Time-domain performance based non-linear state feedback control of constrained linear systems

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This article describes a method to design a non-linear state feedback controller that meets a set of time-domain specifications not attainable by linear state feedback. Using a constrained polynomial interpolation technique, an input signal is computed that satisfies the desired time-domain constraints on the input and state-trajectories. The computed input is constructed by non-linear combinations of the states, such that a non-linear state feedback law is obtained. Stability of the resulting closed-loop polynomial system is analysed using sum-of-squares techniques. An illustrative example is presented, showing that the proposed non-linear controller outperforms the best linear static state feedback. To validate the proposed method, experiments on a fourth-order motion system have been carried out.

Keywords: nonlinear state feedback; constrained systems; polynomial approach; sum-of-squares

1. Introduction

Fundamental limitations on the performance of feedback systems with known linear plant dynamics is a topic that has been widely discussed in the literature. The open-loop system requires sufficient bandwidth and high gain at low frequencies in order to obtain a fast response and good settling behaviour. On the other hand, to suppress residual vibrations and sensor noise, the gain needs to be low at high frequencies. It is well known that this performance trade-off is defined by Bode’s gain/phase relation, Bode (1945), which limits how fast the open-loop gain can cross unity gain while maintaining closed-loop stability. Besides Bode’s gain/phase relation, several other integral relations (both in time as well as frequency domain) restrict the achievable performance that can be obtained by a control system, (Seron, Braslavsky and Goodwin 1997). These limitations are partly linked to the plant dynamics on one side and partly to the plant acting in combination with conventional LTI feedback controllers. This naturally raises the question if these latter limitations can be ameliorated by using non-linear or time-varying instead of LTI feedback. In order to obtain a performance that is superior to an LTI feedback system, several non-linear ‘tricks’ are used in industry (e.g. Heertjes and Steinbuch (2004)).

The goal in this article is to design systematically a non-linear full-state feedback controller that steers a given linear motion system from an initial position to a desired position as fast as possible and meets time-domain constraints that cannot be achieved by linear time-invariant (LTI) full-state feedback. Feed-forward is hence not considered.

Several control design techniques allowing time-domain constraints have been presented in the literature. In Henrion, Tarbouriech and Kucera (2005) a linear controller is constructed such that the closed-loop system yields a desired response and satisfies imposed constraints on the input and output. In predictive control theory, (non)linear controllers can be formulated that satisfy hard constraints on the controls and states. Model predictive control (MPC), (Mayne, Rawlings, Rao and Scokaert 2000), is based on a control action that is obtained by solving an infinite horizon optimisation problem. In Bemporad, Casavola and Mosca (1997), the concept of on-line reference management using a command governor based on predictive control theory such that constraints are fulfilled is addressed. A drawback of predictive control concepts and optimisation based methods in general is that they cannot be implemented on fast motion systems where high sampling rates are required, typically in the order of kHz, because of the required computational effort.

Nonlinear optimal control, see Rekasius (1964), Bass and Webber (1966), Asseo (1969), Moylan and Anderson (1973), Sandor and Williamson (1977b,a), Bernstein (1991), Haddad, Chellaboina and Fausz (1998), provides a systematic synthesis of non-linear
A non-linear full-state feedback law is constructed and the (local) stability of the resulting closed-loop system is analysed. An illustrative example is given in §4 and a comparison is made between the best linear and non-linear full-state feedback controllers. The proposed method is validated on a fourth-order motion system in §5.

2. Computation of the time-optimised input signal for a given control task

To drive a system from a given initial condition to its equilibrium as fast as possible without violating additional constraints, a control signal can be computed that meets the imposed time-domain constraints using a constrained polynomial interpolation technique. This approach, as described in Henrion and Lasserre (2006), is shortly reviewed here for completeness.

2.1 Constrained polynomial interpolation

Consider a linear time-invariant (LTI) system in state-space description

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t),
\end{aligned}
\]

where \(x(t) \in \mathbb{R}^n\) is the state-vector, \(u(t) \in \mathbb{R}^m\) is the control input-vector, \(y(t) \in \mathbb{R}^p\) is the output vector, \(A, B, C\) and \(D\) are matrices of proper dimensions, and the initial state \(x(0) = 0\). The goal is to compute a control signal \(u(t)\) in the time-interval \(t \in [t_u, t_f]\) such that the systems input \(u(t)\), output \(y(t)\) and their respective \(k_i\)-order derivatives \(u^{(k_i)}(t)\), \(y^{(k_i)}(t)\) meet the interpolation and bound constraints

