Dissipation equals production in the log layer of wall-induced turbulence

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Asymptotic analysis is presented of the energy balance equations derived from statistically averaged Navier-Stokes equations pertinent to wall-induced turbulence. Attention is focused on the inertial sublayer, the region outside the viscous sublayer, and the buffer layer where the log-law for mean flow holds. Production and dissipation of turbulence are shown to be equal with a relative error of $O(x_s/L)$, where $x_s$ is the distance from the wall and $L$ is the external length (pipe radius, channel half-height, boundary layer thickness). Diffusion of pressure and kinetic energy together are only of relative magnitude $O(x_s/L)$. Pressure gradient terms are shown to redistribute longitudinal turbulence production in equal portions dissipated in the three orthogonal directions.

$u_i = u_0^i \delta_{i1} + u'_i,$

where $\delta_{i1}$ is the Kronecker delta. Substituting (1) into the $i$th component of the Navier-Stokes equations and subtracting its average (which is the Reynolds equation) yields three equations for the fluctuating velocities. Multiplying each of these equations with $u'_i$ and averaging gives the governing equations for mean fluctuating energy, e.g., Monin and Yaglom (Ref. 6, Sec. 6.2),

$$\frac{1}{2} \frac{\partial}{\partial t} \langle u'_i^2 \rangle + \frac{1}{2} \frac{\partial}{\partial x_i} \langle u'_i u'_j \rangle = - \left\{ u'_i \frac{\partial p}{\partial x_i} - \langle u'_i u'_j \rangle \frac{\partial u}{\partial x_j} \right\} - \epsilon_i + \frac{1}{2} \frac{\partial^2}{\partial x_i^2} \langle u'_i^2 \rangle,$$

where $p$ is pressure normalized with density, $\nu$ is kinematic viscosity, and $\epsilon_i$ is energy dissipation:

$$\epsilon_i = \nu \sum \left\langle \left( \frac{\partial u'_i}{\partial x_\alpha} \right)^2 \right\rangle.$$
\[
\frac{1}{2} \frac{\partial}{\partial x_2} (u_i^* u_j^*) = - \left( u_i^* \frac{\partial p}{\partial x_i} \right) - \left( u_i^* u_j^* \right) \frac{\partial u_i^*}{\partial x_2} \delta_{ij} - \epsilon_i.
\]

Summing the three components and applying continuity, \((\partial / \partial x_i) u_i^* = 0\), yields

\[
\frac{\partial}{\partial x_2} \left( \frac{1}{2} e + p \right) u_2^* = - \epsilon, \quad \epsilon = \sum_i u_i^* \epsilon_i.
\]

We shall now show that diffusion of kinetic energy and pressure, the terms on the left-hand side of Eq. (6), become vanishingly small in the limit \(\text{Re} \to \infty\), compared to production of turbulence, the second term on the right-hand side of Eq. (6). It results in the conclusion that production equals dissipation. The equality holds for the inertial sublayer. This is the region just outside the viscous sublayer and buffer layer along the wall and at the outer edge of the main region in the core. It can be indicated by

\[
\text{Re}^{-1} \ll x_2/L \ll 1.
\]

In practice, \(50\text{Re}^{-1} \ll x_2/L \ll 0.1\). The inertial sublayer is the intermediate region where the mean shear stress can be assumed to be approximately constant with respect to \(x_2\). In this region, the exponential behavior related to the effects of viscosity has vanished while the spatial variation associated with the core region where inertial effects dominate is negligibly small because of the small thickness of the inertial sublayer. In the inertial sublayer, the mean velocity obeys a logarithmic distribution satisfying the equation

\[
\frac{\partial u_2^*}{\partial x_2} = \kappa^{-1} u_2, \quad \kappa = 0.42.
\]

To prove that the terms on the left of Eq. (6) are small in the inertial sublayer, we shall first concentrate on the pressure term. An equation for the pressure is obtained by differentiating the \(i\)th component of the Navier-Stokes equations with respect to \(x_i\), adding up all terms, using continuity, and implementing Eq. (1). This leads to the well-known Poisson equation for pressure (see Ref. 6, p. 30): in our case and notation,

\[
\sum_a \frac{\partial^2 p}{\partial x_a^2} = - \varphi(x,t),
\]

where \(\varphi(x,t) = 2 \frac{\partial u_i^*}{\partial x_2} \frac{\partial u_i^*}{\partial x_i} + \sum_{ij} \frac{\partial^2 u_i^*}{\partial x_i \partial x_j} \).

