A solid-like shell element allowing for arbitrary delaminations

Joris J. C. Remmers*,†, Garth N. Wells and René de Borst

Koiter Institute Delft, Delft University of Technology, P.O. Box 5058, NL-2600 GB Delft, The Netherlands

SUMMARY

In this contribution a new finite element is presented for the simulation of delamination growth in thin-layered composite structures. The element is based on a solid-like shell element: a volume element that can be used for very thin applications due to a higher-order displacement field in the thickness direction. The delamination crack can occur at arbitrary locations and is incorporated in the element as a jump in the displacement field by using the partition of unity property of finite element shape functions. The kinematics of the element as well as the finite element formulation are described. The performance of the element is demonstrated by means of two examples. Copyright © 2003 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The application of layered composite structures in the aerospace and automotive industries has increased significantly over the last decades. Since the material can be tailored to meet special demands, laminated composite structures can be lighter and made with superior characteristics when compared to traditional single-phase materials. However, the use of these materials introduces new failure mechanisms such as delamination, i.e. the debonding of individual layers. In general, the presence of delaminations in the material will lead to a significant reduction of the residual strength.

To arrive at a better understanding of this phenomenon, numerical simulations can be of assistance. In finite element models, a delamination crack is normally modelled using interface elements [1–3]. These elements consist of two surfaces which are connected to the continuum elements that model the adjoining layers of the laminate. A perfect bond prior to delamination growth is simulated with a high dummy stiffness. Debonding is governed by a decohesion relationship, which can be derived on the basis of a damage mechanics theory [1] or can be
formulated in a plasticity framework [2]. These relations have been extended to meet more practical applications such as mixed-mode delamination [4].

In an alternative approach, delamination cracks can be incorporated in continuum elements by exploiting the partition-of-unity property of the finite element shape functions [5]. Initially, this approach was used to simulate stationary cracks in a finite element model by adding the span of the near-tip solution of linear-elastic fracture mechanics to the displacement field [6, 7]. Later, the method was cast in a cohesive framework to model progressive crack growth [8, 9]. In this fashion, a crack is modelled as a jump (a discontinuity) in the displacement field of the continuum elements. The magnitude of the jump is determined by an extra set of degrees of freedom, which are added to existing nodes of the finite element mesh. The separation process can be specified by the same decohesion relations as used with interface elements, including those that describe mixed-mode fracture. The discontinuity, which crosses elements irrespective of the structure of the finite element mesh, can be extended into the next element by simply adding extra degrees of freedom to the nodes that support this element.

The advantages of this new approach are evident. First, the total number of unknowns in the system reduces significantly, since degrees of freedom are only added to the model when the delamination surface propagates, i.e. when the discontinuity is extended. As a consequence, the use of high dummy stiffnesses to simulate the perfect bond is avoided. These high dummy stiffnesses can lead to traction oscillations ahead of the delamination front [10]. Furthermore, since the cohesive surface is incorporated in the continuum element, a consistent extension to a finite strain framework is feasible [11]. Finally, it is possible to model a complete laminate with just one continuum element in thickness direction [12], which allows for the analysis of delamination growth on a structural level.

Herein, this approach is extended to a shell model. A key feature is the choice of a proper element. Conventional volume elements show an overly stiff behaviour when used in thin applications (Poisson thickness locking [13]) due to a constant strain distribution in thickness direction. An alternative is the solid-like shell element [14]. Here, an additional set of internal degrees of freedom is used to add a quadratic term to the displacement field in thickness direction, the internal ‘stretch’ of the element. Hence, the corresponding strain field varies linearly over the thickness instead of being constant and Poisson thickness locking is avoided.

It has been shown that the solid-like shell element can be used for modelling laminate structures by either stacking multiple elements, or by modelling multiple layers within one element [15]. In the latter situation, the element is divided into a number of sub-domains, each of which has different material parameters. When, as is proposed here, this model is extended with the possibility to initiate and propagate delaminations at arbitrary locations within the shell during the loading process, the complete mechanical behaviour of laminated structures can be simulated with just one shell element in thickness direction.

This contribution is ordered as follows. In the next section, a concise description of the kinematic relations for the original solid-like shell element is given. The derivation of the enhanced solid-like shell element with the displacement jump is presented in Sections 3–5. Section 6 discusses the finite element discretisation. The incorporation of the constitutive models and some aspects regarding linearisation of the governing equations are presented in Sections 7 and 8. Key aspects of implementation are addressed in Section 9. The performance of the enhanced element is demonstrated by means of two numerical examples in Section 10, after which some conclusions are drawn.
2. KINEMATICS OF THE SOLID-LIKE SHELL ELEMENT

We consider the thick shell shown in Figure 1. The position of a material point in the shell in the undeformed configuration can be written as a function of the three curvilinear co-ordinates \([\xi, \eta, \zeta]\):

\[
X(\xi, \eta, \zeta) = X_0(\xi, \eta) + \zeta D(\xi, \eta)
\]

where \(X_0(\xi, \eta)\) is the projection of the point on the mid-surface of the shell and \(D(\xi, \eta)\) is the thickness director in this point:

\[
X_0(\xi, \eta) = \frac{1}{2}[X_t(\xi, \eta) + X_b(\xi, \eta)]
\]

\[
D(\xi, \eta) = \frac{1}{2}[X_t(\xi, \eta) - X_b(\xi, \eta)]
\]

The subscripts \((\cdot)_t\) and \((\cdot)_b\) denote the projections of the variable onto the top and bottom surface, respectively. The position of the material point in the deformed configuration \(x(\xi, \eta, \zeta)\) is related to \(X(\xi, \eta, \zeta)\) via the displacement field \(\phi(\xi, \eta, \zeta)\) according to

\[
x(\xi, \eta, \zeta) = X(\xi, \eta, \zeta) + \phi(\xi, \eta, \zeta)
\]

where

\[
\phi(\xi, \eta, \zeta) = u_0(\xi, \eta) + \zeta u_1(\xi, \eta) + (1 - \zeta^2)u_2(\xi, \eta)
\]

In this relation, \(u_0\) and \(u_1\) are the displacements of \(X_0\) on the shell mid-surface, and the thickness director \(D\), respectively:

\[
u_0(\xi, \eta) = \frac{1}{2}[u_t(\xi, \eta) + u_b(\xi, \eta)]
\]

\[
u_1(\xi, \eta) = \frac{1}{2}[u_t(\xi, \eta) - u_b(\xi, \eta)]
\]

and \(u_2(\xi, \eta)\) denotes the internal stretching of the element, which is colinear with the thickness director in the deformed configuration and is a function of an additional ‘stretch’ parameter \(w\):

\[
u_2(\xi, \eta) = w(\xi, \eta)[D + u_1(\xi, \eta)]
\]
In the remainder, we will consider the displacement field $\phi$ as a function of two kinds of variables; the ordinary displacement field $u$, which will be split in a displacement of the top and bottom surfaces $u_t$ and $u_b$, respectively, and the internal stretch parameter $w$:

$$\phi = \phi(u_t, u_b, w)$$ (9)

3. ENHANCED KINEMATIC RELATIONS

The thick shell with constant thickness of Figure 2 is crossed by a discontinuity surface $\Gamma_{d,0}$ which divides the domain into two parts, $\Omega_0^+$ and $\Omega_0^-$. The discontinuity surface is assumed to be parallel to the mid-surface of the thick shell. The displacement field $\phi(\xi, \eta, \zeta)$ can now be regarded as a continuous regular field $\hat{\phi}$ with an additional continuous field $\tilde{\phi}$ that determines the magnitude of the displacement jump [8]. The position of a material point in the deformed configuration can then be written as:

$$x = X + \hat{\phi} + \mathcal{H}_{\Gamma_{d,0}} \tilde{\phi}$$ (10)

where $\mathcal{H}_{\Gamma_{d,0}}$ represents the Heaviside step function, defined as

$$\mathcal{H}_{\Gamma_{d,0}}(X) = \begin{cases} 0 & \text{if } X \in \Omega_0^- \\ 1 & \text{if } X \in \Omega_0^+ \end{cases}$$ (11)

Since the displacement field $\phi$ is a function of the variables $u_t$, $u_b$ and $w$, we need to decompose these three terms as

$$u_t = \hat{u}_t + \mathcal{H}_{\Gamma_{d,0}} \tilde{u}_t$$
$$u_b = \hat{u}_b + \mathcal{H}_{\Gamma_{d,0}} \tilde{u}_b$$
$$w = \hat{w} + \mathcal{H}_{\Gamma_{d,0}} \tilde{w}$$ (12)

Figure 2. Thick shell crossed by a discontinuity $\Gamma_{d,0}$ (heavy line). The vectors $\mathbf{n}_{a,0}$ and $\mathbf{u}_{d,0}$ are perpendicular to the shell surface and the discontinuity surface, respectively.
Inserting Equation (12) into Equations (6)–(8) gives

\[
\mathbf{u}_0 = \hat{\mathbf{u}}_0 + H/0 \mathbf{u}_0 \\
\mathbf{u}_1 = \hat{\mathbf{u}}_1 + H/0 \mathbf{u}_1 \\
\mathbf{u}_2 = \hat{\mathbf{u}}_2 + H/0 \mathbf{u}_2
\]

(13)

where

\[
\hat{\mathbf{u}}_0 = \frac{1}{2}[\hat{\mathbf{u}}_t + \hat{\mathbf{u}}_b], \quad \hat{\mathbf{u}}_0 = \frac{1}{2}[\hat{\mathbf{u}}_t + \hat{\mathbf{u}}_b] \\
\hat{\mathbf{u}}_1 = \frac{1}{2}[\hat{\mathbf{u}}_t - \hat{\mathbf{u}}_b], \quad \hat{\mathbf{u}}_1 = \frac{1}{2}[\hat{\mathbf{u}}_t - \hat{\mathbf{u}}_b] \\
\hat{\mathbf{u}}_2 = \hat{\mathbf{w}}[\mathbf{D} + \hat{\mathbf{u}}_1], \quad \hat{\mathbf{u}}_2 = \hat{\mathbf{w}}[\mathbf{D} + \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_1] + \hat{\mathbf{w}}\hat{\mathbf{u}}_1
\]