\[
\begin{aligned}
&u^{(k_i)}(t) |_{t = t_u} = u_j \\
y^{(k_i)}(t) |_{t = t_u} = y_j \\
&\text{for } j = 0, \ldots, M, \\
&u^{(k_i)}(t) \geq u^{(k_j)}(t) \geq u^{(k_j)}(t) \geq y^{(k_j)}(t) \geq y^{(k_j)}(t) \text{ for } i = 0, \ldots, N,
\end{aligned}
\]

where \(k_i, k_j \geq 0\) are given integers, \(u_j, y_j, u^{(k_i)}_l, u^{(k_j)}_l, y^{(k_i)}_l, y^{(k_j)}_l\) are given real numbers and \(M\) and \(N\) are the number of interpolation and bound constraints, respectively. To this end, system (1) can be expressed as a right coprime polynomial matrix fraction

\[
y(s) = B_r(s)A_r^{-1}u(s).
\]

Under the coprimeness assumption on the pair \((A_r(s), B_r(s))\) there exists a polynomial matrix solution pair \((X_r(s), Y_r(s))\) to the Bézout identity

\[
X_r(s)A_r(s) + Y_r(s)B_r(s) = I.
\]
The internal state, or the flat output (Levine and Nguyen 2003), of system (4) is defined in Laplace-domain by

\[ x(s) = X(s)u(s) + Y(s)y(s), \]  

which makes it possible to represent the input \( u(s) \) and the output \( y(s) \) as linear combinations of the internal state \( x(s) \)

\[ u(s) = A_r(s)x(s) = \left( \sum_k A_{r,k}s^k \right)x(s), \]  
\[ y(s) = B_r(s)x(s) = \left( \sum_k B_{r,k}s^k \right)x(s), \]  

or, equivalently in time-domain

\[ u(t) = \left( \sum_k A_{r,k} \right) \frac{d^k x(t)}{dt^k}, \]  
\[ y(t) = \left( \sum_k B_{r,k} \right) \frac{d^k x(t)}{dt^k}, \]  

where \( A_{r,k} \) and \( B_{r,k} \) are the \( k \)th column of \( A_r \) and \( B_r \), respectively. Therefore, algebraic constraints on the vectors \( u(t) \) and \( y(t) \) can be translated into algebraic constraints on the internal state vector \( x(t) \), which is a polynomial of given degree \( \mu \) of the form

\[ x(t) = \sum_\mu x_\mu t^\mu. \]  

The interpolation constraints (2) can then be written as linear constraints on coefficients \( x_\mu \) of polynomial \( x(t) \)

\[ \sum_k A_{r,k}x^{(k)}(t)|_{t=t_j} = u_j \]  
\[ \sum_k B_{r,k}x^{(k)}(t)|_{t=t_j} = y_j. \]  

Bound constraints (3) as a function of the internal state are denoted by

\[ u^{low}_i \leq \sum_k A_{r,k}x^{(k)}(t) \leq u^{upp}_i \]  
\[ y^{low}_i \leq \sum_k B_{r,k}x^{(k)}(t) \leq y^{upp}_i. \]  

It turns out that finding an internal state vector (9) that satisfies (10) and (11) can be formulated as an LMI problem, Henrion and Lasserre (2006). An input signal can now be computed for a specific control task over a fixed interval \([t_0, t_f]\) that satisfies (10) and (11). Although it was assumed that the initial condition of system (1) was zero during the derivation of this LMI problem, initial conditions other than zero can be imposed through the interpolation constraints (10).

As the constrained polynomial interpolation algorithm is a convex optimisation problem, it will arrive at a feasible solution if one exists.

Since we are interested in driving the system from a given initial condition to its equilibrium as fast as possible, the minimum value of \( t_f \) needs to be determined such that the constraints are still satisfied. However, direct optimisation of the minimum time \( t_f \) is a non-convex problem and is hence performed via a bisection method.

3. Nonlinear state feedback and stability

This section focusses on the design of a state feedback law, based on the computed time-optimised input- and state-signals.

3.1 Construction of the non-linear state feedback

The input signal from the method of the previous section can be applied to the system in order to obtain the optimised response. However, such a feedforward approach is known to be sensitive to small variations in the plant dynamics, disturbances, and to unmodelled dynamics. Furthermore, the input signal is specifically computed for a single initial condition, and feedforward cannot cope with regulation of other initial conditions. Therefore, to make the performance and stability more robust for such variations, a closed-loop configuration is aimed at.