A solution can be constructed adopting the Green’s-function method.\(^8\)

\[
p = \int_X G(x^*,x) \varphi(x^*,t) dx^*,
\]

where integration takes place over the whole domain that is occupied by the fluid. The solution for the Green’s function that satisfies the wall condition \((\partial / \partial x_2)p = 0\) at \(x_2 = 0\) (which follows from symmetry arguments) is

\[
G(x^*,x) = (4 \pi)^{-1} \left[ \left( \frac{x_1^* - x_1}{x_1^* - x_1} \right)^2 + \left( \frac{x_2^* - x_2}{x_2^* - x_2} \right)^2 + \left( \frac{x_3^* - x_3}{x_3^* - x_3} \right)^2 \right]^{-1/2}.
\]

To evaluate the pressure term in Eq. (6), we need to know \(\langle pu_i^* \rangle\), for which we can write

\[
\langle pu_i^* \rangle = \int_X \left( 2 \frac{\partial u_i^*}{\partial x_2} \frac{\partial u_i^*}{\partial x_i} (x^*,t) u_i^*(x,t) \right)
+ \sum_{ij} \frac{\partial^2 u_i^*}{\partial x_i \partial x_j} (x^*,t) u_i^*(x^*,t) u_i^*(x,t) G(x^*,x) dx^*.
\]

The task is to determine the value of these integrals in the inertial sublayer. Therefore, use is made of the results of similarity theory (see Ref. 6, p. 281). For the two-point spatial correlation \(\langle u_2^*(x^*,t) u_2^*(x,t) \rangle\), one can write

\[
\langle u_2^*(x^*,t) u_2^*(x,t) \rangle = \langle u_2^* (x^*) \rangle R_{22} (r)
= \alpha_{2,2} \mu_2^2 B_{2,2} (r) \left[ 1 + O \left( \frac{x_2}{L} \right) \right],
\]

where

\[
x^* - x = r = x_2 r', \quad -\infty \leq r' \equiv + \infty,
-1 \leq r_2' \equiv \infty, \quad \infty \leq r_2' \equiv + \infty.
\]

In Eq. (15), \(\sqrt{\alpha_{2,2}}\) is the constant value of the standard deviation of \(u_2^*\) in the inertial sublayer divided by \(u_2\), \(\sqrt{\alpha_{2,2}} = 1.15\).\(^9\) The term \(O(x_2/L)\) represents the effect of the variations of statistical averages of lateral velocity fluctuations with space coordinate in the core region. As mentioned before, this effect is small on the scale of the inertial sublayer. Furthermore, in accordance with similarity theory, the correlation distance in correlation functions like \(R_{22}\) is assumed to scale with distance from the wall \(x_2\). For the triple space correlation \(\langle u_i^*(x^*,t) u_i^*(x^*,t) u_i^*(x,t) \rangle\), we can write in a similar manner

\[
\langle u_i^*(x^*,t) u_i^*(x^*,t) u_i^*(x,t) \rangle = \alpha_{2,ij} u_i^2 B_{2,ij} (r') \left[ 1 + O \left( \frac{x_2}{L} \right) \right],
\]

where \(\alpha_{2,ij}\) is the constant value of the triple moment \(\langle u_i^*(x,t) u_i^*(x,t) u_i^*(x,t) \rangle\) in the inertial sublayer divided by \(u_i^2\). Disregarding the contributions of relative magnitude \(O(x_2/L)\), Eq. (14) can now be rewritten as...
Outside the viscous sublayer and the buffer layer, viz. \( x_3/L \approx 50/Re \), production and dissipation are equal with a relative error of \( \beta(x_3/L) \approx 1 - 2 \), the difference being due to the combined contribution of diffusion of kinetic energy and pressure. In a region close to the wall but outside the viscous and buffer layer, dissipation and production are practically equal and dominate over pressure and kinetic energy diffusion. With increasing Reynolds number, viscous and buffer layers become smaller and the region where the inertial sublayer approach holds starts closer to the wall. The dominance of dissipation and production then becomes more visible because dissipation and production are proportional to \( x_3^1 \), while the combined contribution of pressure and kinetic energy diffusion is fairly constant. This is what is seen from the results of experiments. Also recent DNS results by Hoyas and Jiménez for 2D channel flow support the above observations. Downloaded values for dissipation and production in the log-layer at \( Re=2003 \) comply with theoretical values and indicated error bounds.

Laufer’s experiments also show that while the combined contribution of diffusion of pressure and kinetic energy is of relative magnitude \( \beta(x_3/L) \) compared to production, the separate contributions tend to be larger and of opposite sign, but still small compared to production. Apart from measurement inaccuracy, the explanation may be the form of the second term in expansions (20) and (21). Because of differentiation of \( \langle pu_2 \rangle \) and \( \frac{1}{2} \langle eu_2^2 \rangle \) with respect to \( x_3 \) in Eq. (6), the second term determines the magnitude of pressure and kinetic energy diffusion. We proposed a form like \( x_3/L \), but other forms like \( (x_3/L) \text{ln}(x_3/L) \) may not be excluded either. It could lead to a slower relative decrease of pressure and kinetic energy diffusion relative to turbulent production. But the outcome of the asymptotic analysis remains the same. Under the limit procedure of the inertial sublayer, the second terms in expansions (20) and (21) vanish in comparison to the first terms. This is sufficient for each of the diffusion terms to vanish compared to production.

The conclusion is that in the inertial sublayer, turbulence production and turbulent dissipation are asymptotically equal. They can be described according to Eqs. (22) and (23); the error terms are in agreement with experimental observations. The error terms in the expressions for pressure and kinetic energy may be different from those presented in Eqs. (20) and (21). They may express a slower limiting behavior, of opposite sign in both equations, but nevertheless become vanishingly small under the limit procedure of the inertial sublayer.

