Dynamic buckling of a shallow arch under shock loading considering the effects of the arch shape

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Abstract

In this paper, the influence of the initial curvature of thin shallow arches on the dynamic pulse buckling load is examined. Using numerical means and a multi-dof semi-analytical model, both quasi-static and non-linear transient dynamical analyzes are performed. The influence of various parameters, such as pulse duration, damping and, especially, the arch shape is illustrated. Moreover, the results are numerically validated through a comparison with results obtained using finite element modeling. The main results are firstly that the critical shock level can be significantly increased by optimizing the arch shape and secondly, that geometric imperfections have only a mild influence on these results. Furthermore, by comparing the sensitivities of the static and dynamic buckling loads with respect to the arch shape, non-trivial quantitative correspondences are found.

Keywords: Dynamic pulse buckling; Semi-analytical models; Shape optimization

1. Introduction

Thin-walled structures possess a favorable stiffness-to-mass ratio and are encountered in a wide variety of applications. If such a thin-walled structure is initially curved and is transversally loaded above some critical value, the structure may buckle so that its curvature suddenly reverses. This behavior, also known as snap-through buckling [1], is often undesirable. The analysis of structures liable to buckling under static loading is a well established topic in engineering science. However, often thin-walled structures are subjected not only to a static load but also to a distinct dynamic load, such as shock/impact loading, step loading or periodic loading. The resistance of structures liable to buckling, to withstand time-dependent loading is often addressed as the dynamic stability of these structures.

The corresponding failure mode is often addressed as dynamic buckling and more specifically as dynamic pulse buckling [2] for the case of pulse loading. Generally, dynamic buckling is related to a large increase in the response resulting from a small increase in some load parameter [3–5]. In the past, many studies have already been performed concerning the dynamic stability of thin-walled structures. Design strategies for such structures under dynamic loading are, however, still lacking. The research described in this paper is intended as a (first) step in deriving such design strategies and deals with thin shallow arches under shock loading.

The dynamic stability problem of structures can be studied by following an energy based approach [6–8], a numerical approach [9–11] or an experimental approach [12–14]. The energy based approach allows to determine a lower bound for the dynamic buckling load without solving the non-linear equations of motion. However, the established lower bound for the dynamic buckling load by the energy approach can be very conservative [15,16]. Furthermore, the energy based approach does not allow to include the effect of damping rigorously, whereas little damping, as present in all real-life structures, can have a significant effect on the dynamic buckling load [9–11].
Dynamic buckling loads for arches with different shapes are earlier compared in [15,17]. In [15], dynamic pulse buckling loads of a circular shaped arch and a sinusoidal shaped arch are compared using an energy based approach and no major differences were found. In [17], the dynamic stability criterion based on energy considerations appeared to be sensitive to the amplitude of the second harmonic in the arch shape and insensitive to the amplitude of the third harmonic in the arch shape. In [18], it is found that for pinned–pinned shallow arches, a circular arch shape is almost optimal with respect to static buckling due to a transversal distributed load. The effects of shape variation on the static buckling of arches subjected to a sinusoidally distributed load are examined in [19] and for arches subjected to a concentrated point load at the center in [19,20]. In [14,21,22], dynamic snap-through of arches is considered due to an axial impact load, due to a prescribed axial motion and for a moving transversal point load, respectively. Shock loaded double-curved shells are considered in [23].

In this paper, a group of thin shallow arches subjected to a short shock-load is examined using numerical means. All arches have the same dimensions like width, height and cross-sectional area but differ in shape. The shock loading of the arch is modelled by a prescribed transversal acceleration of the end points of the arch. The main goal of the research is to study the dynamic pulse buckling of the arch for a wide range of shapes, such that the influence of these shape variations on critical shock loads can be determined. The outline for the paper as follows. The next section will deal with the derivation of the equations of motion. In Section 3, buckling of the arch under a quasi-static acceleration loading will be discussed. The influence of the arch shape and initial imperfections will be illustrated and results will be compared with FEM results. Dynamic buckling of the arch will be discussed in Section 4. The influence of the arch shape, small geometric imperfections, the level of damping and the shock-pulse duration on the critical shock magnitudes will be examined. Furthermore, the sensitivities of the static buckling loads and dynamic pulse buckling loads with respect to the arch shape will be compared. Finally, in Section 5 conclusions will be presented.

