GREEN'S FUNCTIONS FOR A ROTATING TYRE: A SEMI-ANALYTICAL APPROACH


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ABSTRACT

Road traffic noise is becoming an increasingly big problem in densely populated areas. One of the main contributions comes from the tyre/road interaction. For the interior noise, to which the occupants of the vehicle are exposed, tyre/road noise is also an important source, especially at lower frequencies. The vibrations of the tyre up to 500 Hz are transmitted via structural transmission paths to the interior effectively, thus causing acoustic noise. In this frequency range tyre/road noise is dominated by vibrational mechanisms.

One of the aspects that has to be taken into account when modelling tyre vibrations is the effect of rotation. Recently a Finite Element (FE) based methodology has been presented for the modeling of rotating tyres at frequencies up to 500 Hz. A modal approach is used to determine the response of a rotating tyre. In order to solve the contact problem the unit impulse response functions (Green’s functions) of the tyre including rotation are needed.

In the present paper an analytical expression of the Green’s functions including rotation based on this methodology is derived. It is shown that the frequency shift due to rotation is correctly predicted by the derived Green’s functions.
1 INTRODUCTION

Road traffic noise is becoming an increasingly big problem in densely populated areas. One of the main contributions comes from the tyre/road interaction. For the interior noise, to which the occupants of the vehicle are exposed, tyre/road noise is also an important source, especially at lower frequencies. The vibrations of the tyre up to 500 Hz are transmitted via structural transmission paths to the interior effectively, thus causing acoustic noise. In this frequency range tyre/road noise is dominated by vibrational mechanisms [1].

One of the aspects that has to be taken into account when modelling tyre vibrations is the effect of rotation. It has already been shown that if the response of the tyre is transformed from the tyre reference system to a fixed reference system, a split of the eigenfrequencies of the tyre occurs [2], [3], [4]. A straight-forward but computationally costly way to determine the response of the tyre in a fixed reference system is to first calculate the response in the tyre reference system and then transform it to the fixed reference system. An alternative is proposed in [3], where the contact problem is solved first to determine the time history of the contact forces. Then the forces are transformed to the frequency domain where the response of the tyre can be easily determined. In order to include rotation a second transformation to the wave-number domain is applied, where a shift is applied to the frequency [3].

An alternative approach is to first determine the impulse response functions (Green’s functions) of the tyre in the fixed reference frame and then solve the contact problem to obtain the time history of the contact forces directly in the fixed reference frame.

Recently a FE based methodology has been presented for the modeling of rotating tyres at frequencies up to 500 Hz [4]. Using a detailed FE model of a tyre is a feasible approach in this frequency range and with a FE model it is possible to relate tyre design parameters to its vibro-acoustic properties. A modal approach is used, based on modal information of the non-rotating tyre (expressed in the tyre reference frame) extracted from a FE calculation. The natural frequencies and modeshapes of the tyre are determined in the deformed state. This way the influence of the ground contact on the dynamics is taken into account. The influence of the rotation is taken into account using a coordinate transformation. The response of the rotating tyre is determined in a fixed reference frame.

In [4], a receptance matrix for the rotating tyre is derived in the frequency domain and it is said that the Green’s functions can be found from an inverse FFT transformation of the receptance matrix. This is a valid approach but it has the disadvantage that the full time-history of the Green’s functions has to be calculated and stored, which becomes prohibitive in terms of computer effort if long simulation times at medium frequencies are to be achieved.

In the present paper an analytical expression of the Green’s functions including rotation (expressed in the fixed reference frame) based on the above methodology is derived. The advantage of using an analytical expression is that the Green’s functions for a given time instant can be calculated on-line during the computation of the contact forces, which dramatically reduces the storage needs and speeds up the calculations.

2 DYNAMICS OF THE ROTATING TYRE

In this section a short review of the methodology presented in [4] is given. In general the dynamic equations of the tyre in the tyre reference frame can be written as follows,
\[
\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(\alpha + \Omega t, t)
\]  
(1)

where \( \mathbf{M}, \mathbf{D}, \mathbf{K} \) are the mass, damping and stiffness matrices of the system, \( \mathbf{f}(\alpha + \Omega t, t) \) are the forces seen from the tyre reference frame, \( \alpha \) is the angular coordinate in the tyre reference frame and \( \Omega \) is the (constant) rotation velocity of the tyre. Provided the eigenvalues and eigenvectors of the system have been determined the following transformation can be applied

\[
\mathbf{x}(t) = \Phi\eta(t) \quad \Phi: \text{matrix of eigenvectors}
\]  
(2)

to determine the dynamic equations of the system in modal coordinates \( \eta \):

\[
\ddot{\eta}(t) + \mathbf{D}_{\text{mod}}\dot{\eta}(t) + \mathbf{K}_{\text{mod}}\eta(t) = \Phi^T\mathbf{f}(\alpha + \Omega t, t)
\]  
(3)

In Eq. (3) \( \mathbf{K}_{\text{mod}} \) is a diagonal matrix with elements \( k_{ii} = \omega_i^2 \), where \( \omega_i \) are the natural frequencies of the system, and if Rayleigh damping is considered, \( \mathbf{D}_{\text{mod}} \) is a diagonal matrix with elements \( d_{ii} = 2\xi_i\omega_i \), where \( \xi_i \) are the modal damping ratios.

