Synchronization of nonlinear oscillators by bidirectional diffusive coupling with time delay

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Abstract—In this paper we consider synchronization of two identical nonlinear systems bidirectionally coupled by a diffusive coupling with time-delay. Firstly, we show using the small-gain theorem that trajectories of any strictly semi-passive systems converge to a bounded region. Next using this result, we derive a condition for synchronization of the systems by the Lyapunov-Krasovskii theorem. Finally we illustrate the effectiveness of the derived conditions with a numerical example.

Key Words: Synchronization, Time-delay, Coupled oscillator, Bidirectional coupling

1 Introduction

Recently synchronization of coupled oscillators has attracted a lot of attention and has been investigated by a number of researchers in order to clarify the mechanism of synchronization 1, 2, 3). Applications of synchronization to engineering have also been considered and analyzed using control theory 4, 5, 6). In applications to practical systems, however, time-delays caused by the signal transmission affect the behavior of the coupled systems. Although synchronization of coupled systems with delay coupling has been researched from both numerical analysis and theoretical analysis, these works focus on synchronization of systems with the coupling term described by \( K(x_i(t - \tau) - x_j(t - \tau)) \) and there are few results for the case that the delayed output of neighboring systems in the coupling term is described by \( K(x_i(t) - x_j(t - \tau)) \) 7, 8).

In this paper, we consider synchronization of two identical systems coupled by diffusive coupling \( K(x_i(t) - x_j(t - \tau)) \). We show by the small-gain theorem that the trajectories of the systems converge to a bounded set. Then we derive a sufficient condition for synchronization of the systems.

2 Preliminaries

In this section, we define the systems to be considered in this paper and give two lemmas.

Throughout this paper, the symbol \( \| \cdot \| \) denotes the Euclidean norm. For a vector \( v(t) : [0, \infty) \to \mathbb{R}^n \), if \( \|v\|_\infty \triangleq \sup_{t \geq 0} |v(t)| < \infty \), then we denote \( v \in \mathcal{L}_\infty^\infty \).

2.1 Semi-passivity and semi-dissipativity

Consider the nonlinear system,

\[
\dot{x}(t) = f(x, u), \quad y(t) = h(x) \quad (t \geq 0)
\]

with state \( x \in \mathbb{R}^n \), input \( u \in \mathbb{R}^m \), output \( y \in \mathbb{R}^m \), \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \to \mathbb{R}^m \).

According to semi-passivity as defined in 5), we introduce strictly semi-passivity and strictly semi-dissipativity as follows.

Definition 1 (strictly semi-passivity) System (1) is said to be strictly semi-passive, if there exist a \( C^1 \)-class function \( V : \mathbb{R}^m \to \mathbb{R} \), class-\( K_\infty \) functions \( \alpha(\cdot) \), \( \bar{\alpha}(\cdot) \) and \( \alpha(\cdot) \) satisfying

\[
\alpha(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|)
\]

\[
V(x) \leq -\alpha(\|x\|) - H(x) + y^T u
\]

for all \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^m \), where the function \( H(x) \) satisfies

\[
\|x\| \geq \eta \Rightarrow H(x) \geq 0
\]

for a positive real number \( \eta \).

Definition 2 (strictly semi-dissipativity) System (1) is said to be strictly semi-dissipative with respect to the supply rate \( q(u, y) \), if there exist a \( C^1 \)-class function \( V : \mathbb{R}^m \to \mathbb{R} \), class-\( K_\infty \) functions \( \alpha(\cdot) \), \( \bar{\alpha}(\cdot) \) and \( \alpha(\cdot) \) satisfying (2) and

\[
\dot{V}(x) \leq -\alpha(\|x\|) - H(x) + q(u, y)
\]

for all \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^m \), where the function \( H(x) \) satisfies (4).

Remark 1 The system is strictly semi-passive if the supply rate \( q(u, y) = y^T u \) in (5) for all \( u \in \mathbb{R}^m \).

For a strictly semi-dissipative system, we obtain the following lemma in a similar way as the argument of the input-to-state stability (ISS) in 12).