(14)

Note that the enhanced part of the internal stretch parameter \(\mathbf{u}_2\), i.e. \(\hat{\mathbf{u}}_2\), contains regular variables and additional variables. The base vectors at the material point in undeformed configuration \(\mathbf{G}_i\) can be found by differentiating the position vector \(\mathbf{X}\) with respect to the isoparametric coordinates \(\Theta^i = [\xi, \eta, \zeta]\):

\[
\mathbf{G}_x = \frac{\partial \mathbf{X}}{\partial \Theta^x} = \mathbf{E}_x + \zeta \mathbf{D}_x, \quad x = 1, 2
\]

(15)

\[
\mathbf{G}_3 = \frac{\partial \mathbf{X}}{\partial \Theta^3} = \mathbf{D}
\]

(16)

where \((\cdot)_x\) denotes the partial derivative with respect to \(\Theta^x\). \(\mathbf{E}_x\) is the covariant surface vector, which is the projection of the base-vector \(\mathbf{G}_x\) on the mid-surface and is defined as

\[
\mathbf{E}_x = \frac{\partial \mathbf{X}_0}{\partial \Theta^x}
\]

(17)

The base vectors of the shell in the deformed configuration \(\mathbf{g}_i\) are found in a similar fashion:

\[
\mathbf{g}_x = \frac{\partial \mathbf{X}}{\partial \Theta^x} = \mathbf{E}_x + \hat{\mathbf{u}}_{0,x} + \zeta \mathbf{D}_{x} + \zeta \hat{\mathbf{u}}_{1,x} + H/0[\hat{\mathbf{u}}_{0,x} + \zeta \hat{\mathbf{u}}_{1,x}] + \text{h.o.t.} \quad \forall \zeta \neq \zeta_d
\]

(18)

\[
\mathbf{g}_3 = \frac{\partial \mathbf{X}}{\partial \Theta^3} = \mathbf{D} + \hat{\mathbf{u}}_1 - 2\zeta \hat{\mathbf{u}}_2 + H/0[\hat{\mathbf{u}}_1 - 2\zeta \hat{\mathbf{u}}_2] + \text{h.o.t.} \quad \forall \zeta \neq \zeta_d
\]

(19)

Owing to the displacement jump, the base vectors in the deformed configuration \(\mathbf{g}_i\) are not continuous at the discontinuity \(\Gamma_{d,0}\). The higher order terms (h.o.t.) in Equations (18) and (19) contain terms up to the fourth order in the thickness co-ordinate \(\zeta\) and the derivatives of the stretch parameter \(\mathbf{u}_2\) with respect to \(\xi\) and \(\eta\). In the remainder, these terms will be neglected without a significant loss of accuracy of the kinematic model [14]. The metric tensors \(\mathbf{G}\) and \(\mathbf{g}\) can be determined by using the base vectors \(\mathbf{G}_i\) and \(\mathbf{g}_i\) in Equations (15)–(19):

\[
\mathbf{G}_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j, \quad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j
\]

(20)
Elaboration of the expressions yields the following components of the metric tensor in the undeformed configuration:

\[ G_{\alpha\beta} = E_{\alpha} \cdot E_{\beta} + \zeta [E_{\alpha} \cdot D_{\beta} + E_{\beta} \cdot D_{\alpha}] \]  
(21a)

\[ G_{\alpha3} = E_{\alpha} \cdot D + \zeta D_{\alpha} \cdot D \]  
(21b)

\[ G_{33} = D \cdot D \]  
(21c)

and in the deformed configuration (again neglecting the terms which are quadratic in thickness direction):

\[ g_{\alpha\beta} = (E_{\alpha} + \dot{u}_{0,\alpha} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{0,\alpha}) \cdot (E_{\beta} + \dot{u}_{0,\beta} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{0,\beta}) + \zeta [(E_{\alpha} + \dot{u}_{0,\alpha} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{0,\alpha}) \cdot (D_{\beta} + \dot{u}_{1,\beta} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{1,\beta}) + (E_{\beta} + \dot{u}_{0,\beta} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{0,\beta}) \cdot (D_{\alpha} + \dot{u}_{1,\alpha} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{1,\alpha})] \]  
(22a)

\[ g_{\alpha3} = (E_{\alpha} + \dot{u}_{0,\alpha} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{0,\alpha}) \cdot (D + \dot{u}_{1} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{1}) + \zeta [(D + \dot{u}_{1} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{1}) \cdot (D_{\alpha} + \dot{u}_{1,\alpha} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{1,\alpha}) - 2(E_{\alpha} + \dot{u}_{0,\alpha} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{0,\alpha}) \cdot (\dot{u}_{2} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{2})] \]  
(22b)

\[ g_{33} = (D + \dot{u}_{1} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{1}) \cdot (D + \dot{u}_{1} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{1}) - \zeta 4(D + \dot{u}_{1} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{1}) \cdot (\dot{u}_{2} + \mathcal{K}_{\Gamma_{d,0}} \dot{u}_{2}) \]  
(22c)

We can rewrite these relations as

\[ G_{ij} = G_{0ij}^0 + \zeta G_{ij}^1 \]  
(23)

\[ g_{ij} = g_{0ij}^0 + \zeta g_{ij}^1 \]

where \( G_{ij}^0 \) and \( g_{ij}^0 \) correspond to the constant terms in Equations (21) and (22), whereas \( G_{ij}^1 \) and \( g_{ij}^1 \) correspond to the terms that vary linearly with respect to the thickness \( \zeta \). Apart from the covariant base vectors in the undeformed state, we will need the contravariant counterparts to derive the strain field later on. They can be derived as follows:

\[ G^j = (G_{ij})^{-1} G_i \]  
(24)

The contravariant base vector \( G^j \) is related to the contravariant surface vector \( \tilde{E}^k \) via the so-called shell tensor \( \mu^j_k \) \[16\]:

\[ G^j = \mu^j_k \tilde{E}^k; \quad \mu^j_k = (\delta^j_k - \zeta \tilde{G}^j_k) \]  
(25)

In Equation (25), \( \tilde{G}^j_k \) denotes the mixed variant metric tensor which is calculated with the contravariant and the covariant tensor components as follows, see Equation (23) \[14\]:

\[ \tilde{G}^j_k = G^0 \; ^m G_{1 \; mk} \]  
(26)
4. GREEN–LAGRANGE STRAIN FIELD

The Green–Lagrange strain tensor $\gamma$ is defined conventionally in terms of the deformation gradient $F$:

$$\gamma = \frac{1}{2}(F^TF - I)$$

(27)

The deformation gradient $F$ can be written as a function of the covariant basevector in the deformed configuration $g$, and the contravariant basevector in the undeformed reference configuration $G^i$:

$$F = g_i \otimes G^i$$

(28)

Inserting this relation in Equation (27) we can write the Green–Lagrange strain tensor in terms of the contravariant basis $G^i$:

$$\gamma = \gamma_{ij} G^i \otimes G^j$$

where $\gamma_{ij} = \frac{1}{2}(g_{ij} - G_{ij})$

(29)

Substituting Equation (25) in this relation yields

$$2\gamma = (g_{ij} - G_{ij})(\delta^k_j - \zeta G^k_j)(\delta^l_l - \zeta G^l_l)E^k \otimes E^l$$

(30)

After some manipulations, the strain tensor can be written in terms of the membrane mid-surface strain $\varepsilon_{ij}$ and the bending strain $\rho_{ij}$ according to

$$2\gamma = (\varepsilon_{ij} + \zeta \rho_{ij})E^i \otimes E^j$$

(31)

where

$$2\varepsilon_{2\beta} = E_x \cdot (\hat{u}_{0,\beta} + H_{1,0}\hat{u}_{0,\beta}) + E_x \cdot (\hat{u}_{0,\beta} + H_{1,0}\hat{u}_{0,\beta})$$

$$+ (\hat{u}_{0,\beta} + H_{1,0}\hat{u}_{0,\beta}) \cdot (\hat{u}_{0,\beta} + H_{1,0}\hat{u}_{0,\beta})$$

(32a)

$$2\varepsilon_{33} = 2D \cdot (\hat{u}_{1} + H_{1,0}\hat{u}_{1}) + (\hat{u}_{1} + H_{1,0}\hat{u}_{1}) \cdot (\hat{u}_{1} + H_{1,0}\hat{u}_{1})$$

(32b)

$$2\rho_{2\beta} = E_x \cdot (\hat{u}_{1,\beta} + H_{1,0}\hat{u}_{1,\beta}) + E_x \cdot (\hat{u}_{1,\beta} + H_{1,0}\hat{u}_{1,\beta})$$

$$+ (\hat{u}_{1,\beta} + H_{1,0}\hat{u}_{1,\beta}) \cdot (\hat{u}_{1,\beta} + H_{1,0}\hat{u}_{1,\beta})$$

(32c)

$$- G^\beta [E_x \cdot (\hat{u}_{0,\beta} + H_{1,0}\hat{u}_{0,\beta}) + E_x \cdot (\hat{u}_{0,\beta} + H_{1,0}\hat{u}_{0,\beta})$$

$$+ (\hat{u}_{0,\beta} + H_{1,0}\hat{u}_{0,\beta}) \cdot (\hat{u}_{0,\beta} + H_{1,0}\hat{u}_{0,\beta})]$$

(32d)
Note that the strain fields on either side of the discontinuity $\Gamma_{d,0}$ are not necessarily equal. This implies that it is possible to capture phenomena which are restricted to just one layer of the laminate such as delamination buckling \cite{11,12}.

