Classification of periodic orbits for systems with backlash

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Abstract

In this paper systems with backlash are studied for the effect of excitation parameters on the periodic response. These systems are modeled as piecewise linear systems with discontinuity in the net restoring force, caused by additional damping in the contact zone. The periodic orbits are classified by their number of subspace boundary crossings and, alternatively, by the largest Floquet multipliers. Some observations are also presented about the qualitative features such as symmetry breaking bifurcations exhibited by this class of systems.

1. Introduction

Backlash, dead zone or clearance is a common feature of many mechanical systems and can undermine the performance of the system. It can be caused by intended clearance necessary for assembly and operation, but may also be the result of operational wear and tear. Backlash has a large influence on the dynamics and control of systems as power transmissions, robotics and measurement systems. For instance, it can lead to rattle and chaotic motion in gear systems which causes damage and noise. Systems with backlash form a subclass of discontinuous mechanical systems and can be modeled as piecewise linear systems. Here, the spring force has piecewise linear characteristics. The spring force itself is continuous, but the stiffness is discontinuous. The nonlinearity in the damping force, which is also modeled with piecewise linear characteristics, is even more severe, since the damping force itself is discontinuous.

The effect of backlash on dynamics has been investigated in literature which includes bi-linear or piecewise systems as well. For instance, a bi-linear model is used to study the dynamics of compliant off-shore structures for subharmonic resonances and chaos [16]. Periodically forced bi-linear oscillators were studied [14]. The long term response of models with bi-linear stiffness and damping is studied for the existence and stability of boundary crossing periodic orbits [11], which also investigates the phenomena that characterize the response. The most general n-periodic solutions and their stability are also studied for tri-linear systems with harmonic forcing [12]. Chaos in these systems is also analyzed experimentally, where the result are compared to theoretical solutions [17].

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Geared systems form a subclass of systems with backlash and can be modeled as a tri-linear system with time-varying stiffness and damping. The effect of shaft stiffness is numerically studied for changing stiffness as well [10]. Similar systems without the additional shaft are analyzed by using the harmonic balance method. This method is used to study the steady-state forced response analytically for commensurate parametric and external forcing [1]. The same paper also uses experimental results to show the existence of subharmonic resonances. The harmonic balance method is adapted and used to study the sub- and super-harmonic responses, which are compared with experimental data [8]. Chaotic vibration for various nonlinear stiffness characteristics in gear systems with backlash is numerically studied [15].

In addition to geared systems, the dynamics of an elastic beam that moves between stoppers is present in literature. Numerical methods are compared for a elastic beam which is clamped at one end and is limited in deflection on the other end [4]. The dynamics of a piecewise linear beam is influenced by adding a dynamic vibration absorber to suppress the first harmonic resonance [2]. Oscillators in clearance are studied [6] as well. A different approach to systems with backlash is to consider impact. The basic properties of an oscillator in a clearance with impact are analyzed using approximate analytic methods [6]. The chaotic motion of an intermittency type of the impact oscillator is considered [13]. Experimental and numerical analysis of the steady-state behavior of a beam system with impact was discussed [5]. A special issue [7] of Chaos, Solitons and Fractals on dynamics of impact systems contains many articles of interest.

The study in this paper has the following objectives: (1) Classification of the periodic orbits of systems with backlash in the parameter space of excitation frequency and amplitude. (2) Provide observations regarding the effect of backlash on the qualitative features such as symmetry breaking and boundary crossing bifurcations and their relationship with the corresponding Floquet multipliers.

This paper is organized as follows: First a review of an analysis method based on a semi-analytical scheme integrated with the multiple shooting method is presented in Section 2. Next, two systems with backlash are presented in Section 3. We classify the periodic orbits of these systems in the space of excitation parameters in terms of the number of subspace crossings. The results of this analysis are presented in Section 4. Finally, some conclusions and recommendations for further research are presented in Section 5.