The computed input signal is constructed from the system’s state-trajectories to obtain the feedback law. A trivial solution to obtain the required input signal is to use the inverse of the system as the controller acting on the systems output, however, it is well-known that this controller lacks any form of robustness. Furthermore, in many cases, e.g. for systems which exhibit integrating action, this controller is not implementable. Therefore, in this article a state feedback law is proposed. More specifically, a polynomial feedback law is considered since every continuous function can be uniformly approximated as closely as desired by a polynomial function, as stated by the Weierstrass approximation theorem (Kreyzig 1978), and stability properties of polynomial (closed-loop) systems can be evaluated in a systematic way, as will be discussed in §3.2. To compute the polynomial state feedback law, the following least-squares data-fitting problem is considered:

\[ \text{find } \theta \text{ such that } \sum_i (F(\theta, x_i(t), \ldots, x_n(t)) - u(i))^2 < y_{fit}, \]  

where \( u \) is the optimised input signal, \( y_{fit} \) is the desired level of accuracy, index \( i \) is used to specify each data-point, and \( n \) represents the number of states.
The function \( F(\theta, x_1, \ldots, x_n) \) contains all possible monomials in \( n \) states for a given degree \( d \), and with coefficients \( \theta \). For example, if \( n = 2 \), the function \( F(\theta, x_1, x_2) \) is described by

\[
F(\theta, x_1, x_2) = \sum_{p=0}^{d} \sum_{q=0}^{d} \theta_{pq} x_1^p x_2^q. \tag{13}
\]

The resulting closed-loop system is given by

\[
\dot{x} = f(x), \quad x(t_0) = x_0, \quad x(t_f) = x_e, \tag{14}
\]

where \( f(x) \) is a vector-polynomial of degree \( d \). In this article we focus on regulation and hence \( x_e \) equals the origin of system (14).

### 3.2 Estimation of the region of attraction

Closed-loop system (14) is completely described by polynomial differential equations and generally such a system has multiple equilibria. In that case the system does not stabilise at the desired equilibrium at the origin for all initial conditions. The set of all initial conditions that converge to the origin, i.e., the region of attraction, is defined as (Khalil 1992)

\[
R_d := \{ x_0 \in \mathbb{R}^n | \phi(t, x_0) \to 0 \text{ as } t \to \infty \}, \tag{15}
\]

where \( \phi(t, x_0) \) is the solution of (14) that starts at initial state \( x_0 \) at time \( t = 0 \). To assess the stability of such a non-linear system, a general method is to search for Lyapunov functions to prove global or local stability.

A powerful tool for computing Lyapunov functions and estimates of the region of attraction for polynomial systems is the sum-of-squares (SOS) method; see Papachristodoulou and Prajna (2005). A multivariable polynomial \( p(x) = p(x_1, \ldots, x_n) \) is called a SOS, if it is possible to find a set of polynomials \( p_i, i = 1, \ldots, m \) such that

\[
p(x) = \sum_{i=1}^{m} p_i^2(x). \tag{16}
\]

From the above expression it is clear that all SOS-polynomials are non-negative for all \( x \in \mathbb{R}^n \), however \( p(x) \geq 0 \) does not imply that \( p(x) \) is SOS. The SOS-condition (16) is equivalent to the existence of a decomposition of the form

\[
p(x) = Z^T(x)QZ(x), \tag{17}
\]

where \( Q \) is a positive semidefinite matrix and \( Z(x) \) is a properly chosen vector of monomials; see Parrilo (2000). Local asymptotic stability of the origin of (14) can be guaranteed by finding a valid Lyapunov function \( V(x) \) in some region \( D \) of the state space that contains the origin and satisfies (Papachristodoulou and Prajna 2005)

\[
V(x) > 0 \quad \forall x \in D \setminus \{0\} \quad \text{and} \quad V(0) = 0, \tag{18}
\]

\[
\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \quad \forall x \in D. \tag{19}
\]

It is well-known that for linear systems a quadratic Lyapunov-function of the form \( V(x) = x^T P x \) is necessary and sufficient to prove stability. However, for systems with polynomial vector fields, the existence of a quadratic Lyapunov function is only sufficient to prove stability, and hence considering only quadratic Lyapunov functions is generally conservative. To reduce this conservatism, we consider Lyapunov functions that are polynomial in the states, and hence (18) and (19) become polynomial nonnegativity conditions. An additional difficulty lies in the fact that \( V(x) \) needs to be positive definite, not just positive semidefinite. This can be overcome by the introduction of a positive definite shaping function \( \phi(x) \) that ensures \( V(x) > 0 \) instead of \( V(x) \geq 0 \). Local stability of the origin, valid in the region \( D = \{ x \in \mathbb{R}^n | a(x) \leq 0 \} \) (20) containing the origin and with \( a(x) \) any polynomial, can be tested by a SOS program; see Papachristodoulou and Prajna (2005). When asymptotic stability of the origin is established, an estimate of the region of attraction is given by the largest level set of the Lyapunov function \( V(x) \)