The strain field is still defined in the isoparametric frame of axes $E^i$. In order to obtain the strains in the element local frame of reference $l_j$, they must be transformed using:

$$ \gamma_{kl} = (e_{ij} + \zeta \rho_{ij})T^i_k T^j_l, \quad T^i_k = E^i_k $$

The magnitude of the displacement jump $\mathbf{v}$ at the internal discontinuity $\Gamma_{d,0}$ is equal to the magnitude of the additional displacement field at the discontinuity $\zeta$. In the spirit of previous assumptions, we neglect the terms that vary quadratically in the thickness direction, so that

$$ \mathbf{v} = \tilde{\mathbf{u}}_0 + \zeta \mathbf{u}_1 $$

The displacement jump does not have to be transformed, since it is already defined in the element local frame of reference.

5. WEAK FORM OF THE EQUILIBRIUM EQUATIONS

The static equilibrium equations and boundary conditions for the body $\Omega$ without body forces with respect to the undeformed configuration can be written as:

$$ \nabla_0 \cdot \mathbf{P} = 0 \quad \text{in} \quad \Omega_0 $$

$$ \mathbf{n}_{u,0} \cdot \mathbf{P} = \bar{\mathbf{f}} \quad \text{on} \quad \Gamma_{u,0} $$

$$ \mathbf{n}_{d,0} \cdot \mathbf{P} = \mathbf{t} \quad \text{on} \quad \Gamma_{d,0} $$

where $\mathbf{P}$ is the nominal stress tensor, $\bar{\mathbf{f}}$ the applied external load in the reference configuration and $\mathbf{n}_{u,0}$ is the outward unit normal vector to the body; $\mathbf{t}$ is the traction at discontinuity $\Gamma_{d,0}$ with $\mathbf{n}_{d,0}$ the inward unit normal to $\Omega_0^+$, see also Figure 2. Equation (35c) represents the tractions at the discontinuity $\Gamma_{d,0}$, which can be conceived as an internal boundary. The strong governing equations can be written as the weak equations of equilibrium by multiplying Equation (35a) by an admissible displacement variation $\delta \mathbf{\phi}$ and integrating the result over the domain $\Omega_0$:

$$ \int_{\Omega_0} \delta \mathbf{\phi} \cdot (\nabla_0 \cdot \mathbf{P}) \, d\Omega_0 = 0 $$

which must hold for all admissible variation of the displacement field $\delta \mathbf{\phi}$. Following a standard Bubnov–Galerkin approach, the space of admissible displacement variations is taken the same
as the field of actual displacements. Referring to Equation (10), we therefore write
\[ \delta \phi = \delta \hat{\phi} + \mathcal{H}_{\Gamma_0} \delta \tilde{\phi} \] (37)
Substituting this relation into Equation (36) gives
\[ \int_{\Omega_0} \delta \hat{\phi} \cdot (\nabla_0 \cdot \mathbf{P}) \, d\Omega_0 + \int_{\Omega_0} \mathcal{H}_{\Gamma_0} \delta \tilde{\phi} \cdot (\nabla_0 \cdot \mathbf{P}) \, d\Omega_0 = 0 \] (38)
which must hold for all admissible variations \( \delta \hat{\phi} \) and \( \delta \tilde{\phi} \). The equation can be expanded using Gauss’ theorem. The Heaviside function can be eliminated by changing the integration domain \( \Omega_0 \) into \( \Omega_0^+ \) [8]:
\[ \int_{\Omega_0} \nabla_0 \cdot (\mathbf{P} \cdot \delta \hat{\phi}) \, d\Omega_0 - \int_{\Omega_0} \nabla_0 \delta \hat{\phi} : \mathbf{P} \, d\Omega_0 + \int_{\Omega_0^+} \nabla_0 \cdot (\mathbf{P} \cdot \delta \tilde{\phi}) \, d\Omega_0 - \int_{\Omega_0} \nabla_0 \delta \tilde{\phi} : \mathbf{P} \, d\Omega_0 = 0 \] (39)
In this equation, \( \nabla_0 \delta \hat{\phi} \) and \( \nabla_0 \delta \tilde{\phi} \) are the deformation gradients of the regular and the additional parts of the admissible displacement variations [11]. Applying the conditions at the boundary of the domain, Equation (35b), and the condition at the discontinuity, Equation (35c), the above equation can be written as
\[ \int_{\Omega_0} \nabla_0 \delta \hat{\phi} : \mathbf{P} \, d\Omega_0 + \int_{\Omega_0^+} \nabla_0 \delta \tilde{\phi} : \mathbf{P} \, d\Omega_0 + \int_{\Gamma_{u,0}} \delta \vec{v} \cdot \mathbf{t} \, d\Gamma_0 = \int_{\Gamma_{u,0}} \delta \hat{\phi} \cdot \mathbf{t} \, d\Gamma_0 + \int_{\Gamma_{u,0}} \mathcal{H}_{\Gamma_0} \delta \tilde{\phi} \cdot \mathbf{t} \, d\Gamma_0 \] (40)
The terms \( \nabla_0 \delta \hat{\phi} : \mathbf{P} \) and \( \nabla_0 \delta \tilde{\phi} : \mathbf{P} \) can be replaced by the work-conjugate terms \( \delta \vec{\gamma} : \mathbf{\sigma} \) and \( \delta \hat{\gamma} : \mathbf{\sigma} \) [17], where \( \delta \vec{\gamma} \) and \( \delta \hat{\gamma} \) are the regular and additional parts of the variation of the Green–Lagrange strain field and \( \mathbf{\sigma} \) is the second Piola–Kirchhoff stress tensor:
\[ \int_{\Omega_0} \delta \vec{\gamma} : \mathbf{\sigma} \, d\Omega_0 + \int_{\Omega_0^+} \delta \hat{\gamma} : \mathbf{\sigma} \, d\Omega_0 + \int_{\Gamma_{u,0}} \delta \vec{\gamma} \cdot \mathbf{t} \, d\Gamma_0 = \int_{\Gamma_{u,0}} \delta \hat{\gamma} \cdot \mathbf{t} \, d\Gamma_0 + \int_{\Gamma_{u,0}} \mathcal{H}_{\Gamma_0} \delta \tilde{\gamma} \cdot \mathbf{t} \, d\Gamma_0 \] (41)
For completeness, the variation of the strain field \( \delta \gamma \) is now given as a function of the displacement components:
\[ 2 \delta e_{2\beta} = \mathbf{e}_{\beta} \cdot (\delta \hat{\mathbf{u}}_{\beta x} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{\beta x}) + \mathbf{e}_{x} \cdot (\delta \hat{\mathbf{u}}_{0\beta} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{0\beta}) \] (42a)
\[ 2 \delta e_{33} = \mathbf{d} \cdot (\delta \hat{\mathbf{u}}_{0x} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{0x}) + \mathbf{e}_{x} \cdot (\delta \hat{\mathbf{u}}_{1} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{1}) \] (42b)
\[ 2 \delta e_{33} = 2 \mathbf{d} \cdot (\delta \hat{\mathbf{u}}_{1} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{1}) \] (42c)
\[ 2 \delta \rho_{2\beta} = \mathbf{e}_{\beta} \cdot (\delta \hat{\mathbf{u}}_{1\beta} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{1\beta}) + \mathbf{e}_{x} \cdot (\delta \hat{\mathbf{u}}_{0\beta} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{0\beta}) \]
\[ + \mathbf{d} \cdot (\delta \hat{\mathbf{u}}_{0x} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{0x}) + \mathbf{d}_{\beta} \cdot (\delta \hat{\mathbf{u}}_{0\beta} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{0\beta}) \]
\[ - \bar{G}_{\alpha} \mathbf{e}_{\beta} \cdot (\delta \hat{\mathbf{u}}_{\alpha\beta} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{\alpha\beta}) + \mathbf{e}_{x} \cdot (\delta \hat{\mathbf{u}}_{0\beta} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{0\beta}) \]
\[ - \bar{G}_{\beta} \mathbf{e}_{x} \cdot (\delta \hat{\mathbf{u}}_{\alpha x} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{\alpha x}) + \mathbf{e}_{x} \cdot (\delta \hat{\mathbf{u}}_{0\alpha} + \mathcal{H}_{\Gamma_0} \delta \tilde{\mathbf{u}}_{0\alpha}) \] (42d)
In an incremental iterative solution scheme, the strain increment $\Delta \gamma$ is normally needed. It can be derived as

$$
2\delta \rho_{x3} = d \cdot (\delta \hat{u}_{1,x} + \mathcal{H}_{d,0} \delta \hat{u}_{1,z}) + d_{x} \cdot (\delta \hat{u}_{1} + \mathcal{H}_{d,0} \delta \hat{u}_{1})
$$

(42c)

$$
2\delta \rho_{33} = -8u_{2} \cdot (\delta \hat{u}_{1} + \mathcal{H}_{d,0} \delta \hat{u}_{1}) - 4d \cdot d (\delta \hat{w} + \mathcal{H}_{d,0} \delta \hat{w})
$$

(42f)

where

$$
e_{x} = E_{x} + \hat{u}_{0,x} + \mathcal{H}_{d,0} \hat{u}_{0,z}, \quad d = D + \hat{u}_{1} + \mathcal{H}_{d,0} \hat{u}_{1}
$$

(43)

$$
d_{x} = D_{x} + \hat{u}_{1,z} + \mathcal{H}_{d,0} \hat{u}_{1,z}, \quad u_{2} = \hat{u}_{2} + \mathcal{H}_{d,0} \hat{u}_{2}
$$

In an incremental iterative solution scheme, the strain increment $\Delta \gamma$ is normally needed. It can be derived as

$$
2\Delta e_{\beta} = e_{x} \cdot \Delta u_{0,\beta} + e_{\beta} \cdot \Delta u_{0,x} + \Delta u_{0,z} \cdot \Delta u_{0,\beta}
$$