2. Modeling of the arch

The steel arch (see Fig. 1a) has a thickness \(d\), a width \(z\), an initial height \(h\) at the center, a span-width \(L\) and a cross-section with area \(A = zd\) and an area moment of inertia \(I = zd^3 / 12\). The initial (undeformed) shape of the arch is indicated by \(w_0(x, t)\), the shape after (elastic) deformation by \(w(x, t)\) and the axial displacement by \(v(x, t)\) (where \(x\) denotes the axial coordinate and \(t\) denotes time). All geometrical and material properties are considered to be constant over the arch length and are fixed to the values as shown in Table 1 [24]. The internal normal force \(N\) and moment \(M\) in the arch are defined by \(N = EAe\) and \(M = EI\kappa\), with

\[
\begin{align*}
\varepsilon &= v_{,x} + \frac{1}{2} ((w_{,x})^2 - (w_{0,x})^2), \\
\kappa &= -(w_{,xx} - w_{0,xx}),
\end{align*}
\]  
(1)

where \((,\cdot)\) denotes \(\partial / \partial x\). The non-linear kinematic model Eq. (1) is valid for shallow curved, slender beams and moderate displacements [19]. In order to model the shock loading, via a prescribed transversal acceleration, the arch is considered to be pinned–pinned to a movable frame (see Fig. 1b). The boundary conditions for this load-case read as

\[
\begin{align*}
v(0, t) &= v(L, t) = 0 \text{ (m)}, \\
M(0, t) &= M(L, t) = 0 \text{ (N m)}, \\
w(0, t) &= w_0(0, t) = y_p(t) \text{ (m)}, \\
w(L, t) &= w_0(L, t) = y_p(t) \text{ (m)}.
\end{align*}
\]  
(2)

The prescribed transversal acceleration \(\ddot{y}_p(t)\) (where \((\cdot)\) denotes \(\partial^2 / \partial t^2\)) results in a loading equivalent to the uniformly distributed transversal loading as, for example, considered in [18,19]. Damping in the arch is considered as a uniformly distributed viscous force in transversal direction only

\[
F_d = -\beta \dot{w} \ddot{w}_0 \text{ (N/m)}.
\]  
(3)

Under the assumption that rotary and axial inertia terms are negligible with respect to the transversal inertia, the kinetic energy of the arch equals

\[
\mathcal{T} = \frac{1}{2} \rho A \int_0^L \dot{w}^2 \, dx.
\]  
(4)

Table 1

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EI)</td>
<td>0.232</td>
<td>(N m²)</td>
</tr>
<tr>
<td>(h)</td>
<td>38.4 \times 10^{-3}</td>
<td>(m)</td>
</tr>
<tr>
<td>(L)</td>
<td>0.8315</td>
<td>(m)</td>
</tr>
<tr>
<td>(d)</td>
<td>0.803 \times 10^{-3}</td>
<td>(m)</td>
</tr>
<tr>
<td>(A)</td>
<td>2.056 \times 10^{-5}</td>
<td>(m²)</td>
</tr>
<tr>
<td>(\rho)</td>
<td>7850</td>
<td>(kg/m³)</td>
</tr>
</tbody>
</table>

Fig. 1. (a) Arch geometry, (b) pinned–pinned arch with prescribed transversal end-point motion.
Since no axial inertia or axial damping forces are considered, it follows that $N_{,x} = 0$ [8]. Using this fact and $v(0, t) = v(L, t)$, the potential energy may be expressed in terms of $w(x, t)$ solely [8]

$$\mathcal{V} = \frac{N^2 L}{2EA} + \frac{1}{2} \int_0^L (M\dot{\kappa}) \, dx,$$

where

$$N = \frac{EA}{2L} \int_0^L [(w,_{x^2}) - (w,_{0x^2})^2] \, dx.$$

Note that gravity forces are not taken into account (the arch moves in the horizontal plane).

### 2.1. Initial shape

In order to be able to study the effect of shape variations, the curvature of the initial (symmetrical) arch shape is parameterized with a single shape parameter $a$ ($a/h < \frac{1}{8}$, so maximum height at $x = L/2$). For a fair comparison, the shape parametrization leaves the initial height of the arch unchanged $(w_0(L/2, \cdot) = y_p(\cdot) = h)$. Moreover, an imperfection in the form of a small asymmetry with amplitude $e$ is incorporated. In order to study the effect of shape variations, the single mode imperfection shape, a polynomial function is chosen to describe the asymmetry. The parametrization of the arch shape, including the (prescribed) transversal movement $y_p(t)$ of the end points of the arch, reads

$$w(x, t) = (h + a) \sin \left( \frac{\pi x}{L} \right) + a \sin \left( \frac{3\pi x}{L} \right) + e \left[ \frac{36}{L^2 \sqrt{3}} x(x - L/2)(x - L) \right] + y_p(t).$$

The initial shape and imperfection shape are illustrated in Fig. 2. As a reference, the shape-factor $a$ resembling a circular arch shape the most (in a least squares sense), is computed to be $a/h = 0.03649$.

