EXPERIMENTAL RESULTS ON HUYGENS SYNCHRONIZATION

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Abstract: The paper represents some experimental results on synchronization of two metronomes attached to a common support beam. The experiments demonstrate possible in-phase, anti-phase synchronization as well as intermediate seemingly chaotic regimes.

Keywords: Synchronization, mechanical systems.

1. INTRODUCTION

One of the first scientifically documented observations of synchronization is by the Dutch scientist Christiaan Huygens. In the 17th century maritime navigation called for more accurate clocks in order to determine the longitude of a ship. Christiaan Huygens' solution for precise timekeeping was the invention of the pendulum clock (Yoder, 1988) with cycloidal-shaped plates to confine the pendulum suspension. Those plates resulted in isochronous behavior of the pendulum independent of the amplitude and were genius invention of that time. During time Huygens was bound to his home due to illness he observed that two pendulum clocks attached to the same beam supported by chairs would swing in exact opposite direction after some time (Huygens, 1893; Huygens, 1932; Huygens, 1986). A drawing made by Christiaan Huygens is given in figure 1. Disturbances or different initial positions did not affect the synchronous motion which resulted after about half an hour. This effect which Huygens called *"sympathie des horloges"* is nowadays known as synchronization and is characterized by (Pikosvky *et al.*, 2001) as *"an adjustment of rhythms of oscillating objects due to their weak interaction"*. The oscillating objects in Huygens' case are two pendulum clocks and are weakly coupled through translation of the beam.

Many more cases of synchronization have been identified in nature and technology around us. Two striking examples in biology are the synchronized flashing of fireflies (Buck, 1988) or synchronization of neurons in the brain when performing perceptual tasks. Synchronization is also found in technology, for example the frequency synchronization of triode generators. These generators were the basic elements of early radio communication systems (Appleton, 1922). Using synchronization it is possible to stabilize the frequency of a high power genera-

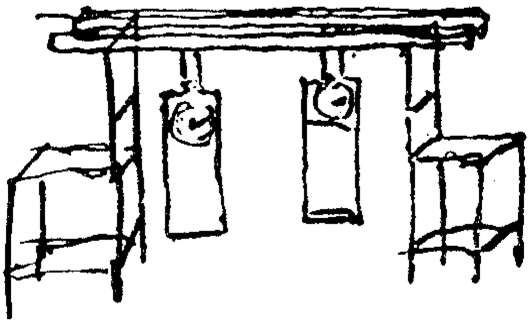

Fig. 1. Drawing by Christiaan Huygens of two pendulum clocks attached to a beam which is supported by chairs. Synchronization of the pendulums was observed by Huygens in this setup.

tors and there are more other applications we are unable to mention in this paper.

Three centuries later the phenomenon of synchronizing driven pendula is, to our best knowledge, repeated twice experimentally. In the first research by Bennett, Schatz, Rockwood and Wiesenfeld (Bennett *et al.*, 2002), one has tried to accurately reproduce the findings of Huygens in an experimental setup consisting of two pendulum clocks attached to a free moving cart. The results of this experiment confirm the documented observations of Christiaan Huygens. A rather simple but interesting experiment is described by Pantaleone (Pantaleone, 2002), where the synchronization of two metronomes is discussed, which are coupled by a wooden board rolling on soda cans. The metronomes in this setup would synchronize most of the time with in phase oscillations.

The research presented in this paper is inspired by the observations of Christiaan Huygens, the work of Bennett et al. and Pantaleone. The main objective is to perform and analyze synchronization experiments with a setup consisting of driven pendula. A particular attention is paid to different syncronization regimes that can be observed in this situation : anti-phase, observed by Huygens, in-phase: observed by Blekhman and explained with the van der Pol model for each pendulum and possible intermediate regimes.

The paper is organized as follows. First the design of the setup and the measurement methods is discussed and the mathematical model describing the setup is be introduced. Then the synchronization experiments are discussed. Finally the conclusions are drawn and recommendations for further research are given.