(44a)

$$
2\Delta e_{x} = e_{x} \cdot \Delta u_{1} + d \cdot \Delta u_{0,x} + \Delta u_{0,z} \cdot \Delta u_{1}
$$

(44b)

$$
2\Delta e_{33} = 2d \cdot \Delta u_{1} + 2\Delta u_{1} \cdot \Delta u_{1}
$$

(44c)

$$
2\Delta e_{\beta} = e_{\beta} \cdot \Delta u_{1,x} + d_{x} \cdot \Delta u_{0,\beta} + \Delta u_{1,x} \cdot \Delta u_{0,\beta}
$$

(44d)

$$
2\Delta e_{3} = d_{x} \cdot \Delta u_{1} + d \cdot \Delta u_{1,x} + \Delta u_{1,x} \cdot \Delta u_{1}
$$

(44e)

$$
2\Delta e_{33} = -4d \cdot \Delta u_{1} - 2w \Delta u_{1} \cdot u_{1} - 2\Delta w d \cdot d
$$

(44f)

where $\Delta u_{1}, \Delta u_{1,x}, \Delta u_{1,z}, \Delta d_{x}$ and $\Delta w$ are defined in the spirit of Equation (43):

$$
\Delta u_{1} = \Delta \hat{u}_{1} + \mathcal{H}_{d,0} \Delta \hat{u}_{1}, \quad \Delta u_{0,x} = \Delta \hat{u}_{0,x} + \mathcal{H}_{d,0} \Delta \hat{u}_{0,x}
$$

$$
\Delta u_{1,x} = \Delta \hat{u}_{1,x} + \mathcal{H}_{d,0} \Delta \hat{u}_{1,x}, \quad \Delta d_{x} = \Delta \hat{d}_{x} + \mathcal{H}_{d,0} \Delta \hat{d}_{x}
$$

$$
\Delta w = \Delta \hat{w} + \mathcal{H}_{d,0} \Delta \hat{w}
$$

(45)

6. FINITE ELEMENT IMPLEMENTATION

The solid-like shell element can be modelled as an eight-node or as a sixteen-node element, see Figure 3. In both cases, a linear distribution of the internal stretch is assumed, so that only four internal degrees of freedom (situated at the four corners of the element mid-surface) are necessary [14]. Each geometrical node $i$ contains three degrees of freedom $a_{i} = [a_{x}, a_{y}, a_{z}]$ that set up the regular displacement field $\hat{u}$ and a set of three additional degrees of freedom $b_{i} = [b_{x}, b_{y}, b_{z}]$ that describes the enhanced displacement
Figure 3. Geometry and location of the nodes for the sixteen-node solid-like shell element. Each geometrical node contains a regular set of three degrees of freedom \([a_x, a_y, a_z]_i\) and an additional set of three degrees of freedom \([b_x, b_y, b_z]_i\). The internal nodes contain one regular and one additional degree-of-freedom \((p_j\) and \(q_j,\) respectively).

field \(\mathbf{\bar{u}}\). Likewise, there are two sets of internal degrees of freedom: \(p_j\) that constructs the regular stretch term \(\mathbf{\hat{w}}\) and \(q_j\) that constructs the additional stretch term \(\mathbf{\tilde{w}}\).

It was shown by Babu\'\v{s}ka et al. [5] that the basis of the finite element shape functions can be enriched with enhanced bases by virtue of the fact that the finite element shape functions \(\Psi_i\) form a partition of unity. Hence, a field \(\mathbf{u}\) can be interpolated in terms of nodal values according to:

\[
\mathbf{u}(\mathbf{X}, t) = \sum_{i=1}^{n} \Psi_i(\mathbf{X}) \left( a_i(t) + \sum_{k=1}^{m} \beta_{ik}(\mathbf{X}) b_{ik}(t) \right) \tag{46}
\]

In this specific case, the function \(\beta_{ik}\) can be replaced by the Heaviside step function \(\mathcal{H}_{\Gamma_{d,0}}\).

The projected displacements at the bottom and top surfaces of the element \(\mathbf{u}_b\) and \(\mathbf{u}_t\) can be constructed using the \(N\) bottom and \(N\) top nodes of the element respectively (consequently, the number of nodes in one element equals \(2N\)). For example, the sixteen-node element has eight bottom and eight top nodes. Hence, the continuous value \(\mathbf{u}_b\) can be constructed from the eight bottom nodes, using the eight quadratic shape functions for a quadrilateral element \(\Psi_i\). In the eight-node element, this value can be constructed using the four bottom nodes and the four linear isoparametric shape functions for a quadrilateral element. Since the internal stretch \(w\) is constructed with four internal degrees of freedom in both elements, we use the four linear shape function \(\Phi_j\). The projection of the displacement vector can be constructed by interpolating the regular and enhanced degrees of freedom at the bottom nodes of the element \((i = 1, N)\):

\[
\mathbf{u}_b = \mathbf{\bar{u}}_b + \mathcal{H}_{\Gamma_{d,0}} \mathbf{\bar{u}}_b = \sum_{i=1}^{N} \Psi_i a_i + \mathcal{H}_{\Gamma_{d,0}} \sum_{i=1}^{N} \Psi_i b_i \tag{47}
\]

where \(\Psi\) represents the conventional matrix which contains the element shape functions for node \(i\) in three directions \(x, y, z\):

\[
\Psi_i = \Psi_i I_3 \tag{48}
\]
where $\mathbf{I}_3$ is the identity matrix. The projected displacements on the top surface $\mathbf{u}_t$ can be interpolated using the degrees of freedom at the top nodes of the element (numbered $i = N + 1, N + 2, \ldots, 2N$) and shape functions $\Psi_i$:

$$
\mathbf{u}_t = \hat{\mathbf{u}}_t + \mathcal{H}_{T,0} \hat{\mathbf{u}}_t = \sum_{i=1}^{N} \Psi_i a_{i+N} + \mathcal{H}_{T,0} \sum_{i=1}^{N} \Psi_i b_{i+N}
$$

(49)

Similarly, the internal degrees of freedom are calculated using the regular and additional internal degrees of freedom $p_j$ and $q_j$, respectively:

$$
\mathbf{w} = \hat{\mathbf{w}} + \mathcal{H}_{T,0} \hat{\mathbf{w}} = \sum_{j=1}^{4} \Phi_j p_j + \mathcal{H}_{T,0} \sum_{j=1}^{4} \Phi_j q_j
$$

(50)

The discretized displacement of the shell mid-surface and the displacement of the thickness director, $\mathbf{u}_0$ and $\mathbf{u}_1$ respectively, can be found using relations (14), (47) and (49):

$$
\hat{\mathbf{u}}_0 = \mathbf{N}_0 a, \quad \hat{\mathbf{u}}_0 = \mathbf{N}_0 b
$$

(51)

$$
\hat{\mathbf{u}}_1 = \mathbf{N}_1 a, \quad \hat{\mathbf{u}}_1 = \mathbf{N}_1 b
$$

(52)

where $\mathbf{N}_0$ and $\mathbf{N}_1$ are defined as

$$
\mathbf{N}_0 = \frac{1}{2}[\Psi_1, \Psi_2, \ldots, \Psi_N, \Psi_1, \Psi_2, \ldots, \Psi_N]^T
$$

(53)

$$
\mathbf{N}_1 = \frac{1}{2}[-\Psi_1, -\Psi_2, \ldots, -\Psi_N, \Psi_1, \Psi_2, \ldots, \Psi_N]^T
$$

(54)

From Equation (50) we define the interpolation matrix for the internal degrees of freedom:

$$
\mathbf{N}_w = [\Phi_1, \Phi_2, \Phi_3, \Phi_4]^T
$$

(55)

Trivially, the derivatives of the displacement vectors $\mathbf{u}_0$ and $\mathbf{u}_1$ with respect to the isoparametric coordinates $\xi$ and $\eta$ can be written as

$$
\hat{\mathbf{u}}_{0,x} = \mathbf{N}_{0,x} a, \quad \hat{\mathbf{u}}_{0,x} = \mathbf{N}_{0,x} b
$$

(56)

$$
\hat{\mathbf{u}}_{1,x} = \mathbf{N}_{1,x} a, \quad \hat{\mathbf{u}}_{1,x} = \mathbf{N}_{1,x} b
$$

(57)

where the matrices $\mathbf{N}_{0,x}$ and $\mathbf{N}_{1,x}$ contain the derivatives of the shape functions

$$
\mathbf{N}_{0,x} = \frac{1}{2} [\Psi_{1,x}, \Psi_{2,x}, \ldots, \Psi_{N,x}, \Psi_{1,x}, \Psi_{2,x}, \ldots, \Psi_{N,x}]^T
$$

(58)

$$
\mathbf{N}_{1,x} = \frac{1}{2} [-\Psi_{1,x}, -\Psi_{2,x}, \ldots, -\Psi_{N,x}, \Psi_{1,x}, \Psi_{2,x}, \ldots, \Psi_{N,x}]^T
$$

(59)

with:

$$
\Psi_{i,x} = \frac{\partial \Psi_i}{\partial x} \mathbf{I}_3
$$

(60)

There are three kinds of admissible variations that need to be discretised, namely those of the Green–Lagrange strain tensor $\delta \mathbf{\hat{\mathbf{S}}}$ and $\delta \mathbf{\hat{\mathbf{S}}}$, that of the displacement jump $\delta \mathbf{\hat{\mathbf{v}}}$, and that of the
displacement field itself, $\delta \hat{\phi}$ and $\delta \tilde{\phi}$. The variations of the Green–Lagrange strain field can be written as

$$\delta \hat{\gamma} = B_u \delta a + B_m \delta p$$  \hspace{1cm} (61)$$

$$\delta \tilde{\gamma} = B_u \delta b + B_m \delta q$$  \hspace{1cm} (62)

with $B_u$ and $B_m$ as defined in Appendix A. From Equation (34) one obtains the discrete variation displacement jump

$$\delta \tilde{\nu} = N_0 \delta b + \tilde{\zeta} N_1 \delta b$$  \hspace{1cm} (63)