To obtain the dynamic equations of the tyre in a fixed reference system the following transformation can be applied.

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta} \quad \text{with} \quad \theta = \alpha + \Omega t
\]  
(4)

In the above equation \( \theta \) is the angular coordinate in the fixed reference frame (the reader is referred to [4] for details). If Eqs. (2) and (4) are used to transform Eq. (1) the dynamic equations of the tyre in a fixed reference frame can be obtained:

\[
\ddot{\eta}(t) + \tilde{\mathbf{D}}(\Omega)\dot{\eta}(t) + \tilde{\mathbf{K}}(\Omega)\eta(t) = \Phi^T\mathbf{f}(\alpha + \Omega t, t)
\]  
(5)

where

\[
\tilde{\mathbf{D}}(\Omega) = 2\mathbf{P}(\Omega, \mathbf{M}, \Phi) + \mathbf{D}_{\text{mod}}
\]  
(6)
\[
\tilde{\mathbf{K}}(\Omega) = \mathbf{S}(\Omega, \mathbf{M}, \Phi) + \mathbf{D}_{\text{mod}}\mathbf{P}(\Omega, \mathbf{M}, \Phi) + \mathbf{K}_{\text{mod}}
\]  
(7)

and the matrices \( \mathbf{S}, \mathbf{P} \) are added stiffness and damping terms due to the rotation.

The set of equations (5) together with Eq. (2) gives the response of the tyre in the fixed reference frame. The data needed to build the matrices \( \tilde{\mathbf{D}}, \tilde{\mathbf{K}} \) are the eigenvalues and eigenvectors of the tyre in the tyre reference system obtained from a FE model and the mass matrix of the FE model. It should be noted that \( \tilde{\mathbf{D}}, \tilde{\mathbf{K}} \) are non-diagonal and non-symmetric matrices, which means that the system of equations (5) is not uncoupled in contrast to Eq. (3). Therefore the set of coordinates \( \eta(t) \) is not a set of modal coordinates of the tyre in the fixed reference frame. In the next section it is shown that a new eigenvalue problem can be formulated for Eq. (5) in order to determine the eigenfrequencies and eigenvectors of the tyre in the fixed reference frame and to find a system of equations analogous to Eq. (3).

3 GREEN’S FUNCTIONS OF THE ROTATING TYRE

The contact interaction between tyre and road is a non-linear problem which has to be solved in the time-domain [5]. This means that the response of the tyre to the contact forces has to be determined in the time-domain. The displacement at a point \( i \) on the tyre can be calculated as the
sum of convolution products between the contact forces at the contact points \(f_j\) and the Green’s functions \(g_{ij}\) of the tyre [5].

\[
x_j(t) = \sum_{j} g_{ij}(t) \otimes f_j(t)
\]  

In this section an expression for the Green’s functions of the tyre in the fixed reference frame will be derived which can be used in Eq. (8) to directly determine the time-domain response of the tyre in the fixed reference frame. The Green’s functions can be obtained by solving the following equation,

\[
\dot{\eta}(t) + \tilde{D}(\Omega)\eta(t) + \tilde{K}(\Omega)\eta(t) = \Phi^T J \delta(t)
\]  

where \(\Phi^T_J\) is the \(j\)th row of \(\Phi\) and \(\delta(t)\) is the Dirac delta function.

The first step is to transform Eq. (9) to a set of 1\(^{st}\) order differential equations:

\[
A\dot{y}(t) + By(t) = \Phi^T J \delta(t) \quad \text{with} \quad y(t) = [\eta(t) \quad \dot{\eta}(t)]
\]  

where

\[
A = \begin{bmatrix} \tilde{D} & I \\ I & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \tilde{K} & 0 \\ 0 & -I \end{bmatrix}, \quad \Phi^T_J = \begin{bmatrix} \Phi^T_J \end{bmatrix}
\]  

Since \(A\) and \(B\) are non-symmetric matrices the right and left eigenvalue problems have to be solved in order to transform Eq. (10) and obtain the following set of uncoupled equations:

\[
a_r \left\{ \dot{\gamma}_r(t) - s_r \gamma_r(t) \right\} = w^T_r \Phi^T_J \delta(t) \quad r = 1,2,\ldots,2n
\]  

where

\[
W^T A V = \text{diag}(\{a_1 \ldots a_{2n}\})
\]

and the following transformation has been applied,

\[
y(t) = V \gamma(t)
\]  

In the above equations \(V\) and \(W\) are matrices of right and left eigenvectors (\(v_r\) and \(w_r\) are columns of \(V\) and \(W\) respectively), \(s_r\) are the corresponding eigenvalues and \(\gamma(t)\) are the modal coordinates of the tyre in the fixed reference frame. It should be stressed that all these quantities are functions of \(\Omega\), the rotational velocity of the tyre. In Eq. (12) \(n\) is the number of modes considered.