### 2.2. Discretization and equations of motion

In order to approximate the continuous problem with a discrete set of equations of motion, the field $w(x, t)$ is discretized as

$$w(x, t) = w_0(x, t) + \sum_{i=1}^n Q_i(t) \sin \left( \frac{i\pi x}{L} \right),$$

satisfying the boundary conditions Eq. (2) a priori. The equations of motion in terms of the generalized coordinates $Q = [Q_1(t), Q_2(t), \ldots, Q_n(t)]^T$ are derived by following a Rayleigh–Ritz approach [25]. Here, the energy integrals Eqs. (4) and (5) are evaluated after substitution of Eqs. (7) and (8). The non-conservative forces due to the viscous damping force Eq. (3) are taken into account by using the following Rayleigh dissipation function [25]

$$\mathcal{R} = \frac{b}{2} \int_0^L (\ddot{w} - \ddot{w}_0)^2 \, dx,$$

which is also evaluated after substitution of Eqs. (7) and (8). Finally, the equations of motion are determined with Lagrange's, equations

$$\frac{d}{dt} \mathcal{F} \cdot Q - \mathcal{F} \cdot \dot{Q} + \mathcal{V} \cdot Q = \mathcal{Q}_{nc},$$

where $(\mathcal{F} Q)$ denotes $\partial/\partial Q$ and $\mathcal{Q}_{nc} = -\mathcal{R} \dot{Q}$. The equations corresponding to the 6-dof ($n = 6$ in Eq. (8)) approximation are given by

$$M \ddot{Q} + C \dot{Q} + K(Q) = -B\ddot{y}_p(t),$$

where $M = (\rho AL/2)I$, $C = (bL/2)I$ with $I$ the identity matrix, $B = (2\rho AL/\pi)[1 \ 0 \ 0 \ 0 \ 0 \ 0]^T$ and

$$K(Q) = \begin{bmatrix}
\frac{\pi^2}{2L} N(Q)(a + h + Q_1) + \frac{6E \pi^4}{L^3} Q_1 \\
\frac{2}{L^2} N(Q)(18e\sqrt{3} + \pi^2 Q_2) + \frac{8E \pi^4}{L^3} Q_2 \\
\frac{2}{L^2} N(Q)(9e\sqrt{3} + 4\pi^2 Q_3) + \frac{128E \pi^4}{L^3} Q_3 \\
\frac{6}{\pi L} N(Q)(2e\sqrt{3} + 3\pi^2 Q_6) + \frac{648E \pi^4}{L^3} Q_6 
\end{bmatrix},$$

where

$$N(Q) = \frac{E A \pi^2}{4L^2} \left[ Q_1(2a + 2h + Q_1) + 18a Q_3 \\
+ \frac{144\sqrt{3}}{\pi^3} e \left( Q_2 + \frac{1}{2} Q_4 + \frac{1}{3} Q_6 \right) \\
+ 4 Q_5^2 + 9 Q_3^2 + 16 Q_4^2 + 25 Q_5^2 + 36 Q_6^2 \right].$$

With the adopted discretization Eq. (8), $Q = 0$ represents the (undeformed) initial shape. As can be noted, coupling of the individual modes is only attained via the non-linear stiffness terms. Moreover, the asymmetrical modes $Q_2, Q_4, \ldots$ are not excited directly by the loading and are only triggered if $e/d \neq 0$ (assuming the initial conditions equal $Q(0) = \dot{Q}(0) = 0$).

### 3. Static buckling

First static buckling of the arch under a constant (time-invariant) acceleration $\ddot{y}_p(t) = P$ (m/s²) is investigated. For this analysis, the evolution of static equilibrium points of Eq. (11), described by

$$K(Q) = -BP$$

(12)
is studied for a quasi-static acceleration $P$. The equilibrium path or ‘load-path’ is computed using a pseudo-arc-length continuation scheme. All presented load-paths are characterized by the following scalar measure:

$$W_{mid}(t) = \frac{w(L/2, t) - w_0(L/2, t)}{\delta h},$$  \hspace{1cm} (13)

where $\delta h$ is the distance between the unloaded upward stable equilibrium position and the unloaded downward stable equilibrium position measured at the mid-point (note that $\delta h$ depends on $a$). Stability of the equilibrium states is assessed by evaluating the eigen-values of $dK(Q)/dQ$.