Another important conclusion is that with Eq. (23) we have obtained an explicit expression for the energy dissipation rate in the inertial sublayer. The value of the Von Kármán constant appearing in the expression is well-established by experiment. An interesting point concerns the possible effect of intensification of streamwise fluctuations, a feature that has attracted quite some attention in recent years. According to Eqs. (14) and (21), streamwise fluctuations can only become important through triple correlations. These terms will contribute in case of appreciable values of skewness. Analysis of measurements by Mochizuki and Nieuwstadt and more re-
cently by Morrison, McKeon, Jiang, and Smits, however, suggest very low values for streamwise skewness. It gives rise to the conclusion that the effect of streamwise intensification on the prevailing energy balance in Eq. (6) is small. In this connection it is also noted that for Gaussian velocity statistics, triple correlations are zero altogether. In that case only the first term in the integral for pressure, cf. Eq. (14), can cause a difference between turbulence production and turbulent diffusion. This term involves spatial correlations of lateral velocities \( u'_i \) only.

The next question we want to address is, how do the magnitudes of the various terms in the three components of the energy balance equations (5) compare to each other? According to Kolmogorov theory, for large \( Re \), energy dissipation \( \epsilon \), splits up in three equal parts, \( \epsilon_i \), \( i=1,2,3 \). Accordingly,

\[
\epsilon_i = \frac{1}{3} \kappa^{-1} u'_i \left( \frac{x_2}{x_2} + O \left( \frac{x_2}{L} \right) \right), \quad Re^{-1} \ll \frac{x_2}{L} \ll 1. \tag{24}
\]

The three components of kinetic energy can be described analogous to Eq. (21) eventually with a modified error term as discussed earlier on. The result is that the three terms of diffusion of kinetic energy on the left-hand side of Eqs. (5) are of a relative magnitude compared to the dissipation terms which vanishes in the leading order balance of the inertial sublayer. Equations (5) thus degenerate to a balance between pressure gradient terms, production terms, and dissipation terms. Using the expression for production, cf. Eq. (22), one thus finds for the pressure gradient terms,

\[
\begin{align*}
\left\langle u'_i \frac{\partial p}{\partial x_k} \right\rangle &= \frac{2}{3} \kappa^{-1} u'_i \left( \frac{x_2}{x_2} + O \left( \frac{x_2}{L} \right) \right), \quad Re^{-1} \ll \frac{x_2}{L} \ll 1, \\
\left\langle u'_i \frac{\partial p}{\partial x_2} \right\rangle &= \left\langle u'_i \frac{\partial p}{\partial x_3} \right\rangle \\
&= -\frac{1}{3} \kappa^{-1} u'_i \left( \frac{x_2}{x_2} + O \left( \frac{x_2}{L} \right) \right), \quad Re^{-1} \ll \frac{x_2}{L} \ll 1. \tag{25a}
\end{align*}
\]

The values of the pressure gradient terms can also be assessed using the previously derived expressions for pressure [cf. Eqs. (12) and (13)]. Evaluation can take place in a manner analogous to the method presented in Eqs. (14)–(19). One then finds for the inertial sublayer, to leading order,

\[
\begin{align*}
\left\langle u'_i \frac{\partial p}{\partial x_k} \right\rangle &= \frac{u^3}{4 \pi \kappa} \int_{r_1}^{+\infty} \int_{r_2}^{+\infty} \int_{r_3}^{+\infty} \frac{dr_1 dr_2 dr_3}{r_1 r_2 r_3} \\
& \times \left\{ 2 \alpha_{22} \frac{\partial B_{22}(r')}{\partial r_1} r'_2 \left[ \frac{1}{f_1^2} + \frac{1}{f_2^2} \right] \\
& + \sum_{ij} \alpha_{i,j} \frac{\partial^2 B_{ij}(r')}{\partial r'_i \partial r'_j} r'_2 \left[ \frac{1}{f_1^2} + \frac{1}{f_2^2} \right] \right\}, \tag{26}
\end{align*}
\]

where variables are defined analogous to Eqs. (15)–(17). The above descriptions show behavior that is consistent with results (25a) and (25b), i.e., the pressure gradient terms in the inertial sublayer are equal to \( \gamma_i u'_i / x_2 \), where the value of \( \gamma_i \) follows from evaluation of the integrals. These values can be shown to be of unit order of magnitude [as in Eq. (18)], but precise values cannot be given because the exact shapes of the spatial correlation functions are unknown. Conversely, the values of \( \gamma_i \) are determined by results (25a) and (25b) and these values can serve as a condition to be imposed on the integral values of the correlation functions according to Eq. (26).

The conclusion is that in the inertial sublayer, the pressure gradient terms redistribute turbulence production in the longitudinal direction in three equal portions dissipated in three orthogonal directions.

The general conclusion is that the results presented reveal the prevailing energy balances in the log-layer of wall-induced turbulence. Results are based on application of inertial-sublayer asymptotics to velocity statistics in exact expressions for kinetic energy and pressure diffusion derived from Navier-Stokes equations.