2. Analysis methodology – a review

In this work, periodic orbits and their stability are analyzed for different forcing parameters (frequency and amplitude) for systems with backlash where the nonautonomous system is given as $\dot{x} = f(t, x)$. This paper considers only period-one periodic orbits, so the period of excitation equals the period of the response, $T = 2\pi/\omega$. The stability of the periodic orbits can be determined by calculating the monodromy matrix [9] and its corresponding Floquet multipliers. Discontinuous systems exhibit discontinuities in the fundamental solution matrix. The fundamental solution matrix exhibits jumps whenever the state changes subspace for discontinuous systems. The effect of this jump on a perturbation can be described by a saltation matrix $S$ [9]. These saltation matrices are used to connect the linear subspaces using the transition property, resulting in the fundamental solution matrix for the entire orbit. It should be further noted that these discontinuities in the state space can result in a jump in the Floquet multipliers and can lead to discontinuous bifurcations [9].

The systems of interest for this study are piecewise linear:

$$\dot{x}(t) = A_v(x(t) - \Delta x_v) + B F(t)$$

where $A_v$ denotes the system matrix in subspace $v$, and the input matrix $B$ is assumed to be equal for all subspaces. A vector constant, $\Delta x$, generally is needed to describe the dynamics in each subspace correctly. These systems are forced using a sine-function, $F(t) = A \sin(\omega t + \phi)$, so an analytical solution can be found in each subspace (Fig. 1). The multiple shooting method [9] is utilized with the analytical solution in each subspace to compute the periodic orbit and uses $N$ shooting points along the periodic solution. These $N$ points are equally spaced in time with constant time interval $h = T/N$.

3. Modeling of system dynamics

Classification of periodic orbits is presented using some observations on a single degree-of-freedom system model as well as a multiple degree-of-freedom model. These models are selected based on the need to understand the underlying phenomenon in this class of systems and also the interaction between the coupling obtained in a multiple degree-of-freedom model representative of a geared power transmission unit.
3.1. Single degree-of-freedom system model

The single degree-of-freedom (SDOF) system with backlash is presented schematically in Fig. 2 and consists of a mass which can move freely between two massless stoppers, (hence it is assumed that the inertia of the stoppers can be neglected) resulting in the following equation of motion:

\[
m\ddot{x} + C(x) + K(x) = F
\]

In Eq. (2), \(m\) is the mass of the system, \(F\) denotes the forcing. The restoring force \(K(x)\) and the damping force \(C(x)\) using the state vector \(x = [x, \dot{x}]^T\) can be described as

\[
K(x) = \begin{cases} 
0, & x \in V_2 \\
k_1(x + b), & x \in V_3 \\
k_2(x - b), & x \in V_1
\end{cases}
\]

(3)

\[
C(x) = \begin{cases} 
c\dot{x}, & x \in V_2 \\
(c + c_1)\dot{x}, & x \in V_3 \\
(c + c_2)\dot{x}, & x \in V_1
\end{cases}
\]

(4)

The state space is divided into three subspaces \(V_i\), \(i = \{1, 2, 3\}\) as is depicted in Fig. 3, based on contact or no-contact with the stoppers and can be described by

\[
V_1 = \{x \in \mathbb{R}^2 | x < -b, k_1(x + b) + c_1\dot{x} \leq 0\}
\]

(5)

\[
V_3 = \{x \in \mathbb{R}^2 | x > b, k_2(x - b) + c_2\dot{x} \geq 0\}
\]

(6)

\[
V_2 = \{x \in \mathbb{R}^2 | x \notin (V_1 \cup V_3)\}
\]

(7)

The nominal system parameters are: \(m = 1\) kg, \(c = 0.05\) Ns/m, \(b = 1\) m, \(k_1 = k_2 = 4\) N/m, \(c_1 = c_2 = 0.5\) Ns/m, \(A = 1\) N. The solutions which are fully enclosed in the backlash gap region may be shifted as long as the stoppers are not reached and hence a constraint is placed for solving the multiple shooting algorithm formulation. This constraint equation ensures that the periodic solution is (roughly) located in the center between the stoppers without influencing the shape of the solution itself.