\[
\Omega_y = \{ x \in \mathbb{R}^n | V(x) \leq \gamma \}, \tag{21}
\]

which is bounded and strictly contained in \( D \). This estimate is in general conservative and in order to obtain good estimates of the region of attraction, one has to iterate between \( V(x) \) and \( a(x) \) as follows. First, initialise the region of interest as a ball of radius \( \zeta \), i.e., \( a(x) = x^T x - \zeta \) and search for a second order Lyapunov function \( V(x) \) while maximising \( \zeta \) via bisection. Then determine the largest level set of \( V(x) \) that is completely contained in the set \( V(x) \leq 0 \) by SOS programming and use the level set \( V(x) - \gamma \) as the new region of interest \( a(x) \). Increase the order of the Lyapunov function by 2 and search for a new Lyapunov function through maximising \( \gamma \). Iterate this procedure until a satisfactory estimate is obtained. If the order of closed-loop system (14) is very high, instead of a single high degree Lyapunov function one can use composite Lyapunov functions in order to keep the number of decision variables low; see Tan and Packard (2006).

If a value of \( \gamma \) is found such that (21) describes a region that includes the initial state values for which
the control task has been designed, it can be concluded that the controller is stabilising for this initial condition.

### 3.3 Nonlinear state feedback synthesis

Beforehand, it is unknown what combination of the states should be used to construct a feedback law that yields satisfactory performance and robustness for different initial conditions. Furthermore, the parameter $\theta$ in (12) influences the region of attraction as well. Since no LMI method is known that simultaneously constructs a feedback law yielding satisfactory performance and optimises the region of attraction, we propose an iterative procedure to synthesise the non-linear state feedback. Start by considering a function $F(\theta, x)$ of small degree $d$, choose the accuracy $\gamma_{th}$ and compute the provable region of attraction via the algorithm in §3.2. If the performance is satisfactory, i.e., the system’s response to the initial condition obtained by the feedback is close to the designed trajectory, but the provable region of attraction is too small, decrease the accuracy (increase $\gamma_{th}$). If both are unsatisfactory, increase the degree $d$ of the feedback. Iterate until both the performance and the region of attraction are satisfactory. Unfortunately, convergence is not guaranteed and the synthesis remains in some sense based on trial-and-error. Furthermore, no guarantees can be given that the performance constraints will not be violated for an arbitrary initial condition within the found region of attraction. Hence, an important topic for future research is to derive a systematic method to analyse the closed-loop with respect to constraint violation.

### 4. Illustrative example

In this section the proposed method is illustrated by an example. Operations on polynomials were performed with the Polynomial Toolbox 2.5 Ltd. (2001), whereas the LMI problems were solved with Yalmip (Loefberg 2004), and SeDuMi 1.1, (Sturm 1999).

#### 4.1 Computation of desired input signal

Consider the following second order motion system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$  \hspace{1cm} (22)

where the state vector $x(t)=[x_1(t), x_2(t)]^T$ contains the position and velocity of a mass in [m] and [(m/s)], respectively. Furthermore, $m=1$ [kg] and the input $u(t)$ is the force $F$ [N] applied to the mass. We assume that the actuator can deliver a force with a maximum of 10 [N]. The goal is to drive the system from its initial condition to the zero equilibrium as fast as possible subject to additional constraints. The interpolation constraints (2) are $x_1(0) = 1, x_2(0) = x_1(t_f) = x_2(t_f) = u(t_f) = 0$, where the final time $t_f$ is to be minimised. To make sure no overshoot occurs and the computed input does not saturate, the following bound constraints (3) are added:

$$0 \leq x_1(t) \leq 1$$

$$-10 \leq u(t) \leq 10.$$  \hspace{1cm} (23)

The order of the polynomial internal state vector (9) still has to be chosen. In general, a higher order of the polynomial results in a faster response since more freedom is provided to minimise the final time $t_f$; however, it also requires more computations. Therefore, a trade-off should be made between the computational burden and the optimality of the solution. The state-trajectories and required input obtained with a seventh order internal state polynomial are depicted in Figure 1. In the same figure, the well-known bang–bang solution is also depicted. This solution is known to be time-optimal for this problem without the constraint $u(t_f) = 0$, and results in the final time $t_f=0.63$ [s]. When this constraint is removed from the polynomial optimisation problem and the order allowed to be increased, it can be shown that the polynomial interpolant approximates the bang–bang solution. The polynomial will never be arbitrarily close to the bang–bang solution, not even if the order is allowed to be infinite. This is due to the well-known