Since the load can only be applied to the geometrical nodes [14], the admissible displacement terms in the external load parts of the equilibrium equations can be replaced by a modified admissible displacement field $\delta \phi^*$ which is defined as

$$\delta \phi^* = \delta \hat{\phi}^* + \delta \tilde{\phi}^* = \delta \hat{u}_0 + \tilde{\zeta} \delta \hat{u}_1 + \mathcal{H}_{T,d,0} (\delta \hat{u}_0 + \tilde{\zeta} \delta \hat{u}_1)$$  \hspace{1cm} (64)

or in discrete form

$$\delta \tilde{\phi}^* = (N_0 + \tilde{\zeta} N_1) \delta a, \quad \delta \hat{\phi}^* = (N_0 + \tilde{\zeta} N_1) \delta b$$  \hspace{1cm} (65)

Substituting Equations (61), (63) and (65) into the equilibrium equation (41) gives

$$\int_{\Omega_0} (B_u \delta a + B_m \delta p)^T \sigma \, d\Omega_0 + \int_{\Omega_0} (B_u \delta b + B_m \delta q)^T \sigma \, d\Omega_0 + \int_{\Gamma_{\text{d},0}} ((N_0 + \tilde{\zeta} d N_1) \delta b)^T t \, d\Gamma_0 = \int_{\Gamma_{u,0}} ((N_0 + \tilde{\zeta} d N_1) \delta a)^T t \, d\Gamma_0 + \int_{\Gamma_{\text{d},0}} \mathcal{H}_{T,d,0} ((N_0 + \tilde{\zeta} N_1) \delta b)^T t \, d\Gamma_0$$  \hspace{1cm} (66)

By taking the variation of one admissible variation $\delta a$, $\delta b$, $\delta p$ or $\delta q$ at the time and setting the others to zero, the following four systems of equations can be derived:

$$\int_{\Omega_0} B_u^T \sigma \, d\Omega_0 = \int_{\Gamma_{u,0}} (N_0 + \tilde{\zeta} N_1)^T t \, d\Gamma_0$$  \hspace{1cm} (67a)

$$\int_{\Omega_0} B_m^T \sigma \, d\Omega_0 + \int_{\Gamma_{\text{d},0}} (N_0 + \tilde{\zeta} d N_1)^T t \, d\Omega_0 = \int_{\Gamma_{u,0}} \mathcal{H}_{T,d,0} (N_0 + \tilde{\zeta} N_1)^T t \, d\Gamma_0$$  \hspace{1cm} (67b)

$$\int_{\Omega_0} B_m^T \sigma \, d\Omega_0 = 0$$  \hspace{1cm} (67c)

$$\int_{\Omega_0} B_m^T \sigma \, d\Omega_0 = 0$$  \hspace{1cm} (67d)

The left-hand sides of these equations constitute the internal force vectors:

$$\mathbf{f}_{\text{int}} = \int_{\Omega_0} B_u^T \sigma \, d\Omega_0$$  \hspace{1cm} (68a)
\[ f_{b}^{\text{int}} = \int_{\Omega_0} B_{b}^{T} \sigma \, d\Omega_0 + \int_{\Gamma_{d,0}} (N_0 + \zeta N_1)^T t \, d\Omega_0 \] (68b)

\[ f_{p}^{\text{int}} = \int_{\Omega_0} B_{p}^{T} \sigma \, d\Omega_0 \] (68c)

\[ f_{q}^{\text{int}} = \int_{\Omega_0} B_{q}^{T} \sigma \, d\Omega_0 \] (68d)

while the right-hand sides represent the external force vectors:

\[ f_{a}^{\text{ext}} = \int_{\Gamma_{u}} (N_0 + \zeta N_1)^T t \, d\Omega_0 \] (69a)

\[ f_{b}^{\text{ext}} = \int_{\Gamma_{u}} \mathcal{H}_{\Gamma_{d,0}} (N_0 + \zeta N_1)^T t \, d\Omega_0 \] (69b)

\[ f_{p}^{\text{ext}} = 0 \] (69c)

\[ f_{q}^{\text{ext}} = 0 \] (69d)

### 7. CONSTITUTIVE RELATIONS

For the bulk material and for the interface standard constitutive relations can be used. At this stage, the treatment is restricted to small strains, but still allows for arbitrarily large displacement gradients. Consequently, a linear relation can be postulated between the increments of the second Piola–Kirchhoff stress \( \Delta \sigma \), and the Green–Lagrange strain, \( \Delta \gamma \), see Equation (44):

\[ \Delta \sigma = D \Delta \gamma \] (70)

where \( D \) is the tangent stiffness matrix of the bulk material.

For the discontinuity, the constitutive relation reads:

\[ \Delta t_{\text{dis}} = C_{\text{dis}} \Delta v_{\text{dis}} \] (71)

where \( \Delta v_{\text{dis}} \) is the incremental displacement jump at the discontinuity as defined in the current orientation of the discontinuity, as denoted by the subscript ‘dis’. Transformation into the element local frame of reference in the undeformed configuration gives:

\[ \Delta t = Q^T \Delta t_{\text{dis}} = Q^T C \Delta v_{\text{dis}} = Q^T C Q \Delta v \] (72)

so that in the undeformed configuration, the tangent stiffness matrix is equal to

\[ C = Q^T C_{\text{dis}} Q \] (73)

The orthogonal transformation matrix \( Q \) performs the transformation of the orientation of the discontinuity in the current configuration into the orientation in undeformed configuration and can be composed using the base vectors in the deformed configuration, Equations (18)
and (19). However, since the triads \( g_i \) at both sides of the discontinuity are normally different, see Figure 4, it is not possible to assign a unique direction for the discontinuity in the deformed position. Therefore, the average direction is taken [11]. The base vectors in deformed position are then equal to

\[
\begin{align*}
\hat{g}_{3,\text{dis}} &= \hat{g}_3 + \frac{1}{2} \tilde{g}_3 = D + \hat{u}_1 + \frac{1}{2} \tilde{u}_1 - 2\zeta \hat{u}_2 - \zeta \tilde{u}_2 \\
\end{align*}
\]

(74b)

which can be used to form the transformation matrix \( Q \) in a standard manner [18]:

\[
Q_{ij} = \cos (g_{i,\text{dis}}, G_j)
\]  

(75)

### 8. LINEARIZATION OF THE GOVERNING EQUATIONS

In order to solve the equations in a Newton–Raphson process, a stiffness matrix is needed:

\[
K \Delta u = f^{\text{ext}} - f^{\text{int}}
\]  

(76)

The stiffness matrix can be found by differentiating the internal force vector, Equation (68). Since both the stresses \( \sigma \) and the geometric matrices \( \mathbf{B}_u \) and \( \mathbf{B}_w \) are a function of the displacements, the stiffness matrix can be decomposed in two terms, the so-called material part and the geometric part:

\[
K = K^{\text{mat}} + K^{\text{geo}}
\]  

(77)

Since

\[
\frac{\partial \sigma}{\partial a} = \frac{\partial \sigma}{\partial \mathbf{B}_u} = \mathbf{DB}_u, \quad \frac{\partial \sigma}{\partial \mathbf{p}} = \frac{\partial \sigma}{\partial \mathbf{q}} = \mathbf{DB}_w
\]  

(78)
we obtain for the material part of the stiffness matrix of the bulk material:

\[
K_{\text{mat}}^{\text{bulk}} = \begin{bmatrix}
\int_{\Omega_0} B_u^T D B_u d\Omega & \int_{\Omega_0^+} B_u^T D B_u d\Omega & \int_{\Omega_0} B_u^T D B_w d\Omega & \int_{\Omega_0^+} B_u^T D B_w d\Omega \\
\int_{\Omega_0} B_w^T D B_u d\Omega & \int_{\Omega_0^+} B_w^T D B_u d\Omega & \int_{\Omega_0} B_w^T D B_w d\Omega & \int_{\Omega_0^+} B_w^T D B_w d\Omega \\
\int_{\Omega_0^+} B_u^T D B_u d\Omega & \int_{\Omega_0^+} B_w^T D B_u d\Omega & \int_{\Omega_0^+} B_u^T D B_w d\Omega & \int_{\Omega_0^+} B_w^T D B_w d\Omega \\
\int_{\Omega_0^+} B_w^T D B_u d\Omega & \int_{\Omega_0^+} B_w^T D B_u d\Omega & \int_{\Omega_0^+} B_w^T D B_w d\Omega & \int_{\Omega_0^+} B_w^T D B_w d\Omega 
\end{bmatrix}
\]

For the discontinuity, the following relation holds:

\[
\frac{\partial \psi}{\partial b} = Q^T C_{\text{dis}} Q (N_0 + \zeta_d N_1)
\]

Substitution there gives the contribution of the discontinuity to the material part of the stiffness matrix:

\[
K_{\text{dis}}^{\text{mat}} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \int_{\Gamma_d,0} (N_0 + \zeta_d N_1)^T Q^T C_{\text{dis}} Q (N_0 + \zeta_d N_1) d\Gamma & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix}
\]

The total material part of the stiffness matrix is formed by their sum; \( K^{\text{mat}} = K^{\text{mat}}_{\text{bulk}} + K^{\text{mat}}_{\text{dis}} \).

