex hollow profile is analysed. In each of the analyses, a decoupled thermo-mechanical analysis was performed, where the deformations of the outflowing surface as well as the deflections of the die were accounted for. See Reference [26] for details on the incorporation of thermal effects.

4.1. Influence of the friction coefficients

In this section, it will be shown that the selected value for the friction coefficient plays a crucial role in the outcome of the simulations. The data justifying these conclusions is obtained by computing the solution fields for a number of test problems with different friction coefficients [27]. The flow fields from these test problems are compared to a reference computation which is performed using a Coulomb friction coefficient \( \mu \) of 0.4 as mentioned in Section 2.2.

For the actual comparison the extrusion profile visualized in Figure 6 is analysed. The radius of the billet is in the order of magnitude of 130 mm, the thicknesses of the thick and the thin legs are 6 and 3 mm, respectively, and the area reduction from billet to profile is close to a factor 40. For each problem, the prescribed ram velocity equals \( 6.3 \times 10^{-3} \) m s\(^{-1}\) and the imposed ram temperature is 733 K.

First, the attention will be focused on the solution fields that are obtained for the reference problem. The streamlines of aluminium are plotted in Figure 6. The notable curvatures in the streamlines near the pocket and the die are indicative for the relatively high-velocity gradients in those areas. The desired and the computed shapes of the profile are plotted in Figure 7 for the reference problem, where the geometrical deviations of the computed profile shape from the nominal shape have been multiplied by a factor 10 for visualization purposes. It can be concluded that the die is not optimally balanced, because the profile tends to flow to the right.
Four test problems are constructed in which the friction coefficient, $\mu$, is altered to respectively, 0.0, 0.2, 0.75, and $\infty$. It should be noted that the first and the last choice of $\mu$ represent a free slip and a full stick boundary condition for the aluminium in the die, respectively.

In Figure 8, the velocities in extrusion direction are presented for the reference problem and the test problems. The velocity contours are plotted in a cross-section of the computed (and consequently deformed) profile shape. In these plots, the outline of the desired profile shape is also depicted as a reference. The deformations of the outflow cross-section have been magnified with a factor 5 to make the shapes more distinguishable. This magnification causes the boundary of the profiles to become somewhat jagged even though they are relatively smooth as can be seen in Figure 9, in which the temperature distribution at the outflow has been plotted in the computed cross-sections without additional scaling.

The large influence of the friction modelling on the flow field of aluminium becomes apparent when Figure 8(a) is compared to Figures 8(b)–(e). For the full stick assumption with $\mu = 0$, the outflow velocities are depicted in Figure 8(e); the full stick assumption is physically unrealistic and it can be observed that it also renders an unrealistic flow field. For the friction coefficients $\mu = 0.0$, 0.2 and 0.75, the balance of the flow field is worse than in the reference problem. This can be concluded from the distorted cross-sections in Figures 8(b)–(e).
In Figure 8, the velocities in extrusion direction. The difference between the computed and the nominal shape is magnified with a scaling factor 5: (a) $\mu = 0.4$; (b) $\mu = 0$; (c) $\mu = 0.2$; (d) $\mu = 0.75$; and (e) $\mu = \infty$.

In Figure 9, the temperatures in a cross-section of the profiles are plotted. The temperature contours are depicted in the computed profile shape without additional scaling while the nominal profile shape is outlined as a reference. The differences in the temperature fields between the reference problem and the problems with $\mu = 0$ and 0.2 are relatively small. On the other hand, the temperature does increase considerably as the friction coefficient $\mu$ is increased, where it should be noted that the temperatures for $\mu = \infty$ exceed the maximum of the temperature scale by as much as 50 K. Increasing the friction coefficient leads to temperature increases because the shear rates inside the bearing become higher. The temperatures, however, do not decrease if the friction coefficient is lowered below 0.4. This indicates that in these cases the temperature of the profile is mainly determined by the deformations that aluminium experiences before it enters the bearing.

4.2. Hollow profile

In this section, the extrusion of a complex hollow profile is simulated. The profile consists of two hollow chambers as can be seen in Figure 10 and is relatively thin-walled, while the wall thickness varies with a factor 2 within the cross-section of the profile. The combination of the multiple chambers, the thin walls, and the wall thickness variations complicates the design of
Figure 9. Temperatures in a cross-section of the profile. Both the computed (solid) and the nominal (outline) shape are depicted: (a) $\mu = 0.4$; (b) $\mu = 0$; (c) $\mu = 0.2$; (d) $\mu = 0.75$; and (e) $\mu = \infty$.

Figure 10. Hollow profile with two chambers.

The dies for hollow profiles are considerably more sophisticated than those for flat profiles, which in turn results in rather complex computational domains. The meshing is performed...
in a number of steps which are described in detail in References [26, 28]. The mesh of the aluminium domain, containing 320,000 elements, is depicted in Figure 11. Within the computation less than 10 iterations problems were necessary to calculate the converged solution fields.

In Figure 12, the temperature field is plotted in the computed profile shape. It can be observed that, contrary to flat profiles, aluminium heats up considerably before entering the pocket and the bearing. This heating occurs in the legs, where the shear rates are relatively high. Also, it can be concluded that the computed profile hardly deviates from its nominal shape. This corresponds to the well-balanced profiles that were extruded in practice with this die design.

The fact that the profile is well-balanced is further illustrated in Figure 13, where the velocities at which aluminium exits the die are plotted in a cross-section of the deformed profile. The differences between the computed shape and the nominal shape have been magnified with a factor 10.
Figure 12. Temperature field in the profile.  Figure 13. Velocity field in a cross-section of the deformed profile. The deformations of the profile are magnified with a factor 10.

As can be observed, the only significant deviation from the nominal shape, apart from rigid body movements, occurs in the part of the profile that separates the two chambers. The middle of this part is slightly off-centred to the left, which is precisely what is observed in the real extruded profile. It can also be concluded that the velocity of aluminium in extrusion direction is practically constant throughout the cross-section. It should be noted that this prediction of an almost constant exit velocity is a non-trivial result, considering the flow imbalances that were observed in the results of the previous section.

5. CONCLUSION

An improved method for reconstructing the substitute normal in a node on a curved surface has been presented. This method uses the flow field in the vicinity of the node to establish the direction of the normal. Numerical experiments on a relatively coarse mesh indicate that this method renders an accurate flow field. Because the normals and the flow field are mutually dependent, an iterative method is required. This iterative method can be incorporated into the iterative procedure that is needed to account for the non-linear friction equations. That way the increase of the number of iterations, due to the a priori unknown normals, remains relatively small.

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