![Figure 1](image.png)

Figure 1. State-trajectories $x_1(t), x_2(t)$ and required input $u(t)$ corresponding to the polynomial optimisation problem (solid) and the bang–bang solution (dashed).
Gibbs phenomenon. Indeed, oscillations near the discontinuities caused by using a continuously differentiable function to approximate a jump discontinuity are prohibited by the bound constraints on the input \( u \) and will, therefore, never occur.

From the response, it can be seen that no overshoot occurs and the initial position and velocity as well as the conditions at the optimised final time \( t_f = 0.76 \) [s] are satisfied. Furthermore, for this time-optimised control objective, the actuator signal does not violate the imposed saturation constraint. It can be concluded that all interpolation and bound constraints are satisfied.

### 4.2 Construction of the non-linear state feedback

A non-linear state feedback law can now be obtained by mapping the desired state trajectories on the computed input signal. In this case, a satisfying mapping was found consisting of all possible monomials of degree 1 up to 4, described by

\[
f(x) = -297.63x_1 - 31.92x_2 + 203.56x_1^2 + 87.34x_1x_2 - 6.58x_2^2 + 132.35x_1^3 + 142.99x_1^2x_2 + 12.74x_1^2x_2^2 - 9.22x_2^3 - 48.34x_1^4 - 199.35x_1^3x_2 + 7.92x_1^2x_2^2 - 3.74x_1x_2^3 - 1.17x_2^4.
\]

This mapping is depicted in Figure 2(a). The resulting closed-loop system is completely described by the polynomial differential equations

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f(x).
\end{align*}
\]

Since the resulting closed-loop system is of second order, stability can easily be assessed by looking at the phase portrait. The phase portrait and equilibria of the resulting closed-loop is shown in Figure 2(b).

### 4.3 Stability and performance analysis

Since (25) has four equilibria, the origin is not a global attractor and hence the region of attraction of the origin should be estimated. To obtain this estimate, the method as described in §3.2 is used. Starting by searching for the largest level set of a second order Lyapunov function in a circular area around the origin, the first estimate of the region of attraction is depicted in Figure 3(a). In this figure, the dark grey area denotes the region where \( V > 0 \), the light grey area is the region where \( V \leq 0 \), and the solid line is the largest level set of \( V(x) \) contained in the area where \( V < 0 \), i.e. an approximation of the region of attraction. The arrows denote the direction of the vector field of the closed-loop system and the dashed line is the nullcline \( f(x) = 0 \). The estimates of the region of attraction with the subsequent higher order Lyapunov functions are depicted in Figure 3(b)–(d). When an eighth order Lyapunov function is used, the provable region of attraction includes the initial state \( x(0) \), and hence the designed feedback law is stabilising for the posed regulation problem. The system’s response (taking into account the bounds on the input) with respect to the initial conditions \( x_0 = [1 0] \), \( x_0 = [0.7 0] \), and \( x_0 = [0.4 0] \) (all within the region of attraction) are depicted in Figure 4. The performance is measured by the standard...
regulation indicators fall-time, settling-time and overshoot of the response, defined by

\[ T_{\text{fall}} = \min(T) \quad y(t) \leq 0.1 \quad \text{for} \quad t \geq T \]
\[ - \min(T) \quad y(t) \leq 0.9 \quad \text{for} \quad t \geq T \] \quad [s]
\[ T_{\text{settle}} = \min(T) \quad |y(t)| \leq 0.02 \quad \text{for} \quad t \geq T \] \quad [s]
\[ O_s = |\min(y(t))| \cdot 100\% \]

As can be seen, the interpolation and bound constraints are satisfied while the responses exhibit satisfactory settling times and overshoot as shown in Table 1. While robust stability with respect to different initial conditions can be analysed through the computed region of attraction, it cannot be guaranteed that robust performance, i.e. satisfactory performance for each initial condition, is obtained for this region.