Lemma 1 Suppose that system (1) is strictly semi-dissipative with respect to the supply rate \( \beta(\|u\|) \) for a class-\( K \) function \( \beta(\cdot) \). Then there exist class-\( K \) functions \( \rho(\cdot) \), \( \gamma(\cdot) \) and a real number \( \eta > 0 \) such that the response \( x(t) \) of (1) with the initial state \( x(0) = x_0 \) satisfies

\[
\|x\|_\infty \leq \max_{t \to \infty} \{\rho(\|x_0\|), \gamma(\|v\|_\infty), \rho(\eta)\}
\]

\[
\limsup_{t \to \infty} \|x(t)\| \leq \max(\gamma(\limsup_{t \to \infty} \|u(t)\|), \rho(\eta))
\]

for any input \( u \in \mathcal{L}_\infty^\infty \) and any \( x_0 \in \mathbb{R}^n \). Here functions \( \gamma(\cdot) \) and \( \rho(\cdot) \) are given by

\[
\rho(\cdot) = \alpha^{-1} \circ \bar{\alpha}(\cdot)
\]

\[
\gamma(\cdot) = \alpha^{-1} \circ \bar{\alpha} \circ \alpha^{-1} \circ \kappa \beta(\cdot)
\]

(\( r \geq 0 \))

where \( \kappa > 1 \), functions \( \alpha(\cdot) \), \( \bar{\alpha}(\cdot) \) and \( \alpha(\cdot) \) satisfy (2) and (5) and \( \eta \) satisfies (4).
2.2 Small-gain theorem

Consider the following two systems
\[ \dot{x}_i(t) = f_i(x_i, u_i), \quad y_i(t) = C_i x_i \quad (t \geq 0) \] (9)
where state \( x_i \in \mathbb{R}^n \), input \( u_i \in \mathbb{R}^m \), output \( y_i \in \mathbb{R}^m \), \( C_i \in \mathbb{R}^{m \times n} \) and \( f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) with initial conditions
\[ x_i(\theta) = \phi_i(\theta), \quad -\tau \leq \theta \leq 0 \]
\[ x_i(0) = \phi_i(0) = x_{0i} \] (10)
for \( i = 1, 2 \), respectively. Suppose that the systems (9) are strictly semi-dissipative. Then from Lemma 1, each system have the properties of (6) and (7). Now we consider the case in which these two systems are coupled by the following inputs,
\[ u_1(t) = y_2(t - \tau) = C_2 x_2(t - \tau) \]
\[ u_2(t) = y_1(t - \tau) = C_1 x_1(t - \tau). \] (11)

Define class-K functions as
\[ \pi_1(r) = \gamma_1(\sigma_{\text{max}}(C_2) \cdot r) \]
\[ \pi_2(r) = \gamma_2(\sigma_{\text{max}}(C_1) \cdot r) \quad (r \geq 0). \] (12)
where \( \gamma_i(\cdot) \) are defined as (8) and \( \sigma_{\text{max}}(\cdot) \) denotes the maximum singular value of a matrix.

Lemma 2 For two systems (9), (10) coupled by (11), if the functions \( \pi_1(\cdot) \) and \( \pi_2(\cdot) \) in (12) satisfy
\[ \pi_1 \circ \pi_2(r) < r \quad \text{for all } r > 0, \] (13)
then the trajectories \( x_1(t) \) and \( x_2(t) \) satisfy
\[ \limsup_{t \to \infty} \| x_1(t) \| \leq \max\{\pi_1 \circ \rho_2(\eta_2), \rho_1(\eta_1)\} \] (14)
\[ \limsup_{t \to \infty} \| x_2(t) \| \leq \max\{\pi_2 \circ \rho_1(\eta_1), \rho_2(\eta_2)\} \] (15)
where \( \rho_i(\cdot) \) are defined as (8) and \( \eta_i \) satisfy (4).

Sketch of proof: By extending the time domain \([-\tau, \infty)\), this lemma can be proved in a similar way as Theorem 10.6.1 in (12).

3 Boundedness of the trajectories

3.1 Problem formulation

Consider the following two identical systems
\[ \begin{align*}
\dot{x}_i(t) &= A x_i + f(x_i, u_i) + B u_i \\
y_i(t) &= C x_i \\
x_i(\theta) &= \phi_i(\theta), \quad -\tau \leq \theta \leq 0 \\
x_i(0) &= \phi_i(0) = x_{0i}
\end{align*} \] (16)
where \( x_i \in \mathbb{R}^n \), \( y_i \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^n \), \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times n} \), \( C \in \mathbb{R}^{n \times n} \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a smooth function.