The load-path of the arch with $a/h = e/d = 0$, using the 6-dof model Eq. (11), is depicted in Fig. 3a. Starting at the unloaded initial state ($W_{mid} = 0$), the slope of the obtained load-path for the quasi-static increasing load $P$ varies and $P$ reaches a maximum, i.e. the limit-point $LP$ \cite{1} (the corresponding load is denoted with $P_{LP}$). Due to the fact that all symmetric modes are directly forced by the loading (see Eq. (11)), the arch shape will change during loading, see Fig. 3b. At a significantly lower load than the limit-point load $P_{LP}$, the initial load-path loses stability at the bifurcation point $B$ \cite{1} (the corresponding load is denoted with $P_B$). In the secondary load-path, which bifurcates from the initial load-path at point $B$, the first harmonic asymmetric arch shape becomes dominant. Three load-paths for the situation where $e/d \neq 0$, are depicted in Fig. 4a. The deformed arch shapes for $e/d = 0.5$ during loading are shown in Fig. 4b. Clearly, the geometric imperfection initiates asymmetric deformations and introduces a new limit-point in the load-path. For $e/d \to 0$, the location of the limit-point for $e/d \neq 0$ tends to the location of the bifurcation point $B$ as found for $e/d = 0$ (indicated in Fig. 4a with $\ast$). In applications the arch will
Table 2
Convergence bifurcation load ($P_B$) and limit-point load ($P_{LP}$) for $a/h = e/d = 0$

<table>
<thead>
<tr>
<th>n (Eq. (8))</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_B$ (m/s²)</td>
<td>35.323</td>
<td>35.219</td>
<td>35.219</td>
<td>35.219</td>
</tr>
<tr>
<td>$P_{LP}$ (m/s²)</td>
<td>$2.3 \times 10^4$</td>
<td>58.913</td>
<td>58.908</td>
<td>58.907</td>
</tr>
</tbody>
</table>

never be purely symmetric. Consequently, if in practise the arch would be quasi-statically loaded, the arch will exhibit snap-through buckling to a downward configuration ($W_{mid} \approx -1$) at a load-level close to $P_B$ (depending on the actual imperfection) and, via an asymmetrical buckling mode.

The influence of the retained number of dof’s in the semi-analytical model on the bifurcation load $P_B$ and the limit-point load $P_{LP}$ is shown in Table 2. The results for the semi-analytical model do not change dramatically if more than 6 dofs are used. As shown for the 6-dof model in Fig. 3b, the deformed arch shape shows clearly the presence of the third harmonic. The 2-dof model does not include this mode, resulting in a highly overestimated $P_{LP}$, see Table 2.

The semi-analytical model is based on a number of assumptions. It does not include the effects of shear, the effects of axial and rotary inertia and some higher order terms are neglected in Eq. (1) due to the shallowness assumptions [19]. For validation of these assumptions, the results for the semi-analytical model with 6-dof are compared with results obtained using a FE model which includes these effects, see Fig. 5. In this figure, the load-paths for two arches are compared; one for the perfect arch ($e/d = 0$) to validate the limit-point load for the perfect arch ($LP$ in Fig. 3a) and one for an arch with a small geometric asymmetry ($e/d = 0.1$) to validate the limit-point load for the imperfect arch ($LP_{e/d=0.1}$ in Fig. 4a). Both the FE model for the perfect arch and the FE model for the imperfect arch consist of 20 three-node Timoshenko beam elements known as element type 45 [26]. In all FEM analyzes kinematic relations are used which are valid for large displacements and moderate rotations.

The good agreement between the FEM results and the semi-analytical results supports the assumptions made for the semi-analytical model. It is noted that the negligible effect of shear also follows from the fact that FE analyzes based on the less sophisticated Euler/Bernoulli beam theory (which corresponds closer to the adopted kinematics given by Eq. (1)), show practically the same results.

As noted the bifurcation-buckling load $P_B$ for $e/d = 0$ dominates the static stability behavior. Unfortunately, the bifurcation load $P_B$ can hardly be influenced by varying the arch shape, as illustrated in Fig. 6. The bifurcation load can be increased by 6% (with respect to $a/h = 0$) by setting the shape-factor to $a/h = 0.097$. For comparison with the results for dynamic pulse buckling in the next section, also the sensitivity of the secondary buckling load $P_{LP}$ for $e/d = 0$ with respect to the arch shape parameter $a$ is examined. The limit-point load $P_{LP}$ shows a distinct maximum at which the corresponding snap-through mode switches between the $w$-shape and the $m$-shape. With respect to the arch with $a/h = 0$, $P_{LP}$ can be increased by 40% by setting the shape-factor to $a/h = 0.0384$ (a shape very close to the circular arch, see Section 2.1). However, observing the results of the quasi-static analysis, this secondary buckling load has no practical relevance.