The geometric part of the stiffness matrix is found by differentiating the incremental change of the variational strains, Equation (42) with respect to the displacements:

\[
2d(\delta e_{\alpha\beta}) = (\delta u_{0,\alpha} + \mathcal{H}_{\Gamma_d,0} \delta u_{0,\alpha}) \cdot (d\dot{u}_{0,\beta} + \mathcal{H}_{\Gamma_d,0} d\dot{u}_{0,\beta}) + (\delta u_{0,\beta} + \mathcal{H}_{\Gamma_d,0} \delta u_{0,\beta}) \cdot (d\dot{u}_{0,\alpha} + \mathcal{H}_{\Gamma_d,0} d\dot{u}_{0,\alpha})
\]

\[
2d(\delta e_{\alpha\beta}) = (\delta u_{0,\alpha} + \mathcal{H}_{\Gamma_d,0} \delta u_{0,\alpha}) \cdot (d\dot{u}_{1} + \mathcal{H}_{\Gamma_d,0} d\dot{u}_{1}) + (\delta u_{0,\beta} + \mathcal{H}_{\Gamma_d,0} \delta u_{0,\beta}) \cdot (d\dot{u}_{0,\alpha} + \mathcal{H}_{\Gamma_d,0} d\dot{u}_{0,\alpha})
\]

\[
2d(\delta e_{\alpha\beta}) = 2(\delta u_{1} + \mathcal{H}_{\Gamma_d,0} \delta u_{1}) \cdot (d\dot{u}_{1} + \mathcal{H}_{\Gamma_d,0} d\dot{u}_{1})
\]

\[
2d(\delta e_{\alpha\beta}) = (\delta u_{1,\alpha} + \mathcal{H}_{\Gamma_d,0} \delta u_{1,\alpha}) \cdot (d\dot{u}_{0,\beta} + \mathcal{H}_{\Gamma_d,0} d\dot{u}_{0,\beta}) + (\delta u_{1,\beta} + \mathcal{H}_{\Gamma_d,0} \delta u_{1,\beta}) \cdot (d\dot{u}_{0,\alpha} + \mathcal{H}_{\Gamma_d,0} d\dot{u}_{0,\alpha}) + (\delta u_{0,\alpha} + \mathcal{H}_{\Gamma_d,0} \delta u_{0,\alpha}) \cdot (d\dot{u}_{1,\beta} + \mathcal{H}_{\Gamma_d,0} d\dot{u}_{1,\beta})
\]
where
\[ A \]

yield the geometric part of the stiffness matrix:

The incremental change of the virtual strains must be multiplied with the current stresses to yield the geometric part of the stiffness matrix:

\[
K_{\text{geo}} = \begin{bmatrix}
\int_{\Omega_0^+} A_{uu} \, d\Omega & \int_{\Omega_0^+} A_{uu} \, d\Omega & \int_{\Omega_0^+} A_{uw} \, d\Omega & \int_{\Omega_0^+} A_{uw} \, d\Omega \\
\int_{\Omega_0^+} A_{uu} \, d\Omega & \int_{\Omega_0^+} A_{uu} \, d\Omega & \int_{\Omega_0^+} A_{uw} \, d\Omega & \int_{\Omega_0^+} A_{uw} \, d\Omega \\
\int_{\Omega_0^+} A_{uw} \, d\Omega & \int_{\Omega_0^+} A_{uw} \, d\Omega & 0 & 0 \\
\int_{\Omega_0^+} A_{uw} \, d\Omega & \int_{\Omega_0^+} A_{uw} \, d\Omega & 0 & 0
\end{bmatrix}
\]

where \( A_{uu}, A_{uw} \) and \( A_{wu} \) are defined in Appendix B. Consistent linearisation of the surface integrals related to the tractions at the discontinuity, Equation (68b) will result in additional, non-symmetric, terms in the geometric part of the stiffness matrix due to the transformation matrix \( Q \), Equation (72). However, since these contributions are relatively small, they will have only a minor impact the iteration process [11]. Owing to the additional computational effort required to solve a non-symmetric system, these terms will be neglected.

9. IMPLEMENTATION ASPECTS

9.1. Enhancement of geometrical and internal nodes

The magnitude of the displacement jump at the discontinuity is governed by an additional set of degrees of freedom which are added to the existing nodes of the model. Figure 5 shows

the activation of these additional sets of degrees of freedom for a given (static) delamination surface in the model. Both the geometrical and the internal nodes are enhanced when the corresponding element is crossed by the discontinuity. This implies that each geometrical node now contains three additional degrees of freedom next to the three regular ones, giving six degrees of freedom in total. Each internal node has one extra degree of freedom added to the single regular degree of freedom. The discontinuity always stretches through an entire element. This avoids the need for complicated algorithms to describe the stress state in the vicinity of a delamination front within an element. As a consequence, the discontinuity ‘touches’ the boundary of an element. The geometrical and internal nodes that support this boundary are not enhanced in order to assure a zero crack tip condition [8].

At variance with conventional interface elements, a criterion is needed for the placement of the discontinuity upon propagation. This criterion is based on the stress state at the delamination front, which can be monitored by adding temporary sample points. When the criterion exceeds a threshold value, the discontinuity is extended into the new element. The corresponding nodes of this element are enhanced with an additional set of degrees of freedom.

9.2. Condensation of the internal degrees of freedom

Since the internal degrees of freedom are not able to support an external loading, it was suggested by Parisch to eliminate them on element level by condensation [14]. The complete set of the linearised equilibrium equations for a single element can be written as

\[
\begin{pmatrix}
K_{aa} & K_{ab} & K_{ap} & K_{aq} \\
K_{ba} & K_{bb} & K_{bp} & K_{bq} \\
K_{pa} & K_{pb} & K_{pp} & K_{pq} \\
K_{qa} & K_{qb} & K_{qp} & K_{qq}
\end{pmatrix}
\begin{pmatrix}
da \\
db \\
dp \\
dq
\end{pmatrix}
= \begin{pmatrix}
f_{\text{ext}a} \\
f_{\text{ext}b} \\
0 \\
0
\end{pmatrix}
- \begin{pmatrix}
f_{\text{int}a} \\
f_{\text{int}b} \\
f_{\text{int}p} \\
f_{\text{int}q}
\end{pmatrix}
\] (84)

where \(da\) etc. denote the incremental degrees of freedom. Since the applied force at the internal degrees of freedom are equal to zero, we can write the incremental internal degrees
of freedom $d\mathbf{p}$ and $d\mathbf{q}$ as follows:

$$
\begin{bmatrix}
    d\mathbf{p} \\
    d\mathbf{q}
\end{bmatrix} = - \begin{bmatrix}
    K_{pp} & K_{pq} \\
    K_{qp} & K_{qq}
\end{bmatrix}^{-1} \begin{bmatrix}
    K_{pa} & K_{pb} \\
    K_{qa} & K_{qb}
\end{bmatrix}
\begin{bmatrix}
    d\mathbf{a} \\
    d\mathbf{b}
\end{bmatrix}
+ \begin{bmatrix}
    f_{\text{int}}^a \\
    f_{\text{int}}^b
\end{bmatrix}
$$

(85)

The internal degrees of freedom can be eliminated by substituting this relation into Equation (84). Further derivation yields the condensed stiffness matrix:

$$
\tilde{\mathbf{K}} = \begin{bmatrix}
    K_{aa} & K_{ab} \\
    K_{ba} & K_{bb}
\end{bmatrix} - \begin{bmatrix}
    K_{ap} & K_{aq} \\
    K_{bp} & K_{bq}
\end{bmatrix} \begin{bmatrix}
    K_{pp} & K_{pq} \\
    K_{qp} & K_{qq}
\end{bmatrix}^{-1} \begin{bmatrix}
    K_{pa} & K_{pb} \\
    K_{qa} & K_{qb}
\end{bmatrix}
$$

(86)

and the condensed internal forces vector

$$
\tilde{\mathbf{f}}_{\text{int}} = \begin{bmatrix}
    f_{\text{int}}^a \\
    f_{\text{int}}^b
\end{bmatrix} - \begin{bmatrix}
    K_{ap} & K_{aq} \\
    K_{bp} & K_{bq}
\end{bmatrix} \begin{bmatrix}
    K_{pp} & K_{pq} \\
    K_{qp} & K_{qq}
\end{bmatrix}^{-1} \begin{bmatrix}
    f_{\text{int}}^p \\
    f_{\text{int}}^q
\end{bmatrix}
$$

(87)

It is emphasized that the additional degrees of freedom at geometrical nodes that describe the displacement jump cannot be condensed. The magnitude of the displacement jump must be continuous across element boundaries. The degrees of freedom that describe this jump are therefore global and cannot be solved on the element local level. An exception is made for the additional internal degrees of freedom $q$. Since the regular internal degrees of freedom $p$ are already discontinuous across element boundaries, there is no need for the additional internal degrees of freedom to be continuous. In practice, an enhanced eight-node solid-like shell element has a total of 56 degrees of freedom; two times three in each geometrical node plus two times four internal degrees of freedom. These last two sets are eliminated by condensation reducing the contribution of the element to the global solution vector to 48 degrees of freedom.

### 9.3. Shear locking

It was observed by Parisch [14] that the eight-node element suffers from the same shear locking phenomenon as general four-node shell elements. A remedy to this problem is the assumed natural strain approach [19]. Here, the transverse shear strains $\varepsilon_{x3}$ in Equation (32b) are evaluated at four alternative positions, the tying points A–D in Figure 6 along the edges

![Figure 6. Location of tying points in eight-node solid-like shell element.](image_url)
of the elements:

\[
\begin{align*}
\varepsilon_{23} &= \chi_C \varepsilon_{23}^C + \chi_A \varepsilon_{23}^A + \mathcal{N}_{d} \left( \chi_C \tilde{\varepsilon}_{23}^C + \chi_A \tilde{\varepsilon}_{23}^A \right) \\
\varepsilon_{13} &= \chi_B \varepsilon_{13}^B + \chi_D \varepsilon_{13}^D + \mathcal{N}_{d} \left( \chi_C \tilde{\varepsilon}_{23}^C + \chi_A \tilde{\varepsilon}_{23}^A \right)
\end{align*}
\] (88)

In this equation \( \chi_S \) denote the corresponding linear shape functions. The contributions of the assumed natural strain terms to the discretised variational strain and the geometric part of the stiffness matrix can be found in Appendix C.