Throughout this paper, we make the following assumptions.
Assumption 1: The matrix \( C \) is nonsingular.
Assumption 2: The systems (16) are strictly semi-passive.

We assume that these two systems are bidirectionally coupled by the following inputs,
\[ u_i(t) = K (y_i(t) - y_j(t - \tau)), \] (17)
for \( i, j = 1, 2 \) and \( i \neq j \), where \( K \in \mathbb{R}^{n \times n} \) is a negative semi-definite matrix and \( \tau > 0 \) is a constant time delay. In what follows, we denote \( y_i \triangleq y_i(t - \tau) \).

We define synchronization as follows.

Definition 3 If, no matter what initial conditions are taken, the states of systems \( x_1 \) and \( x_2 \) satisfy
\[ \| x_1 - x_2 \| \to 0 \quad \text{as } t \to \infty, \]
then these systems are synchronized asymptotically.

The goal of this work is to derive conditions for the systems (16) to synchronize.

3.2 Boundedness of the trajectories

Firstly, we show that (strictly) semi-passive systems (16) coupled by (17) can be transformed into strictly semi-dissipative systems. For the systems (16), there exist a \( C^1 \)-class function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \), class-\( K \)-functions \( \alpha(\cdot), \bar{\alpha}(\cdot) \) and \( \epsilon(\cdot) \) satisfying
\[ \alpha(\| x_1 \|) \leq V(x_1) \leq \bar{\alpha}(\| x_1 \|) \] (18)
\[ \dot{V}(x_1) \leq -\epsilon(\| x_1 \|) - H(x_1) + y_1^T u_1 \] (19)
for all \( x_1 \in \mathbb{R}^n, u_i \in \mathbb{R}^m, y_i \in \mathbb{R}^m, i = 1, 2 \). In the case of \( i = 1 \), substituting (17) into (19), we obtain
\[ \dot{V}(x_1) \leq -\epsilon(\| x_1 \|) - H(x_1) + y_1^T K (y_1 - y_2) \]
\[ \leq -\epsilon(\| x_1 \|) - H(x_1) + \frac{1}{2} y_1^T K y_1 - \frac{1}{2} y_2^T K y_2 \]
\[ \leq -\sigma_{\text{max}}(\gamma_1(\sigma_{\text{max}}(C_1) \cdot r)) \] (20)
where
\[ \alpha(r) = \epsilon(r) - \frac{1}{2} \lambda_{\text{max}}(K) \sigma_{\text{min}}^2(C) r^2 \quad (r > 0). \] (21)
For \( i = 2 \), similarly we obtain
\[ \dot{V}(x_2) \leq -\sigma_{\text{max}}(\gamma_2(\sigma_{\text{max}}(C_2) \cdot r)) \] (22)
Since the matrix \( K \) is negative semi-definite, the functions \( \alpha(\cdot) \) and \( \beta(\cdot) \) in (21) belong to class-\( K_{\infty} \).

Therefore the systems (16) coupled by (17) can be seen as two strictly semi-dissipative systems with respect to \( \beta(\| y_{1r} \|) \) coupled by inputs \( v_1 = y_2 \) and \( v_2 = y_{1r} \).
As a consequence, using Lemma 2 for semi-dissipative systems, we obtain the following theorem.

Theorem 1 Define a class-\( K \) function as \( \pi(r) \triangleq \gamma(\sigma_{\text{max}}(C) \cdot r) \) for \( r \geq 0 \), where \( \gamma(\cdot) \) is defined as (8). If the function \( \pi(\cdot) \) satisfies
\[ \pi(r) < r \quad \text{for all } r > 0, \] (23)
then the trajectories of the systems (16) converge to the bounded set
\[ \Omega = \{ x \in \mathbb{R}^n \mid \| x \| \leq \rho(\eta) \}. \] (24)
\[\Sigma \Delta \left[ \begin{array}{ccc} PM_1 + M_1^T P + Y + Y^T + Q & PM_2 - Y + W^T & -\tau Y \\ -\tau Y^T & -\tau W & \tau M_2^T Z \\ M_2^T P - Y^T + W & -\tau M_2^T Z \\ \tau Z M_1 & -\tau W & -\tau Z \end{array} \right] < 0 \] (A)