![Fig. 1. Simulation method-with the effect of subspaces.](image1)

![Fig. 2. Single degree-of-freedom system with backlash.](image2)
labeled. The stoppers are again assumed to be massless resulting in the following equations of motion:

\[ m_1 \ddot{x}_1 + k_1 x_1 + c_1 \dot{x}_1 - k_2 (x_2 - x_1) - c_2 (\dot{x}_2 - \dot{x}_1) = F \]  
\[ m_2 \ddot{x}_2 + k_2 (x_2 - x_1) + c_2 (\dot{x}_2 - \dot{x}_1) - K(x) - C(x) = 0 \]  
\[ m_3 \ddot{x}_3 + K(x) + C(x) - k_3 (x_4 - x_3) - c_4 (\dot{x}_4 - \dot{x}_3) = 0 \]  
\[ m_4 \ddot{x}_4 + k_4 (x_4 - x_3) + c_4 (\dot{x}_4 - \dot{x}_3) = 0 \]

The state vector is \( \mathbf{x} = [x_1, \dot{x}_1, x_2, \dot{x}_2, x_3, \dot{x}_3, x_4, \dot{x}_4]^T \). The restoring force \( K(x) = K(x_3 - x_2, x_3 - \dot{x}_2) \) and the damping force \( C(x) = C(x_3 - x_2, \dot{x}_3 - \dot{x}_2) \) between masses \( m_2, m_3 \) are given by

\[
K(x) = \begin{cases} 
0, & \mathbf{x} \in V_2 \\
k_3 (x_3 - \dot{x}_2), & \mathbf{x} \in V_3 \\
k_2 (x_3 - x_2 - b), & \mathbf{x} \in V_1 
\end{cases}
\]

\[
C(x) = \begin{cases} 
(c_3 + c_{31})(\dot{x}_3 - \dot{x}_2), & \mathbf{x} \in V_3 \\
(c_3 + c_{32})(\dot{x}_3 - \dot{x}_2), & \mathbf{x} \in V_1 
\end{cases}
\]

The state space is divided into three subspaces \( V_1, V_2 \) and \( V_3 \), as shown in Fig. 5:

\[
V_1 = \{ \mathbf{x} \in \mathbb{R}^8 \mid x_3 - x_2 < -b, k_{31}(x_3 - x_2 + b) + c_{31}(\dot{x}_3 - \dot{x}_2) \leq 0 \} 
\]
\[
V_3 = \{ \mathbf{x} \in \mathbb{R}^8 \mid x_3 - x_2 > b, k_{32}(x_3 - x_2 - b) + c_{32}(\dot{x}_3 - \dot{x}_2) \geq 0 \} 
\]
\[
V_2 = \{ \mathbf{x} \in \mathbb{R}^8 \mid \mathbf{x} \notin (V_1 \cup V_3) \}
\]

3.2. Multiple degree-of-freedom system model

The multiple degree-of-freedom system (MDOF) with backlash is depicted in Fig. 4 and consists of four masses, labeled \( m_i, i = 1, 2, 3, 4 \). These masses are interconnected by linear springs \( k_i \) and dampers \( c_i \). The backlash gap has a width of \( 2b \). The stoppers are again assumed to be massless resulting in the following equations of motion:

\[
m_1 \ddot{x}_1 + k_1 x_1 + c_1 \dot{x}_1 - k_2 (x_2 - x_1) - c_2 (\dot{x}_2 - \dot{x}_1) = F \\n\]
\[
m_2 \ddot{x}_2 + k_2 (x_2 - x_1) + c_2 (\dot{x}_2 - \dot{x}_1) - K(x) - C(x) = 0 \\n\]
\[
m_3 \ddot{x}_3 + K(x) + C(x) - k_3 (x_4 - x_3) - c_4 (\dot{x}_4 - \dot{x}_3) = 0 \\n\]
\[
m_4 \ddot{x}_4 + k_4 (x_4 - x_3) + c_4 (\dot{x}_4 - \dot{x}_3) = 0 \\n\]