The performance should therefore be analysed by means of simulation. As can be observed from the phase portrait, although the feedback is stabilising for the initial state \( x_0 = [1 \ 0] \), the closed-loop is unstable when a starting position is used that is slightly larger than 1. If a larger region of attraction is desired, another design should be made. Following the procedure of §4.2 a ninth order internal state vector is used (resulting in a final time \( t_f = 0.75 \)) and the states are mapped to a third order feedback, resulting in a global stabilising feedback. This feedback is given by

\[ f(x) = -4.83x_1 - 22.13x_2 + 7.86x_1^2 \]
\[ + 19.19x_1x_2 - 12.42x_2^2 - 13.51x_1^3 \]
\[ - 1.41x_1^2x_2 + 4.69x_1x_2^2 - 2.14x_2^3 \] \quad (27)

Figure 3. Estimates of the region of attraction (solid) for increasing order of the Lyapunov functions together with the nullcline \( f(x) = 0 \) (dashed): (a) second order Lyapunov function; (b) fourth order Lyapunov function; (c) sixth order Lyapunov function; (d) eighth order Lyapunov function.
and a Lyapunov function proving global stability is
\[
V(x) = 1.43x_1^2 + 0.07x_1x_2 + 0.03x_2^2.
\] (28)

The phase portrait of the resulting closed-loop and the response to various initial conditions are depicted in Figure 5. As can be seen from 5(a), the origin is a global attractor. The regulation performance of this controller is given in Table 2. From this table it is clear that the performance of other initial conditions than the one the design was made for is worse when compared to Table 1, especially when the settling time is considered. Thus although a larger region of attraction is obtained, i.e. the entire state-space, the regulation performance is worse than in the previous design.

### 4.4 Linear vs. non-linear state feedback

A closed-loop system has been obtained, which guarantees that starting from initial condition \( x_0 = [1 \ 0] \), the system returns to the origin. The computed input signal and desired state-trajectories (see Figure 1) cannot be mapped onto a static linear state feedback. It is interesting to see what the actual benefit of the non-linear state feedback controller (24) is over a static linear state feedback controller. The linear state feedback controller must satisfy the input-constraint \( |u(t)| = | - Kx(t)| \leq 10 \) [V] and is optimised for minimum fall-time, settling-time and overshoot. In order to find the best linear state feedback controller that satisfies the input constraint and has the smallest fall-time, settling-time and overshoot, trade-off curves can be created (Boyd and Barratt 1991). The trade-off plot between fall-time and overshoot is depicted in Figure 6(a) and was obtained by varying the pole-locations of the closed-loop system \( \dot{x}(t) + (A + BK)x(t) \), where only the controllers that

![Figure 4. Response to various initial conditions: \( x_0 = [1 \ 0] \) (solid), \( x_0 = [0.7 \ 0] \) (dash-dot), \( x_0 = [0.4 \ 0] \) (dotted), computed optimised signal \( x_1(t) \) (dashed).](image)

![Figure 5. Phase portrait and regulation performance: (a) phase portrait of the resulting closed-loop system, together with the nullcline \( f(x) = 0 \); (b) responses to various initial conditions.](image)

Table 1. Performance measures for various initial conditions.

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( T_{\text{fall}} )</th>
<th>( T_{\text{settle}} )</th>
<th>( O_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1.0 \ 0]</td>
<td>0.38</td>
<td>0.62</td>
<td>0.10</td>
</tr>
<tr>
<td>[0.7 \ 0]</td>
<td>0.30</td>
<td>0.51</td>
<td>0.15</td>
</tr>
<tr>
<td>[0.4 \ 0]</td>
<td>0.23</td>
<td>0.39</td>
<td>0.26</td>
</tr>
</tbody>
</table>
satisfy the actuator constraint are admitted. The best static linear state feedback controller was determined by minimising the euclidian norm of the three time-domain specifications, resulting in an optimal solution with respect to the three performance constraints. The simulated responses to initial conditions $x_0=[1 \ 0]$, $x_0=[0.7 \ 0]$, and $x_0=[0.4 \ 0]$, together with the corresponding inputs are shown in Figure 6(b). Implementation of the optimal linear controller for $x_0=[1 \ 0]$ results in a minimal fall-time of 0.78 [s], minimal settling-time of 1.19 [s] and a minimal overshoot of 1.6 [%]. The proposed non-linear controller yields for the same initial condition a fall-time of 0.38 [s], a settling-time of 0.62 [s] and 0.10 [%] overshoot. From these figures it can be concluded that the proposed non-linear state feedback controller outperforms its linear counterpart, at least for the chosen initial conditions within the found region of attraction.