\[M_1 = A + BK C + D(x_2), \quad M_2 = BK C \]

\[\zeta(t, \alpha) = [e(t)^T \quad e(t - \tau)^T \quad \dot{\epsilon}(\alpha)^T]^T \]

\[\Lambda = \left[ \begin{array}{cccc} PM_1 + M_1^T P + Y + Y^T + \tau M_2^T Z M_1 + Q & PM_2 - Y + W^T + \tau M_2^T Z M_2 & -\tau Y \\ -\tau Y^T & -\tau W & \tau M_2^T Z M_1 \\ M_2^T P - Y^T + W + \tau M_2^T Z M_1 & -\tau W & -\tau Z \end{array} \right] \] (B)

Proof: (23) implies that (13) holds. Therefore, using Lemma 2 and (23), the trajectories of the systems (16) satisfy

\[\limsup_{t \to \infty} \|x_i(t)\| \leq \max \{\pi \circ \rho(\eta), \rho(\eta)\} = \rho(\eta). \quad (25)\]

This means the trajectories converge to \(\Omega\). \(\square\)

4 Synchronization

Using the above results, we derive a sufficient condition for synchronization of coupled systems.

The dynamics of the error \(e(t) \triangleq x_1(t) - x_2(t)\) is given by

\[\dot{e} = (A + BK C) e + BK C e_r + \phi(e, x_2), \quad (26)\]

where \(\phi(e, x_2) = f(e + x_2) - f(x_2)\). If the error system (26) has \(e = 0\) as an asymptotically stable equilibrium, the two systems (16) synchronize.

Linearization of (26) around the origin is given by

\[\dot{e} = (A + BK C + D(x_2)) e + BK C e_r, \quad (27)\]

where

\[D(x_2) \triangleq \left( \frac{\partial \phi(e, x_2)}{\partial e} \right) \bigg|_{e=0}. \quad (28)\]

It is well known that if \(e = 0\) of (27) is asymptotically stable, then \(e = 0\) of (26) is also asymptotically stable \(^9\). Considering the stability of (26), we obtain the following theorem.

Theorem 2 Assume \(\Omega \subset \mathbb{R}^n\) is a nonempty bounded set where the trajectories of the systems (16) converge to. For all \(x_2 \in \Omega\), if there exist positive definite matrices \(P, Q, Z \in \mathbb{R}^{n \times n}\) and matrices \(Y, W \in \mathbb{R}^{n \times n}\) satisfying LMI (A) at the top of this page, then \(e = 0\) of (26) is asymptotically stable.

Proof: For the linearized error system (27), we consider the following Lyapunov-Krasovskii functional \(^{10}\),

\[V(e_1) = e(t)^T P e(t) + \int_{-\tau}^{\tau} \dot{\epsilon}(\alpha)^T Z \dot{\epsilon}(\alpha) d\alpha + \int_{t-\tau}^{t} e(\alpha)^T Q e(\alpha) d\alpha \]

Differentiating \(V(e_1)\) along the solution of (27), we obtain

\[\dot{V}(e_1) = \frac{1}{\tau} \int_{-\tau}^{\tau} \zeta(t, \alpha)^T \Lambda \zeta(t, \alpha) d\alpha, \quad (30)\]

where \(\zeta(t, \alpha)\) and \(\Lambda\) are defined as (B) at the top of this page.

Since the linear matrix inequality (A) implies \(\Lambda < 0\) from the Schur complement equivalence, we obtain

\[\dot{V}(e_1) < -a \|e(t)\|^2 \quad (31)\]

where \(a = \lambda_{\text{min}}(-\Lambda) > 0\). Thus, by the Lyapunov-Krasovskii stability theorem \(^{11}\), we can conclude that \(e = 0\) of (26) is asymptotically stable. \(\square\)

5 Example

Consider the following Lorenz systems coupled by (17) with \(B = C = I\) and \(\tau = 0.01\).