The state vector is \( \mathbf{x} = [x_1, \dot{x}_1, x_2, \dot{x}_2, x_3, \dot{x}_3, x_4, \dot{x}_4]^T \). The restoring force \( K(x) = K(x_3 - x_2, x_3 - \dot{x}_2) \) and the damping force \( C(x) = C(x_3 - x_2, \dot{x}_3 - \dot{x}_2) \) between masses \( m_2, m_3 \) are given by

\[
K(x) = \begin{cases} 
0, & \mathbf{x} \in V_2 \\
k_3 (x_3 - \dot{x}_2), & \mathbf{x} \in V_3 \\
k_2 (x_3 - x_2 - b), & \mathbf{x} \in V_1 
\end{cases}
\]

\[
C(x) = \begin{cases} 
(c_3 + c_{31})(\dot{x}_3 - \dot{x}_2), & \mathbf{x} \in V_3 \\
(c_3 + c_{32})(\dot{x}_3 - \dot{x}_2), & \mathbf{x} \in V_1 
\end{cases}
\]

The state space is divided into three subspaces \( V_1, V_2 \) and \( V_3 \), as shown in Fig. 5:

\[
V_1 = \{ \mathbf{x} \in \mathbb{R}^8 \mid x_3 - x_2 < -b, k_{31}(x_3 - x_2 + b) + c_{31}(\dot{x}_3 - \dot{x}_2) \leq 0 \} 
\]
\[
V_3 = \{ \mathbf{x} \in \mathbb{R}^8 \mid x_3 - x_2 > b, k_{32}(x_3 - x_2 - b) + c_{32}(\dot{x}_3 - \dot{x}_2) \geq 0 \} 
\]
\[
V_2 = \{ \mathbf{x} \in \mathbb{R}^8 \mid \mathbf{x} \notin (V_1 \cup V_3) \}
\]

Fig. 3. Subspaces of the SDOF system with backlash in Fig. 2.

Fig. 4. Multiple degree-of-freedom system with backlash.
The nominal system parameters are $m_1 = m_4 = 0.5\, \text{kg}$, $m_2 = m_3 = 1\, \text{kg}$, $k_1 = k_2 = k_4 = 0.2\, \text{N/m}$, $c_1 = c_2 = c_4 = 0.02\, \text{Ns/m}$, $c_3 = 0.05\, \text{Ns/m}$, $k_{11} = k_{22} = 0.5\, \text{Ns/m}$, $c_{11} = c_{22} = 0.05\, \text{Ns/m}$, $b = 1\, \text{m}$, $A = 1\, \text{N}$.

4. Results on boundary classification

For each system model the period-one periodic response is analyzed for the parameter space of excitation amplitudes and excitation frequencies and a boundary classification is presented in terms of number of the subspace crossings and is related to a jump in the largest absolute value of the Floquet multipliers. A boundary crossing is counted every time the periodic solution changes subspace. Variation of excitation parameters results in changes in the number of boundary crossings. Each time the number of boundary crossings changes, a corner collision bifurcation [3] takes place. A boundary crossing results in a discontinuity, which can potentially lead to damage and degradation of the system at hand. Thus, a boundary crossing is an essential feature of systems with backlash which can be utilized to classify the inherent dynamics.