Remark 1: So far, only regulation of the states to the zero equilibrium has been considered. To be able to perform pick-and-place tasks where the goal is to drive the system from a given start point to a given endpoint, an error-space approach can be used. Consider, for example, the problem of driving system (22) from initial condition $x(0)=[0, 0]^T$ to final position $x(t_f)=[1, 0]^T$ as fast as possible. Analogously to the procedure from §4.1, a time-optimised input signal $u$ is computed that satisfies the bound constraints (23) and the interpolation constraints $x_1(0)=x_2(0)=x_2(t_f)=u(t_f)=0$ and $x_1(t_f)=1$. The main difference is that in the pick-and-place control setup, the error is regulated to the zero equilibrium instead of the states. Therefore, the computed input signal $u(t)$ is constructed by a non-linear feedback of the form

$$F(\theta, e_1, e_2) = \sum_{p=0}^{d} \sum_{q=0}^{d} \theta_{pq} e_1^p e_2^q,$$

where $e_1(t)=r_1(t)-x_1(t)$, $e_2(t)=r_2(t)-x_2(t)$, $r_1(t)=1$ and $r_2(t)=0$. The closed-loop system in terms of the error is given by

$$\dot{e}_1(t) = -x_2(t)$$

$$\dot{e}_2(t) = -f(e(t)),$$

and stability can be verified by searching for Lyapunov-functions that are a function of the error.

5. Practical example

In this section experimental results on a fourth-order dual rotary motion system are presented. The system consists of two masses that are connected by a flexible bar, see Figure 7. The first mass is driven by a motor and the positions of both masses are measured by encoders. In order to obtain the required state-space model, frequency

![Figure 6. Comparison linear and non-linear state feedback: (a) trade-off plot between fall-time and overshoot: linear state feedback controller (dots), best linear state feedback controller (star), best non-linear state feedback controller (cross); (b) best linear state feedback controller (dashed), non-linear state feedback controller (solid).](image)
response measurements have been performed, followed by parametric identification. The resulting state-space representation is given by

$$
\dot{x}(t) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-4646 & -2.9452 & 4646 & -2.9452 \\
0 & 0 & 0 & 1 \\
57013 & 3.6138 & -57013 & -3.6138
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
4894.8 \\
0 \\
0
\end{bmatrix} u(t),
$$

with $x(t) = [\theta_1(t) \dot{\theta}_1(t) \theta_2(t) \dot{\theta}_2(t)]^T$ the positions and velocities of the two masses in [rad] and [(rad/s)] respectively and the input $u(t)$ in [V]. The frequency response function of the transfer from the input to $x_1$ and $x_3$, together with the identified model are depicted in Figure 8(a) and (b), respectively. In order to design a non-linear state feedback that drives the system from its initial position to the zero equilibrium, an input is computed with the constrained polynomial interpolation technique. The following interpolation constraints are imposed: $x_1(0) = x_3(0) = 100$ and $x_2(0) = x_4(0) = x_1(t_f) = x_3(t_f) = x_2(t_f) = x_4(t_f) = u(t_f) = \dot{u}(t_f) = 0$, where the final time $t_f$ needs to be optimised. The rotational speed of the motor is limited to 450 [(rad/s)] and since overshoot and actuator saturation are undesirable, the following bound constraints are formulated

$$
-450 \leq x_2(t) \leq 450 \text{ [rad/s]} \\
0 \leq x_1(t) \leq 100 \text{ [rad]} \\
-2.5 \leq u(t) \leq 2.5 \text{ [V]},
$$

A 6th-order internal state polynomial was generated, with an optimised final time of $t_f = 0.44$ [s]. The state-trajectories have been mapped by the second order non-linear function

$$
f(x) = -0.231 x_1 - 1.93 \cdot 10^{-2} x_2 - 0.232 x_3 \\
+ 1.472 \cdot 10^{-3} x_1^2 \\
+ 9.376 \cdot 10^{-2} x_1 x_2 + 4.168 \cdot 10^{-5} x_2^2 \\
+ 1.472 \cdot 10^{-3} x_1 x_3 \\
+ 9.376 \cdot 10^{-3} x_2 x_3 + 1.472 \cdot 10^{-3} x_3^2.
$$

This non-linear state feedback law is implemented on the experimental setup. Only the positions of the two masses can be measured and since full-state feedback is considered, the two velocities have to be reconstructed. For this purpose a model-based linear full-state observer is designed to obtain an estimate of both velocities. The results of this experiment, together with the designed state-trajectories and input signal, are depicted in Figure 9. As can be seen, the system is regulated to the origin from its initial position in the
optimised final time $t_f$, without overshoot and the actuator and velocity constraints are satisfied.