\[\begin{align*}
\dot{x}_{i1}(t) &= \sigma(x_{i2} - x_{i1}) + u_{i1} \\
\dot{x}_{i2}(t) &= r x_{i1} - x_{i2} - x_{i1} x_{i3} + u_{i2} \\
\dot{x}_{i3}(t) &= -b x_{i3} + x_{i1} x_{i2} + u_{i3}
\end{align*}\]

where \(\sigma = 10, \quad r = 28, \quad b = 8/3, \) and set coupling gain \(K\) as \(K = \text{diag}\{-k - k - k\}\) for \(k > 0\). We consider a storage function \(V(\tilde{x}_i) = \frac{1}{2} \tilde{x}_i^T \tilde{x}_i\), where \(\tilde{x}_i = [x_{i1} \quad x_{i2} \quad x_{i3} - \sigma - r]^T\). Then the derivative along the trajectory of (32) is given by

\[\dot{V}(\tilde{x}_1) = -\alpha \|\tilde{x}_1\|^2 - H(\tilde{x}_1) + \beta(\|\tilde{x}_2\|)\]

for \(i = 1\), where the functions \(\alpha(\cdot), \beta(\cdot)\) and \(H(x)\) are defined at the top of the next page and \(0 < \epsilon < 1\.

Then \(\gamma(\cdot)\) in (8) is given by

\[\gamma(r) = \sqrt{\frac{k}{2} + r} \quad (33)\]
\[ H(\tilde{x}_i) = (\sigma - \epsilon)\tilde{x}_{1i}^2 + (1 - \epsilon)\tilde{x}_{2i}^2 + (b - \epsilon)\left(\tilde{x}_{13} - \frac{b - 2\epsilon}{2(b - \epsilon)}(\sigma + r)\right)^2 - \frac{b^2(\sigma + r)^2}{4(b - \epsilon)} \]

\[ \alpha(||\tilde{x}_i||) = \left(\frac{k}{2} + \epsilon\right)||\tilde{x}_i||^2, \quad \beta(||\tilde{x}_i||) = \frac{k}{2}||\tilde{x}_i||^2 \quad (i = 1, 2) \]

\(\rho(\cdot)\) is an identity map. So the bounded set is given by \(\Omega = \{ x \in \mathbb{R}^3 ||x|| \leq 39.4 \} \). This means that the trajectories \(x_1(t)\) and \(x_2(t)\) converge to the set \(\Omega = \{ x \in \mathbb{R}^3 (x_1^2 + x_2^2 + (x_3 - \sigma - r)^2)^{\frac{1}{2}} \leq 39.4 \}. \) (34)

While, the error dynamics for these systems is given as follows.

\[ \dot{e}(t) = \begin{bmatrix} -\sigma - k & \sigma & 0 \\ r & -1 - k & 0 \\ 0 & 0 & -b - k \end{bmatrix} e + \phi(e, x_2) + Ke_\tau \]

where

\[ \phi(e, x_2) = \begin{bmatrix} 0 \\ -e_1e_3 - x_21e_3 - x_23e_1 \\ e_1e_2 + x_21e_2 + x_22e_1 \end{bmatrix}. \]

The error dynamics linearized around \(e = 0\) is given by

\[ \dot{e}(t) = \begin{bmatrix} -\sigma - k & \sigma & 0 \\ r - x_23 & -1 - k & -x_21 \\ x_22 & x_21 & -b - k \end{bmatrix} e + Ke_\tau. \] (35)

Setting \(k = 6\), LMI condition (A) for (35) holds for all \(x_2 \in \Omega\).

Fig.1 shows the behavior of system (32). In this figure, the ball illustrates the estimated boundary of the set \(\Omega\) given by (34). We know that the trajectories converge to the set \(\Omega\). Fig.2 shows the behavior of the error \(e(t) = x_1(t) - x_2(t)\). We know that the error converges to zero, which means the synchronization of these systems is accomplished.

6 Conclusion
In this paper, we have considered a condition for synchronization of two nonlinear oscillators coupled by a bidirectional diffusive coupling with time-delay. By regarding semi-passive systems coupled by diffusive coupling with time-delay as semi-diffusive systems coupled by delayed outputs, we derived a condition for boundedness of the systems by the small-gain theorem. Under the obtained condition, we derived a sufficient condition for synchronization of the systems by the Lyapunov-Krasovskii theorem.

References