4.1. SDOF system

The dynamics of the single degree-of-freedom system (Fig. 2) is characterized by the amplitude–frequency diagram in Fig. 6. This figure shows the amplitude of the periodic solution for a range of forcing frequencies $\omega$ for a nominal excitation amplitude, $A = 1\, \text{N}$. Stable branches are indicated by solid lines, while unstable branches are shown in dashed lines. The branches are calculated using the multiple shooting algorithm in combination with path following. It is clear that multiple solutions for the same excitation frequency exist near the primary peak (see points labeled A, B and C). Fig. 7 shows the magnitude of the Floquet multipliers corresponding to the response diagram in Fig. 6. The branch where the response amplitude is smaller than one shows a Floquet multiplier equal to one, which is caused by the absence of a restoring force.

The total backlash between the stoppers is $2b$, so a linear solution exists for displacement amplitudes up to $b$ (provided that the periodic solution is symmetric with respect to $x = 0$), which is around $\omega = 1\, \text{rad/s}$. As stated previously, for certain excitation frequencies, this symmetric system exhibits multiple solutions (stable and unstable) for the same system parameters. Which stable solution is reached will depend on the initial conditions. In addition, this system exhibits symmetric and asymmetric periodic responses. Further, the largest Floquet multiplier will pass the unit circle when the stability is lost or recovered due to a change in a system parameter, e.g., the excitation frequency. Depending on the type of bifurcation the passing of the unit circle of the largest Floquet multiplier may occur via a discontinuous jump (due to a change in the number of boundary crossings, an explanation will be provided later) or continuously (symmetry breaking bifurcations). The number of boundary crossings is computed for different forcing frequencies and amplitudes, yielding Figs. 8 and 11. Figs. 8 and 11 are the result of different initial conditions. So in the construction of Figs. 8 and 11 path following was not utilized. The initial condition (for the simulation) for Fig. 8 is based on the dynamics in subspace $V_2$. On the contrary, the initial condition for Fig. 11 is chosen to be in a region where there is contact with a stopper. Using the dynamics in subspace $V_2$, a corner collision boundary can be analytically computed such that the amplitude of the periodic response is equal to one and the periodic solution just touches (but does not cross) the subspaces $V_1$ and $V_3$ [3]. This is depicted by the dashed line in Figs. 8 and 11. Further, it should be noted that the boundary
crossing results presented in this paper are such that an unstable solution is presented only if there are no coexisting stable solutions. No stable period-one solutions exist in the region labeled d in Fig. 8. However, if multiple stable solutions coexist, then depending on the initial conditions one of those stable solutions will be found and depicted. This approach is the primary reason for the differences in the information presented in Figs. 8 and 11. Specifically, to the right side of the corner collision boundary in Fig. 8 the stable low amplitude solutions are found whereas in Fig. 11
stable large amplitude solutions are found and presented. The differences between the two figures will be discussed in more detail later.

The essential trend in Fig. 8 is that the number of crossings increases for decreasing frequency. For low excitation frequencies, the dynamics of the system in contact with the stoppers is faster than the change in external forcing direction. The numerically calculated corner collision boundary matches well with the analytical boundary. The small discrepancy is likely to be caused by the simulated solution not to be exactly in the center between the stoppers.

Fig. 8. Classification of periodic orbits in \((A, \omega)\) space: boundary crossings for system in Fig. 2.

Fig. 9. Classification of periodic orbits in \((A, \omega)\) space: Floquet multipliers for system in Fig. 2.
For all periodic orbits that are classified by their number of boundary crossings, the Floquet multipliers are calculated and their maximum absolute value is depicted in Figs. 9 and 12. All Floquet multipliers with an absolute value equal to or higher than one (corresponding to, respectively, marginally stable and unstable period-one solutions) are set to one (white regions) for clarity. Among others, these figures clearly show the boundaries describing the number of boundary crossings. This can be explained by considering monodromy matrices [9]. Each time a switching boundary is crossed in the phase plane, the monodromy matrix exhibits a discontinuity or jump. This jump (which is described by

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Fig. 10. Periodic orbits at the labels in Fig. 8.
a saltation matrix) also affects the Floquet multipliers, so a change in the number of boundary crossings causes a sudden change in Floquet multipliers.