It can be easily shown that the solution of Eq. (12) has the following form:

\[
\gamma_r(t) = \frac{w_r^T \Phi^T_J}{a_r} e^{s_r t} \quad r = 1,2,\ldots,2n
\]  

By combining Eqs. (2) and (14) a relationship can be found between the response at a given point of the tyre in the fixed reference frame and the modal coordinates of the tyre in the fixed reference frame.

\[
x(t) = \Phi V \gamma(t) \quad \text{with} \quad \Phi = [\Phi \quad 0]
\]  

If Eq.(15) is substituted in Eq. (16) an expression for the Green’s function, expressed in the fixed reference frame, can be obtained. In Eq. (17) the \(\Omega\) dependency of \(g_{ij}\) has been explicitly included to stress the fact that the Green’s functions determined by Eq. (17) are expressed in the fixed reference frame and that they will change if the rotational velocity changes. It is also important to note that the only data needed to apply Eq. (17) are the eigenvalues and eigenvectors of the tyre in
the tyre reference frame, which can be readily obtained from a FE model of the tyre, and the mass matrix from the FE model.

\[
g_y(\Omega, t) = \sum_{k=1}^{n} \Phi_k \left( \sum_{r=1}^{n} v_{kr} \frac{\Phi_{jp}}{a_j} e^{i \omega_j t} \right) \quad r = 1, 2, \ldots, 2n
\]  

(17)

In the following section it will be shown that the Green’s functions of the rotating tyre calculated according to the proposed methodology show the frequency split predicted in the literature [2], [3].

4 RESULTS

The effect of rotation on the Green’s functions will be illustrated with a very simplified 3D model of the tyre. The cross-section of this model is shown in Fig. 1. It must be said that the model parameters have been chosen arbitrarily, they have not been fitted to match experimental data (the values used for this example can be found in [4]). It has been shown in [2] that the effect of rotation on the natural frequencies of the tyre can be predicted by:

\[
\omega_j^* = \omega_j \pm m\Omega
\]  

(18)

where \(\omega_j^*\) are the eigenfrequencies of the rotating tyre and \(m\) is the number of nodal diameters of the corresponding eigenmodes.

![Cross-section of the simplified 3D model](image)

**Fig. 1.** Cross-section of the simplified 3D model

![First 10 natural frequencies(Hz) as a function of the rotational velocity (rad/s).](image)

**Fig. 2.** First 10 natural frequencies(Hz) as a function of the rotational velocity (rad/s).

Following the methodology described in Section 3, the eigenfrequencies of the rotating tyre can be determined from \(\omega_j^* = \text{imag}(s_j)\). The result is shown in Fig. 2, where the eigenfrequencies of the first 10 modes are plotted as a function of the rotational velocity. The frequency shift due to the rotational velocity shown in Fig. 2 is in agreement with Eq. (18).

Next the Green’s functions between response and excitation in the normal direction at one point (Fig. 1) have been calculated with Eq. (17) for 2 different values of the rotational velocity \(\Omega = 0 \text{ and } 100 \text{ rad/s}\). The result is shown in Fig. 3(a). The first 50 modes and a damping factor \(d_j = 100Ns/m\) have been used in the calculation. In Fig. 3(b) the corresponding admittances, obtained from an FFT of the Green’s functions are shown. The numbers 1 and 2 indicate the number of nodal diameters of the mode associated to that resonance frequency.
From Fig. 3(a) it can be concluded that the Green’s functions for rotational velocity 0 and 100 rad/s differ considerably both in frequency and amplitude. The amplitude decreases due to the rotational velocity which is caused by the increase of damping due to rotation (see Eq. (6)). Furthermore, it can be concluded from the admittances plotted in Fig. 3(b) that the frequency shift due to rotation is correctly predicted by Eq. (17).

5 SUMMARY

In this paper an analytical expression for the Green’s functions of a tyre expressed in a fixed reference frame has been derived. The advantage of this approach compared to the inverse FFT is that the Green’s functions for a given time instant can be calculated on-line, which dramatically reduces the storage needs and speeds up the calculations. The data needed to actually compute these Green’s functions are the eigenfrequencies and eigenmodes of the tyre in the tyre reference frame and the mass matrix obtained from a standard FE calculation. Calculations performed on a simplified model show that the damping increases due to rotation and that the frequency shift is correctly captured in the derived Green’s functions.

REFERENCES