4. Dynamic buckling

In this section, dynamic buckling of the arch under shock loading is examined. First the loading and the approach used to analyze the dynamic buckling of the arch are discussed. Next, the dynamic buckling of both the perfect arch and the imperfect arch are examined for various parameters. Finally, for generalization of the results, some arches with different dimensions are considered.
4.1. Loading and approach

The shock loading of the arch is modelled by a half-sine acceleration pulse, characterized by the pulse duration $T_p$ and the maximum acceleration $P$

$$\ddot{y}_p(t) = \begin{cases} P \sin \left( \frac{\pi t}{T_p} \right) & \text{if } 0 \leq t \leq T_p, \\ 0 & \text{if } t > T_p. \end{cases}$$  \hspace{1cm} (14)

After the arch is briefly loaded by the shock-load pulse, the arch is no longer subjected to external forces. Assuming that $\mathbf{Q}(t) = 0$ (no deformations occur) during the short interval $0 \leq t \leq T_p$, the only forces acting on the arch during this interval are the inertia forces due to the prescribed acceleration $\ddot{y}_p(t)$. Assuming $\mathbf{Q}(0) = 0$, the velocities just after the shock-pulse can be determined by using the impulse-momentum theorem, i.e.

$$\mathbf{Q}(T_p) = M^{-1} \int_0^{T_p} B \ddot{y}_p(t) \, dt = M^{-1} B(2/\pi) \lambda$$

where $\lambda$ is called the dynamic buckling load [3,4]. In order for various values of the load parameter, $W$, to be determined by using the impulse-momentum theorem, i.e. $\mathbf{Q}(T_p) = 0$, the velocities just after the shock-pulse can be determined by using the impulse-momentum theorem, i.e. $\mathbf{Q}(T_p) = M^{-1} \int_0^{T_p} B \ddot{y}_p(t) \, dt = M^{-1} B(2/\pi) \lambda$ with only a single load parameter

$$\lambda = PT_p \text{ (m/s).}$$  \hspace{1cm} (15)

Under the considered assumptions, the shock is imparted instantaneously into the structure as kinetic energy only. Relating this amount of kinetic energy to the level of potential energy at some saddle equilibrium point and by neglecting the effect of damping, a lower bound for the dynamic pulse buckling load can be derived, see for example [7,8,27,28]. However, application of the energy approach is not trivial for multi-dof systems (such as considered in this paper) and the established lower bound for the dynamic buckling load by the energy approach can be very conservative [15,16]. Therefore, here the critical shock loads are determined using the equations of motion approach [8]. The equations of motion are (numerically) solved for various values of the load. The load at which there exist a sudden jump in the response for small variation of the load parameter, is called the dynamic buckling load [3,4].

In order to evaluate the time response, some scalar measure must be chosen. Here, the following measure is adopted:

$$\tilde{\mathbf{W}}_{\text{mid}} = \max_{0 \leq t \leq T} |W_{\text{mid}}(t)|,$$  \hspace{1cm} (16)

with $W_{\text{mid}}(t)$ defined by Eq. (13). The parameter $P$ is selected as load parameter to be varied. However, various (fixed) pulse durations $T_p$ will be considered. The dynamic pulse buckling load is denoted with $P_p$. For the numerical integration of Eq. (11) in combination with Eq. (14), an integration routine based on an eighth order Runge–Kutta scheme with automatic step-size control is used. For all results a relative tolerance of TOL = 1 $\times$ 10$^{-8}$ is used.

Given the fact that little damping is taken into account, the energy imparted by the shock pulse decays. Since a certain amount of energy is required for escape from the initial well, the time-span for dynamic buckling to occur is limited. Similar as in [16] and for the considered levels of damping, numerical simulations show that the occurrence for dynamic buckling of the pulse loaded arch (practically) always takes place in the time-span $0 \leq t \leq 3T_1$, where $T_1$ is the period corresponding to the lowest eigen-frequency $f_1$. This time-span is, therefore, used for the numerical integration in all results, unless stated otherwise. Note that the time-span for dynamic buckling may become longer for lower levels of damping and that it can become considerably longer for step-loaded structures (also with small damping) [5,13,29].

For the arch under consideration, the period corresponding to the lowest eigen-frequency for $a/h = 0$ and $e/d = 0$ appears to be $T_1 \approx 0.1$ (s), see Table 3. Note that the first eigen-frequency of the arch remains nearly constant for the considered range of the shape-factor $a$, see also [18,30]. The eigen-frequencies are determined by linearizing Eq. (11) around the unloaded upward configuration ($\mathbf{Q} = 0$) and are in good correspondence with FEM results, see Table 3.

The damping ratios of the first two linear vibrational eigen-modes (see Table 3) for the considered values of the viscous damping parameter $b$, are listed in Table 4.

<table>
<thead>
<tr>
<th>$b$ (N s/m$^2$)</th>
<th>1</th>
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<th>4</th>
</tr>
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<tbody>
<tr>
<td>$f_1$ (Hz)</td>
<td>10.88</td>
<td>0.0453</td>
<td>0.0905</td>
</tr>
<tr>
<td>$f_2$ (Hz)</td>
<td>24.51</td>
<td>0.0201</td>
<td>0.0402</td>
</tr>
</tbody>
</table>

Table 4 Damping ratios of eigen-modes as listed in Table 3, for several values of $b$