9.4. Numerical integration

The construction of the element internal force vector and stiffness matrix requires a proper integration of three domains; \( \Omega_0, \Omega_0^+ \) and \( \Gamma_{d,0} \). In this case, all three domains are standard geometrical entities (two six-sided volumes and a rectangular surface) and can therefore be integrated numerically with standard Gauss integration schemes.

10. NUMERICAL EXAMPLES

Two examples are presented to demonstrate the performance of the element. In the first example, attention is focused on the accuracy of the element for a decreasing thickness in geometrically linear applications. In the second example, the performance of the element in a geometrically and physically nonlinear analysis is illustrated by means of the simulation of a delamination buckling test.

10.1. A peel test

To test the performance of the new element for a decreasing thickness, a peel test has been performed. The double cantilever beam shown in Figure 7 consists of two layers of the same material with Young’s modulus \( E = 100.0 \, \text{N/mm}^2 \) and Poisson ratio \( \nu = 0.0 \). The beam has delaminated over its entire length. The test is performed for two different models. The first model contains ten eight-node enhanced solid-like shell elements (SLS+8); the second model is built with just five 16-node enhanced solid-like shell elements (SLS+16). In both cases, only one element in thickness direction is used. The delamination is modelled by a traction free discontinuity.

Figure 7. Geometry of a delaminated double cantilever beam under peel loading.
The linear out-of-plane displacements are given as functions of the ratio of layer thickness and beam length in Figure 8. The results are normalised by the exact solution that follows from the theory of beam deflections. The eight-node enhanced solid-like shell element gives nearly exact results for aspect ratios up to 2000. The performance of the sixteen-node solid-like shell element is even better. Indeed, it appears that the bending properties of the enhanced element are not affected by the enhancement and give the same performance as the underlying element [14]. The enhanced element effectively acts as two elements.

10.2. Delamination buckling of a cantilever beam

Next, a combination of delamination growth and structural instability is considered [12,20]. The double cantilever beam in Figure 9 has an initial delamination length of \( a_0 = 10 \) mm and is subjected to an axial compressive load \( 2P \). Two small perturbation forces \( P_0 \) are applied to trigger the desired buckling mode. Both layers are made of the same material with Young’s modulus \( E = 135 000 \) N/mm\(^2\) and Poisson’s ratio \( \nu = 0.18 \). Owing to symmetry in both the geometry of the model and the applied loading, delamination propagation can be modelled with an exponential mode-I decohesion law:

\[
\tau_{\text{dis}}^n = t_{\text{ult}} \exp \left( -\frac{t_{\text{ult}}}{G_c} v_{\text{dis}}^n \right),
\]

where \( \tau_{\text{dis}}^n \) and \( v_{\text{dis}}^n \) are the normal traction and displacement jump, respectively. The ultimate traction \( t_{\text{ult}} \) is equal to \( 50 \) N/mm\(^2\), the fracture toughness is \( G_c = 0.8 \) N/mm. The shear tractions
are equal to zero. The delamination surface is extended when the normal stress in thickness direction ahead of the delamination front \((\sigma_{33})\) exceed the ultimate traction \(t_{\text{ult}}\).

The critical load for local buckling of the beam, prior to delamination growth can be calculated analytically using the equation for a single cantilever beam with length \(a_0\) and thickness \(h\). For the given material parameters, the buckling load is

\[
P_{\text{cr}} = \frac{\pi^2 Eh^3}{48a_0^2} = 2.22 \text{ N}
\]

The finite element mesh used for the analysis is composed of eight-node enhanced solid-like shell elements and is shown in Figure 10. Again, it consists of only one element in thickness direction. In order to capture delamination growth correctly, the mesh is locally refined.

Figure 10 shows the lateral displacement \(u\) of the beam as a function of the external force \(P\). The load-displacement response for a specimen with a perfect bond (no delamination growth) is given as a reference. The numerically calculated buckling load is in agreement with the analytical solution. Steady delamination growth starts at a lateral displacement \(u = 4\) mm, which is in agreement with previous simulations [20].

11. CONCLUSIONS

In this contribution, a new concept for the simulation of delamination growth in thin layered composite structures is presented. The delamination crack is incorporated in a solid-like shell element [14] by means of the partition-of-unity property of finite element shape functions. The approach has a number of advantages. First, the displacement jump is only activated as the delamination propagates, which already results in a reduction of the total number of degrees of freedom. Moreover, it is possible to model a laminate and capture delamination, with just one element in the thickness direction. With conventional techniques, at least double the number of elements is needed. These properties allow the use of coarse meshes and to analyse delamination phenomena on a structural level.

A delamination buckling example underlines the excellent performance of the element in combined geometrically and physically non-linear analyses.
APPENDIX A. DERIVATION OF THE $B$ MATRIX

The variational strains as derived in Equation (42) can be written in the following matrix notation:

$$
\delta \gamma = H^1 \delta \hat{u}_1 + H^0_0 \delta \hat{u}_{0,1} + H^1_1 \delta \hat{u}_{1,1} + H^0_2 \delta \hat{u}_{0,2} + H^1_2 \delta \hat{u}_{1,2} + H^w \delta \hat{w} \\
+ \mathcal{K}_{\Gamma_{\delta}} (H^1_1 \delta \hat{u}_1 + H^0_1 \delta \hat{u}_{0,1} + H^1_1 \delta \hat{u}_{1,1} + H^0_2 \delta \hat{u}_{0,2} + H^1_2 \delta \hat{u}_{1,2} + H^w \delta \hat{w}) 
$$

(A1)

where

$$
H^1 = \begin{bmatrix}
0 \\
0 \\
d^T \\
0 \\
e^T_2 \\
e^T_1
\end{bmatrix}, \quad H^0_0 = \begin{bmatrix}
0 \\
0 \\
0 \\
-4w d^T \\
0 \\
0
\end{bmatrix}
$$

(A2)

$$
H^1_1 = \begin{bmatrix}
e^T_1 \\
0 \\
0 \\
e^T_2 \\
0 \\
d^T
\end{bmatrix}, \quad H^1_2 = \begin{bmatrix}
e^T_1 \\
0 \\
0 \\
e^T_2 \\
0 \\
d^T
\end{bmatrix}, \quad H^w = \begin{bmatrix}
0 \\
0 \\
0 \\
-2d \cdot d \\
0 \\
0
\end{bmatrix}
$$

(A3)

$$
H^2 = \begin{bmatrix}
e^T_1 \\
0 \\
0 \\
e^T_2 \\
0 \\
d^T
\end{bmatrix} + \zeta \begin{bmatrix}
d^T_1 - [2e^T_1 \tilde{G}^1_1 + e^T_2 \tilde{G}^2_1] \\
0 \\
0 \\
-\tilde{e}^T_2 \tilde{G}^1_2 \\
0 \\
0
\end{bmatrix}
$$

(A4)
\[ \mathbf{H}^I_2 = \begin{bmatrix} 0 & -\mathbf{e}_1^T \mathbf{G}_1^I \\ \mathbf{e}_2^T \quad \mathbf{d}_2^T - [\mathbf{e}_1^T \mathbf{G}_2^I + 2\mathbf{e}_2^T \mathbf{G}_2^I] \\ 0 & 0 \\ \mathbf{e}_1^T \quad \mathbf{d}_1^T - [\mathbf{e}_1^T \mathbf{G}_1^I + 2\mathbf{e}_2^T \mathbf{G}_2^I] \\ 0 & 0 \end{bmatrix} \]  

(A5)

The variational displacement vectors can be obtained from the nodal degrees of freedoms via the shape functions as defined in Equations (53)–(59).

\[ \delta \mathbf{u}_i = \mathbf{N}_i \delta a, \quad \delta \mathbf{u}_i = \mathbf{N}_i \delta b \]  

(A6)

\[ \delta \mathbf{u}_{0,z} = \mathbf{N}_{0,z} \delta a, \quad \delta \mathbf{u}_{0,z} = \mathbf{N}_{0,z} \delta b \]  

(A7)

\[ \delta \mathbf{u}_{1,z} = \mathbf{N}_{1,z} \delta a, \quad \delta \mathbf{u}_{1,z} = \mathbf{N}_{1,z} \delta b \]  

(A8)

\[ \delta \mathbf{w} = \mathbf{N}_w \delta p, \quad \delta \mathbf{w} = \mathbf{N}_w \delta q \]  

(A9)

Substituting these equations into (A1) gives

\[ \delta \gamma = \mathbf{B}_u \delta a + \mathbf{B}_w \delta p + \mathbf{H}_{\gamma,0}(\mathbf{B}_u \delta b + \mathbf{B}_w \delta q) \]  

(A10)

where \( \mathbf{B}_u \) and \( \mathbf{B}_w \) are defined as

\[ \mathbf{B}_u = \mathbf{H}_I^I \mathbf{N}_1 + \mathbf{H}_I^0 \mathbf{N}_{0,1} + \mathbf{H}_I^1 \mathbf{N}_{1,1} + \mathbf{H}_I^0 \mathbf{N}_{0,2} + \mathbf{H}_I^1 \mathbf{N}_{1,2} \]  

\[ \mathbf{B}_w = \mathbf{H}_w \mathbf{N}_w \]  

(A11)

Note that the matrices \( \mathbf{B}_u \) and \( \mathbf{B}_w \) are identical for the regular displacement terms \( a \) and \( p \) as well as the additional terms \( b \) and \( q \). The matrices \( \mathbf{B}_u \) and \( \mathbf{B}_w \) are still defined in the isoparametric frame of reference and must be transformed in the element local system \( I_j \), using the transformation tensor \( T^I_{j} \) in Equation (33).