This number of boundary crossings is not the only qualitative difference in the periodic solutions. For example, Fig. 9, when compared to Fig. 8, shows an extra boundary between labels A and B. This suggests a change in the characteristics of the periodic orbit (i.e., a bifurcation), although the number of crossings does not change. This extra boundary is due to the change in periodic orbit from being symmetrical and stable at A to asymmetrical and stable at B (see Fig. 10a–b), without a change in the number of boundary crossings which results in the largest Floquet multiplier becoming one at the symmetry breaking bifurcation point between A and B. The asymmetric periodic orbit at point C (Fig. 10c) does show extra boundary crossings as noticed in Figs. 8 and 9.

Fig. 10d–f shows the periodic orbits at points D, E and F, which show an increase in the number of crossings as the forcing frequency decreases. The periodic orbit at point E is again asymmetric. This could also have been concluded from the number of crossings. The number of boundary crossings for this periodic orbit is 10, which means the stoppers are hit five times (so an uneven number of times) in a period, thus resulting in an asymmetrical periodic orbit. It has to be noticed that an even number of stopper hits (so a number of boundary crossings which is a multiple of four) does not imply that the periodic orbit is symmetrical, as can be observed by considering the periodic orbit at point B in Fig. 10b.

![Fig. 10. (continued)](image)

![Fig. 11.](image)

A. Shukla et al. / Chaos, Solitons and Fractals 41 (2009) 131–144 139
For low excitation frequencies ($\omega < 0.2 \, \text{rad/s}$), the amplitude of the forcing appears to have a larger influence on the number of crossings. Fig. 10g and h shows this influence for $\omega = 0.15 \, \text{rad/s}$. The periodic orbit at G shows higher harmonics that are entirely in subspace $V_1$ or $V_3$, so contact with a stopper is not lost. For a lower forcing amplitude, the force is too small to maintain this dynamics and contact with the stopper will be lost, causing an increase in the number of boundary crossings. Periodic orbit H in Fig. 10h is an example.

![Classification of periodic orbits in $(A, \omega)$ space: Floquet multipliers for system in Fig. 2 – $x_0 = [3, 0]^T$.](image1)

![Classification of periodic orbits in $(A, \omega)$ space: boundary crossings for the MDOF system.](image2)
Fig. 14. Classification of periodic orbits in $(A, \omega)$ space: Floquet multipliers for the MDOF system.

Fig. 15. Periodic orbits at labels A $(\omega = 0.99 \text{ rad/s})$, B $(\omega = 0.90 \text{ rad/s})$ and C $(\omega = 0.85 \text{ rad/s})$, respectively, in Fig. 13 with 6, 4 and 6 boundary crossings, respectively.
As stated before, the evaluation of boundary crossings and the Floquet multipliers is repeated in Figs. 11 and 12, respectively, with an initial condition with greater probability of contact with the stoppers. When we compare Figs. 8 and 11, the solutions and their characteristics are almost identical when the response exhibits more than four boundary crossings. The classification is different towards the right side of the corner collision boundary (compare Figs. 8 and 11). The region where the periodic solutions cross the boundaries four times is increased. A fractal like boundary can be seen where this region ends. Beyond that boundary two situations occur: first, there are periodic orbits that do not cross any boundary, as was observed earlier in relation to the periodic solutions completely in subspace $V_2$ and second, periodic orbits with two boundary crossings are found. This means that only one stopper is touched and that the periodic solution is asymmetric. Although it is rather difficult to see, Fig. 12 shows that these periodic orbits are stable, i.e., non-white regions in region a in Fig. 11 (corresponding to the solutions with two boundary crossings) are also non-white in Fig. 12. Depending on the initial condition of the multiple shooting algorithm, either this solution or the linear non-touching solution is found. Using the insights obtained so far, similar observations, regarding the classification and nonlinear dynamics of periodic orbits, are presented next for the MDOF system.