4.2. Perfect arch ($e = 0$)

As a reference, first the results for arches without an imperfection ($e/d = 0$) are discussed. For this case, the asymmetrical modes ($Q_2, Q_4, Q_6$) are not triggered (see Section 2.2) and are, therefore, removed from Eq. (11). First the influence of damping is discussed for a pulse duration of $T_p = 10$ (ms) and $a/h = 0.04$. As illustrated in Fig. 7, increasing the amount of damping results in an increasing dynamic pulse buckling load [9–11]. The dynamic pulse buckling loads ($P_p$) can clearly be distinguished by the sudden jumps in the graphs. However, for the case with the lowest level of damping ($b = 1$ (N s/m$^2$)), the boundary between the region where no dynamic buckling occurs and the region where dynamic buckling does occurs, is not indicated by a single sudden jump in the response measure, but by a small transition region. In this region the occurrence of dynamic buckling is extremely sensitive to small variations in the load parameter as illustrated in Fig. 8. Note that the time-span in Fig. 8 is extended from $T = 0.3$ ($\approx 3T_1$) to 2.0 (s) and the sensitivity is thus not due to a too short integration time. Furthermore, the complex transitions are also
found using a model with three additional symmetrical modes ($Q_7 \sin(7\pi x/L)$, $Q_9 \sin(9\pi x/L)$ and $Q_{11} \sin(11\pi x/L)$), see Fig. 9. Indeed, the sudden jumps do not occur exactly at the same place. However, the differences between the load values where the sudden jumps occur do not differ more than 3% and qualitatively the transitions are similar. It is noted that the complex transition only occurs in the region around $a/h \approx 0.05$ (which will appear to be approximately the optimal shape of the arch with respect to dynamic pulse buckling) and only for a low level of damping (see also Fig. 7). Similar load-parameter sensitivities in transient analyses are reported in [29,31,32].

The influence of the arch shape on the dynamic buckling load is illustrated in Fig. 11 for various pulse durations and $b = 2$ (N s/m²). Clearly, the shape of the arch has a distinct influence on the dynamic pulse buckling load for the perfect arch (the dynamic buckling load increases approximately 50% by changing the arch shape from $a/h = 0$ to 0.04). The arch shape during dynamic buckling is illustrated in Fig. 10. Similar as the secondary static buckling mode (see $P_{LP}$ in Fig. 6), the arch buckles for $a/h = 0$ via a $w$-shape and for $a/h = 0.08$ via an $m$-shape. Furthermore, in Fig. 11 the dynamic buckling loads are presented in terms of the product $PT_p = \lambda$ (see Eq. (15)) allowing to examine the mutual relation between $P$ and $T_p$ with respect to the dynamic pulse buckling load. As can be noted, the stability boundaries for the various pulse durations do not coincide perfectly and, therefore, if the parameters $P$ and $T_p$ are varied independently, the dynamic pulse buckling load does not scale exactly with the parameter $\lambda$. Still, the results for the various pulse durations match qualitatively, that is,
for each value of $T_p$ the dynamic buckling load shows a comparable sensitivity to variations in the arch shape.

The optimal arch shape with respect to the secondary static buckling load $P_{LP}$ (a/h ≈ 0.04, see Fig. 6) and the optimal arch shape with respect to dynamic pulse buckling load $P_p$ (a/h ≈ 0.05, see Fig. 11) do not exactly match. Nevertheless, there seems to be a correspondence between the sensitivities of these two critical loads with respect to the arch shape parameter $a$. Although the (symmetric) buckling mode corresponding to the secondary static buckling $P_{LP}$ and the (symmetric) deformations occurring during the dynamic pulse buckling for $e/d = 0$ show similarities, the correspondence in sensitivity of these two critical loads with respect to $a$ is not trivial. After all, in the case of dynamic pulse buckling, transient inertia and damping forces are taken into account which are absent in the quasi-static buckling case.

4.3. Imperfect arch ($e \neq 0$)

Next, dynamic buckling of imperfect arches will be examined. For this analysis, the complete 6-dof model Eq. (11) will be used, since asymmetrical deformations will occur for $e/d \neq 0$. In Fig. 12, the influence of a small imperfection on the dynamic buckling load is illustrated. The dynamic pulse buckling load shows a mild sensitivity to small geometric imperfections. The arch shape during dynamic buckling for $P$ just exceeding $P_p$ is illustrated in Fig. 13. Clearly, for $e/d \neq 0$ the arch shape shows asymmetric deformations during dynamic buckling.