Some slight differences occur between the experimental and calculated system signals, as is obvious from Figure 9(b). The main reason lies in the fact that the system is represented by a fourth order linear model, while in reality there is some unmodelled dynamics and some friction, even though it may be small. Furthermore, an observer is implemented to provide an estimate of the velocities. Although this gives a good approximation, it will not exactly resemble the actual velocity profile. These observations support the idea that the closed-loop approach is robust for small variations. Another reason for a closed-loop approach was mentioned, namely robustness with respect to different initial conditions. In Figure 10(a) the responses to initial conditions $x_0 = [100 \ 0 \ 100 \ 0]$ (solid), $x_0 = [60 \ 0 \ 60 \ 0]$ (dashed), and $x_0 = [20 \ 0 \ 20 \ 0]$ (dotted) are depicted. As can be seen, the performance is satisfactory since other initial conditions than the one designed for are regulated to zero while fulfilling the imposed constraints. In case negative initial conditions are used, the proposed feedback law (33) results in an unstable closed-loop due to the even terms. To obtain a stabilising feedback law for both positive and

Figure 9. Experimental results: (a) calculated system signals $x_1$, $x_2$, $u$ (dashed) and experimental results (solid), see (b) for difference; (b) difference between calculated system signals $x_1$, $x_2$, $u$ and experimental results.

Figure 10. Response to different initial conditions.
negative initial conditions the even terms of (33) are pre-multiplied by the sign of the first state of the system resulting in the control law

\[
f(x) = -0.231x_1 - 1.93 \cdot 10^{-2}x_2 - 0.232x_3
\]

\[+ 1.472 \cdot 10^{-3}\text{sign}(x_1)x_1^2 + 9.376 \cdot 10^{-3}\text{sign}(x_1)x_1x_2
\]

\[+ 4.168 \cdot 10^{-5}\text{sign}(x_1)x_2^2 1.472 \cdot 10^{-3}\text{sign}(x_1)x_1x_3
\]

\[+ 9.376 \cdot 10^{-5}\text{sign}(x_1)x_2x_3 + 1.472 \cdot 10^{-3}\text{sign}(x_1)x_3^2.
\]

This non-linear feedback is equivalent to (33) in case positive valued initial conditions are present and is also able to regulate the system when it is subject to negative valued initial conditions as is shown in Figure 10(b).

6. Conclusions and recommendations

In this article, a time-domain performance based approach for non-linear state feedback controller design for linear systems has been presented. The application of the method focussed on regulating the system to its origin as fast as possible without overshoot. The nominal design is made for one specific initial condition but the resulting non-linear controller is able to regulate the system for other initial conditions as well, as long as they are within the region of attraction of the closed-loop system. However, although the closed-loop system is guaranteed to be stable for these other initial conditions, performance can in general not be guaranteed a priori and should be analysed by simulations. The presented simulation and experimental results showed that satisfactory performance for a range of initial conditions can be achieved, and that the proposed non-linear state feedback controller can outperform its linear counterpart.

Although only regulation was considered, pick-and-place tasks where the goal is to drive the system from a given start point to a given endpoint while taking into account time-domain specifications are easily handled by considering an error-space approach. Furthermore, the designed state feedback regulator can also be applied in a disturbance setting. Indeed, if the system is subject to non-persisting external disturbances (like shocks), the controller will regulate the system to its origin as long as the disturbance does not bring the system in a state outside the region of attraction. Therefore, one should design the controller for a set of initial conditions which contains the states to which the expected disturbances can drive the system.

At this moment however, the method should be regarded as a first step towards actual use in applications since there are some issues that should be resolved before it can be applied to real-life problems. First, as stated in §3.3, the method lacks a procedure to systematically analyse the performance (including the violation of bounds) after a feedback law is obtained. Secondly, it would be interesting to combine the computation of the non-linear state feedback with the construction of a Lyapunov function during the design, such that stability is guaranteed a priori. Ideally, the two issues mentioned above are blended into one approach. This would result in a method to compute a non-linear polynomial state feedback law that a priori guarantees stability and performance for a maximised set of initial conditions, while taking the specified time-domain constraints into account. A third potential problem is the computational tractability of the method when the plant dynamics become more complicated. This is due to the fact that the constraint polynomial interpolation and the stability analysis are performed using LMIs. The currently available LMI solvers are not always able to come up with feasible solutions for large problems, even if those problems are feasible. As the feedback law is computed a priori, i.e. off-line, the LMI computation does not hamper real-time application of the resulting controller. The polynomial feedback law itself is computationally cheap since only additions and multiplications of the state signals are performed. Another topic for future research is to investigate the possibility to track different reference signals (other than pick-and-place actions) with the proposed method.

References