### 4.2. MDOF system

The dynamics (as given by Eqs. (8)–(11)) of the MDOF system (Fig. 4) is characterized by the number of boundary crossings of a periodic orbit as depicted in Fig. 13. The MDOF system also exhibits both symmetric and asymmetric solutions. The dashed line denotes combinations of forcing frequency and amplitude where the amplitude of the periodic orbit (in terms of $x_3 - x_2$) is $b$, calculated for the linear dynamics in the backlash region. As a result, periodic solutions that do not touch the stoppers can occur at the right side of that boundary. However, Fig. 13 is created using an initial condition in a contact region.

In region b with four boundary crossings, a V-shaped region c of periodic orbits with six crossings can be found in Fig. 13. The periodic orbits at labels A and C are in the latter V-shaped region c, orbit B is in the region between these labels. Orbits A, B and C correspond to a forcing frequency of $\omega = 0.99, 0.90, 0.85$ rad/s, respectively, at a forcing amplitude $A = 1.5$ N. They are depicted in Fig. 15 containing phase plots in terms of $x_3 - x_2$ and show a higher har-

![Fig. 16. Periodic orbits at labels D, E, F and G, respectively, in Fig. 13 with 4, 8, 4 and 4 boundary crossings, respectively.](image-url)
monic near or on the boundary. For excitation frequencies slightly higher than the frequency at label A ($\omega = 0.99$ rad/s), this higher harmonic is entirely in the backlash region. When the excitation frequency is decreased, the amplitude of the response increases. This increase in amplitude causes the higher harmonic to be pushed over the boundary, resulting in a small region with six crossings (orbit A in Fig. 15). Periodic orbit B shows the situation where the entire higher harmonic is in contact with a stopper, as is also the case for higher excitation and response amplitudes (for example see orbit G in Fig. 16). Decreasing the excitation frequency further starting from orbit B the amplitude of the response decreases again and the higher harmonic is again pushed over the boundary, now in opposite direction, see orbit C.

The same effect can be observed for lower frequencies, as is depicted in Fig. 16 for frequencies $\omega = 0.55, 0.40, 0.25$ rad/s at an excitation amplitude $A = 2$ N. When the amplitude of the periodic response is high, the higher harmonics are totally in the contact regions as can be observed for orbits D and F. A higher number of boundary crossings can be observed in orbit E in the right upper graph in Fig. 16 as the higher harmonic also crosses the boundary.

The boundaries between regions labeled a and b, b and c, c and d, as shown in Fig. 13, are consistently identified by both the number of boundary crossings (Fig. 13) as well as the largest Floquet multiplier indicating bifurcations (Fig. 14). However, there are some features which cannot be identified by a jump in largest Floquet multiplier: for example, the boundary between regions d and e is not clearly described by the largest Floquet multiplier (Fig. 14). This is a topic for further research.

5. Conclusions

In this paper, a single degree-of-freedom system and a multiple degree-of-freedom system with backlash are studied for their period-one responses under periodic excitation. The systems with backlash are modeled as piecewise linear systems. Characteristic phenomena such as stability and symmetry are discussed for these periodic orbits. Further, the responses are classified by their number of subspace boundary crossings and the absolute value of the largest Floquet multiplier in the parameter space of forcing frequency and amplitude. The inter-relationship between the boundary crossing and the jump in Floquet multipliers is demonstrated and used for the classification. It is shown that the Floquet multipliers undergo a sudden change when the number of boundary crossings of a periodic orbit changes and therefore give the same classification. In addition, the Floquet multipliers indicate some additional characteristics, such as symmetry breaking bifurcations.

Future work in the classification of periodic orbits for systems with backlash might focus on the effect of periodically varying stopper stiffness and damping, which is characteristic for gear systems. The coupling between the interplay of more than one backlash region in a multiple degree-of-freedom case is also of interest and part of future work for the authors. Finally, experimental validation is an essential and much needed component of classification of nonlinear system response.

References