The influence of the arch shape on the dynamic pulse buckling load for $e/d = 0$ and $e/d = 1$ is compared in Fig. 14. For the depicted range of the shape-factor $a$, the dynamic buckling load decreases only with maximum 10% for the moderate imperfection of $e/d = 1$ (compared to the case $e/d = 0$). The distinct maximum in the dynamic buckling load as found around $a/h \approx 0.05$ seems insensitive to the geometric imperfection. Similar stability boundaries, also with a maximum close to $a/h \approx 0.05$, are found at the lower level of damping $b = 1$ (N s/m²). Consequently, also for the practical situation where $e/d \neq 0$, the arch shape has a distinct influence on the dynamic buckling load. Furthermore, since the dynamic stability boundaries do not change dramatically due the presence of imperfections they show again a clear qualitative correspondence with the secondary static buckling load for the perfect arch $P_{LP}$ (see Fig. 6). For the imperfect arch this correspondence is even less expected as for the perfect case, since now the arch buckles dynamically via an asymmetric mode, whereas the secondary static buckling corresponds to a symmetrical buckling mode. This correspondence, however, could be very useful for shape parameter sensitivity studies based on quasi-static analyzes. After all, compared to the non-linear transient
dynamic buckling analysis, the quasi-static analysis is computationally significantly less expensive. However, more research on the found correspondence will be necessary before it can be generalized.

For validation, dynamic pulse buckling results computed with the 6-dof semi-analytical model (Eq. (11)) are compared with FEM results. For the non-linear dynamical transient FE analyzes, the same FE models are used as discussed in Section 3. However, now obviously also inertial and damping forces are taken into account. In the FE model, viscous damping is introduced via Rayleigh damping, i.e. the damping matrix is composed as \( C = \alpha K + \beta M \), where \( K \) and \( M \) are the (linear) stiffness and consistent mass matrix of the FE model, respectively. Obviously, the Rayleigh damping model is different from the damping model (Eq. (3)) as incorporated in 6-dof model (Eq. (11)). In order to obtain a comparable level of damping in the FE model, \( \alpha \) and \( \beta \) are tuned [33] so that \( \tilde{\zeta}_{1,2} \) (the damping ratios of the first two vibrational eigen-modes) of the FE model are equal to \( \tilde{\zeta}_{1,2} \) of the 6-dof model for \( b = 1 \) (Ns/m²) (see Table 4) resulting in \( \alpha = 9.85 \times 10^{-7} \) (s) and \( \beta = 6.092 \) (s⁻¹). Both models predict a slight increase in the dynamic pulse buckling load if a small imperfection \( e/d = 0.1 \) is incorporated in the arch shape, see Table 5. The responses in time for values of \( P \) near the dynamic pulse buckling load for \( e/d = 0.1 \) are compared in Fig. 15. Note that in the FE model, next to shear also rotary and axial inertia are included which are not included in the 6-dof model. Given the good agreement between the semi-analytical results and the FEM results, it is shown that these effects are indeed negligible.

### 4.4. Other arches

As a first step towards generalization of the results found so far, the influence of the arch shape on the dynamic pulse buckling load is examined for a number of variations in the arch dimensions. Only the initial height \( h \) and the thickness \( d \) will be varied while leaving the other parameters unchanged to the values as listed in Table 1. For comparison, the height and thickness are scaled as \( \tilde{h} = h/h^* \) and \( \tilde{d} = d/d^* \), where \( h^* \) and \( d^* \) are the height and thickness of the original arch (see Table 1), respectively. Furthermore, only one pulse duration is considered \( (T_p = 10 \text{ ms}) \) and the relative damping ratios \( \tilde{\xi} \) are set to the values for the original arch for \( b = 1 \) (Ns/m²) (see Table 4).

Considering \( e/d = 0 \), the sensitivities of the bifurcation buckling load \( P_b \) and the secondary limit-point load \( P_{LP} \) with respect to the arch shape parameter \( a \) are compared in Fig. 16a for the arch with double initial height \( (\tilde{h} = h/2, \tilde{d} = d) \), for the arch with double thickness \( (\tilde{h} = 1, \tilde{d} = 2) \) and for the original arch \( (\tilde{h} = 1, \tilde{d} = 1) \). The sensitivities of \( P_b \) and \( P_{LP} \) with respect to the shape parameter \( a \) are qualitatively similar. Quantitatively, for a double initial height, the critical loads (approximately) double, whereas for a double thickness, the critical loads (approximately) quadruplicate. Doubling the initial height of the arch does hardly affect the first two eigen-frequencies, whereas doubling the thickness of the arch doubles the first two eigen-frequencies. For both cases the corresponding eigen-modes are not affected. The dynamic stability boundaries for the three arches under consideration are compared in Fig. 16b for \( e/d = 0.5 \). From these comparisons, the following observations can be made: (1) for all three (imperfect) arches, the arch shape has a distinct influence on the dynamic buckling load with a maximum around \( a/h = 0.05 \) and (2) the dynamic pulse

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**Table 5**

Dynamic buckling loads \( P_p \) as computed for 6-dof model and FE model for \( a/h = 0, b = 1 \) (Ns/m²) and \( T_p = 20 \) (ms)