APPENDIX B. STRESS DEPENDENT PART OF THE STIFFNESS MATRIX

The relations for the incremental change of the virtual strains in Equation (82) can be written as a function of the nodal degrees of freedom \( a \) and \( b \) and the internal degrees of freedom \( p \) and \( q \) by using the shape functions as defined in Equations (53)–(55), (58), and (59).

\[ 2d(\delta e_{2\beta}) = (\delta a + \mathcal{H}_{\gamma,0} \delta b)^T \mathbf{N}_{0,\beta}^I \mathbf{N}_0 \beta (\delta a + \mathcal{H}_{\gamma,0} \delta b) \]  

\[ + (\delta a + \mathcal{H}_{\gamma,0} \delta b)^T \mathbf{N}_{0,\beta}^I \mathbf{N}_0 \beta (\delta a + \mathcal{H}_{\gamma,0} \delta b) \]  

(B1a)
\[ 2d(\delta v_{23}) = (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{0,2}^T N_1 (da + \mathcal{H}_{T_d \theta} db) \]
\[ + (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{0,2}^T N_1 (da + \mathcal{H}_{T_d \theta} db) \] (B1b)

\[ 2d(\delta e_{33}) = 2(\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{1,2}^T N_1 (da + \mathcal{H}_{T_d \theta} db) \] (B1c)

\[ 2d(\delta \rho_{2\beta}) = (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{0,\beta}^T N_{1,2} (da + \mathcal{H}_{T_d \theta} db) \]
\[ + (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{0,\beta}^T N_{1,2} (da + \mathcal{H}_{T_d \theta} db) \]
\[ + (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{0,\beta}^T N_{1,2} (da + \mathcal{H}_{T_d \theta} db) \]
\[ - (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T C_{12}^T N_{1,\beta} (da + \mathcal{H}_{T_d \theta} db) \]
\[ - (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T C_{12}^T N_{1,\beta} (da + \mathcal{H}_{T_d \theta} db) \]
\[ - (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{0,\beta}^T C_{12} (da + \mathcal{H}_{T_d \theta} db) \]
\[ - (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{0,\beta}^T C_{12} (da + \mathcal{H}_{T_d \theta} db) \] (B1d)

\[ 2d(\delta \rho_{23}) = (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{1,2}^T N_1 (da + \mathcal{H}_{T_d \theta} db) \]
\[ + (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{1,2}^T N_1 (da + \mathcal{H}_{T_d \theta} db) \] (B1e)

\[ 2d(\delta \rho_{33}) = (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{1,2}^T dN_u (da + \mathcal{H}_{T_d \theta} db) \]
\[ + (\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{1,2}^T dN_u (da + \mathcal{H}_{T_d \theta} db) \]
\[ + w(\delta a + \mathcal{H}_{T_d \theta} \delta b)^T N_{1,2}^T N_1 (da + \mathcal{H}_{T_d \theta} db) \] (B1f)

where

\[ C_x = \bar{G}_x N_{0,1} + \bar{G}_x N_{0,2} \] (B2)

The incremental change of the virtual strains still refers to the covariant components. However, the stress components are referred to the local system \( \mathbf{l} \). Hence, for the set-up of the stress dependent part of the stiffness matrix \( \mathbf{K}^{\text{geo}} \) the factor \( \omega^{ij} \) is introduced as

\[ \omega^{mn} = T_k^m T_p^o \sigma_{kl} \] (B3)

Elaboration of Equation (B3) and Equations (B1a)–(B1d) give the following matrices:

\[ \mathbf{A}_{mn} = \omega^{11} N_{0,1}^T N_{0,1} + \omega^{22} N_{0,2}^T N_{0,2} + \omega^{33} N_1^T N_1 + \omega^{12} (N_{0,1}^T N_{0,2} + N_{0,2}^T N_{0,1}) \]
\[ + \omega^{23} (N_{0,2}^T N_1 + N_1^T N_{0,2}) + \omega^{13} ((N_{0,1})^T N_1 + (N_1)^T N_{0,1}) \]
\begin{align*}
+ \zeta \omega^1 (N_{1,1}^T N_{0,1} + N_{0,1}^T N_{1,1} - C_{1}^T N_{0,1} - N_{0,1}^T C_1) \\
+ \zeta \omega^2 (N_{1,2}^T N_{0,2} + N_{0,2}^T N_{1,2} - C_2^T N_{0,2} - N_{0,2}^T C_2) \\
- \zeta \omega^3 \omega N_1^T N_1 + \zeta \omega^3 (N_{1,2}^T N_{1,2} + N_{1,2}^T N_{0,1} + N_{1,1}^T N_{0,2}) + \zeta \omega^3 (N_{1,1}^T N_1 + N_1^T N_{1,1}) \\
+ \zeta \omega^1 (N_{1,1}^T N_{0,2} + N_{1,2}^T N_{0,1} + N_{0,1}^T N_{1,2} + N_{0,2}^T N_{1,1}) \\
- C_{1}^T N_{0,2} - C_{2}^T N_{0,1} - N_{0,1}^T C_2 - N_{0,2}^T C_1)
\end{align*}

\begin{align*}
A_{uv} &= - \zeta \omega^3 4 N_0^T d^T N_1 & (B4a) \\
A_{uw} &= - \zeta \omega^3 4 N_1^T d^T N_w & (B4b) \\
A_{vw} &= 0 & (B4c)
\end{align*}

Integration yields of these terms over the domain yields the geometric part of the stiffness matrix:

\begin{equation}
\mathbf{K}_{\text{geo}} = \begin{bmatrix}
\int_{\Omega} A_{uu} \, d\Omega & \int_{\Omega} A_{uu} \, d\Omega & \int_{\Omega} A_{uw} \, d\Omega & \int_{\Omega} A_{uw} \, d\Omega \\
\int_{\Omega} A_{uu} \, d\Omega & \int_{\Omega} A_{uu} \, d\Omega & \int_{\Omega} A_{uw} \, d\Omega & \int_{\Omega} A_{uw} \, d\Omega \\
\int_{\Omega} A_{uw} \, d\Omega & \int_{\Omega} A_{uw} \, d\Omega & 0 & 0 \\
\int_{\Omega} A_{uw} \, d\Omega & \int_{\Omega} A_{uw} \, d\Omega & 0 & 0 \\
\end{bmatrix}
\tag{B5}
\end{equation}

**APPENDIX C. ASSUMED NATURAL STRAINS**

It was noted by Parisch [14] that the eight-node shell element, which can be compared with an original four-node shell element suffers from shear locking; the shear stiffness is overestimated. As proposed by Bathe et al. [19], an assumed natural strain approach is applied to overcome this problem. Here, the strain terms \( \varepsilon_{23} \) are replaced by an alternative strain, determined in four tying points located at the mid-surface of the element as shown in Figure 6. The strains are defined as

\begin{align*}
\varepsilon_{23} &= \chi c \varepsilon_2^C + \chi A \varepsilon_2^A + \mathcal{Z} T_{1,0} \left( \chi c \varepsilon_2^C + \chi A \varepsilon_2^A \right) \\
\varepsilon_{13} &= \chi B \varepsilon_1^B + \chi D \varepsilon_1^D + \mathcal{Z} T_{1,0} \left( \chi c \varepsilon_2^C + \chi A \varepsilon_2^A \right)
\end{align*}

\tag{C1}
The virtual strains in tying point \( S \) can be written directly in terms of the regular and enhanced degrees variational degrees of freedom \( \delta a \) and \( \delta b \):

\[
2\delta e_{23}^S = B_{u}^{\text{ANS}} \begin{bmatrix}
\delta a_J \\
\delta a_K \\
\delta a_L \\
\delta a_M \\
\end{bmatrix} + \mathcal{K}_{\Gamma,\alpha} B_{u}^{\text{ANS}} \begin{bmatrix}
\delta b_J \\
\delta b_K \\
\delta b_L \\
\delta b_M \\
\end{bmatrix}
\]

(C2)

where the superscripts \( J, K, L \) and \( M \) represent the nodes which span the side on which the sampling point \( S \) is located, see also Table CI. The matrix \( B_{u}^{\text{ANS}} \) replaces the terms that are related to the membrane strains \( \varepsilon_{13} \) and \( \varepsilon_{23} \) in Equation (A11) and is equal to

\[
B_{u}^{\text{ANS}} = \frac{1}{4} [(e - d), (-e - d), (e + d), (-e + d)]^S
\]

(C3)

where

\[
e^S = \frac{1}{4}(-x_J - x_K + x_L + x_M); \quad d^S = \frac{1}{4}(x_J - x_K + x_L - x_M)
\]

(C4)

where for example, \( x_J \) is the position of node \( J \) in deformed configuration:

\[
x_J = X_J + a_j + \mathcal{K}_{\Gamma,\alpha} b_j
\]

(C5)

Furthermore, the terms in the geometric part of the stiffness matrix \( K^{\text{geo}} \) that corresponds to the shear stresses \( \omega_{13} \) and \( \omega_{23} \) are replaced by

\[
K^{\text{geo,ANS}} = \begin{bmatrix}
\int_{\Omega} A_{uu}^{\text{ANS}} d\Omega & \int_{\Omega^+} A_{uu}^{\text{ANS}} d\Omega & 0 & 0 \\
\int_{\Omega} A_{uu}^{\text{ANS}} d\Omega & \int_{\Omega^+} A_{uu}^{\text{ANS}} d\Omega & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(C6)

where

\[
A_{uu}^{\text{ANS}} = \omega_{23}(\chi_B \tilde{H}_2^B + \chi_D \tilde{H}_2^D) + \omega_{13}(\chi_A \tilde{H}_{13}^A + \chi_C \tilde{H}_{13}^C)
\]

(C7)
where $\mathbf{H}_{33}^{S}$ represents the mapping matrices and is obtained by expanding the matrix $\mathbf{H}_{33}^{S}$ to the correct nodal degrees of freedom according to Table CI [14].

\[
\mathbf{H}_{33}^{S} = \frac{1}{8} \begin{bmatrix}
-I_3 & 0 & 0 & I_3 \\
0 & I_3 & -I_3 & 0 \\
0 & -I_3 & I_3 & 0 \\
I_3 & 0 & 0 & -I_3
\end{bmatrix} \tag{C8}
\]

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