<table>
<thead>
<tr>
<th>( e/d [-] )</th>
<th>( 0 )</th>
<th>( 0.1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-dof ( P_p )</td>
<td>211.4</td>
<td>212.9</td>
</tr>
<tr>
<td>FEM ( P_p )</td>
<td>206.9</td>
<td>208.8</td>
</tr>
</tbody>
</table>

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**Fig. 15.** Comparison of time responses of 6-dof model and FE model for \( a/h = 0, b = 1 \) (Ns/m²), \( e/d = 0.1 \) and \( T_p = 20 \) (ms).
As can be noted, for decreasing relative pulse duration, explained by the fact that for the arch with
are approximately the same, whereas their static buckling loads show a factor two in difference. This last observation may be
approximately scales with the parameter $d$. As shown, the dynamic pulse buckling load approx-
imal arch shape with respect to the secondary static buckling shifts from the opti-
Snap-through buckling of perfect sinusoidal arches ($a/h=0$) due to shock loading is only possible for $h/r > 4$ (where $r^2 = 1/A = d^2/12$) [6,8]. After all, for initial heights $h/d < 4/\sqrt{12} \approx 1.15$, there is only one unloaded stable equilibrium state, making a sudden jump to the secondary stable equilibrium state at $W_{\text{mid}} = -1$ impossible. It is clear that by introducing an imperfection, the second stable equilibrium state may already cease to exist for initial heights $h/d > 4/\sqrt{12}$. For example, for $e/d = 0.5$ and $a/h = 0$, the secondary stable equilibrium state of the unloaded arch ceases to exist at $h/d = 3.6$ ($h \approx 0.075$). In Fig. 17, $P_B$ and $P_{LP}$ for $e/d = 0$ and the dynamic stability boundaries for $e/d = 0.5$ are compared for arches with thickness $d = 1$ and nine initial heights equidistantly distributed in the range $0.08 \leq h \leq 0.38$. In this figure, the maxima for $P_B$, $P_{LP}$ and $P_p$ are indicated with $\diamond$, $\times$ and $\circ$, respectively. In the considered region of initial heights, the first two eigen-frequencies and eigen-modes remain nearly the same as for the original arch. As can be noted, for decreasing $h$, the optimal arch shape with respect to dynamic pulse buckling shifts from the optimal arch shape with respect to the secondary static buckling load $P_{LP}$ towards the optimal arch shape with respect to the first static buckling load $P_B$. Consequently, for the consid-
ered dimensions and level of imperfection, the correspondence between the secondary static buckling load and the dynamic pulse buckling load seems to be restricted to arches with $h/d > 20$ ($h > 0.4$). However, for generalization of this result more research is required.

5. Conclusions

The objective of this paper was to study the influence of the arch shape on the dynamic pulse buckling load for thin shallow arches under shock loading and to compare parameter sensitivities of static and dynamic buckling loads. Based on an approximate non-linear kinematic model, a multi-dof model of the arch is derived. The model includes a shape-factor by which the arch shape (the initial curvature) can be varied while keeping the initial height of the arch unchanged and an imperfection parameter which controls the amplitude of an asymmetry in the arch shape. By comparing quasi-static responses, modal analysis results and non-linear dynamical transient analysis results of this model with finite element modeling results based on Timoshenko beam theory, the model is validated.

First static snap-through buckling of the arch under a quasi-static varying acceleration is considered. The primary static buckling load of the arch for this load-case corresponds to an asymmetrical buckling mode and can hardly be influenced by varying the arch shape. The arch shape has significant influence on the secondary buckling load corresponding to a limit-point in the load-path of the perfect arch.

The dynamic response of the arches under shock loading is studied by numerically solving the equations of motion. The dynamic buckling load is determined for various levels of damping, shock pulse durations, imperfection amplitudes, and a wide range of arch shapes. Depending on the level of damping, the imperfection amplitude and the arch shape, the occurrence of
dynamic buckling can be extremely sensitive to small variations in the load parameter. Small geometric imperfections have only a mild effect on the dynamic pulse buckling load and do not significantly change the sensitivity of the dynamic pulse buckling load with respect to the arch shape parameter. Consequently, the shape of the arch (with or without an imperfection) has a significant influence of the dynamic pulse buckling load.

Although the optimal arch shape with respect to the secondary static buckling load and the optimal arch shape with respect to dynamic pulse buckling load do not match exactly, there exists a quantitative correspondence between the arch shape sensitivities of these two critical loads. This correspondence is not trivial but could be very useful for shape parameter sensitivity studies of the dynamic pulse buckling load using quasi-static analyses. After all, compared to the non-linear transient dynamic buckling analysis, the quasi-static analyses are computationally significantly less expensive. The correspondence between the secondary static load and the dynamic pulse buckling load is also found for arches with other dimensions. However, as shown, more research on the found correspondence will be necessary before it can be generalized.

References