2. EXPERIMENTAL SETUP

The experimental setup consists of two metronomes coupled by a platform which can translate horizontally. The metronomes are made by Wittner, type Maelzel (series 845). The platform is suspended by leaf springs, which allows a frictionless horizontal translation. A photograph of the experimental setup is given in figure 2.

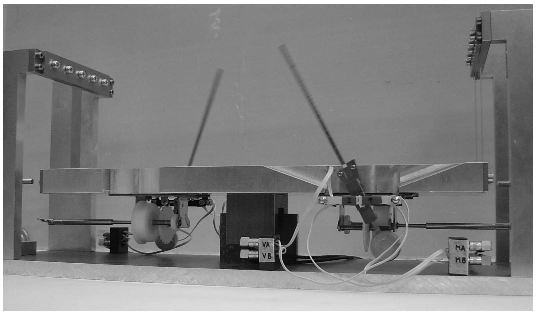

Fig. 2. Photograph of the setup.

2.1 Metronomes

The metronomes which are normally used for indicating a rhythm for musicians consist of a pendulum and a driving mechanism, called the escapement. The energy lost due to friction is compensated by this escapement. The escapement consists of a spring which loads on a toothed wheel. These teeth push alternately one of the two cams on the axis of the pendulum. Each time the teeth hit a cam a tick is produced, the typical sound of mechanical metronomes. The frequency of the metronomes can be adjusted with a contra weight attached to the upper part of the pendulum. Variation of the frequency between 2.4 rad/s and 10.8 rad/s is possible with the weight attached, without it the frequency of the metronomes increases to 12.3 rad/s. The amplitude of the metronomes' oscillations cannot be influenced, however at increasing frequencies the amplitude decreases.

2.2 Platform

The platform does not only act as a support for the metronomes but because of its horizontal translation it couples the dynamics of both metronomes as well. In order to keep the equations of motion of the total system simple a suspension with linear stiffness and damping is desired. As long as the translation of the platform is not too large (mm range) the use of leaf springs makes a frictionless translation possible with linear stiffness and damping properties (Rosielle and Reker, 2000). In order to calculate the necessary dimensions for leaf springs the following estimates have been used. The stiffness of a leaf spring can be estimated by assuming it behaves as two bars clamped at one side. For small deflections the stiffness of bar clamped at one side is given by (Fenner, 1989)

$$k = \frac{Ehb^3}{4L^3}, \tag{1}$$

where E is the elastic modulus, h the width, b the thickness and L the length of the bar. The leaf spring has a stiffness equal to (1) since both halves of the leaf spring take half the deflection and half the force. Thus equal to the stiffness of a single sided clamped beam. The eigenfrequency of the platform in rad/s is estimated by

$$\Omega = \sqrt{\frac{Ehb^3}{4(l/2)^3 M} + \frac{g}{l}} \tag{2}$$

where l is the length of the leaf spring, M the mass of the platform and g the constant of gravity. For the setup the following dimensions for the leaf springs are chosen: l=80 mm, b=0.4 mm and h=15+15+10 mm (three leaf springs). The constants are $E = 200 \cdot 10^9$ kg/m/s^2 and g=9.81 m/s^2. The platform has a mass of 2.35 kg which results in an eigenfrequency of 31.2 rad/s.

2.3 Measurements

In the setup the angle of the metronomes and the translation of the platform are of interest. Since alteration of the dynamics of both the metronomes and the platform should be avoided, contactless measurement methods have been chosen. All signals are recorded using a Siglab data acquisition system, model 20-42. First the measurement of the metronomes is discussed, secondly that of the translation of the platform.

The angle of the metronomes is measured using a sensor based on the anisotropic magnetoresistance (AMR) principle (*Applications of magnetic position sensors*, 2002), (*Linear / angular / rotary displacement sensors*, 2003). The resistance of AMR materials changes when a magnetic field is applied. Above a minimal field strength the magnetization of the material aligns with the external field and the following relation holds for the resistance R

$$R \sim \cos^2\theta \tag{3}$$

where θ is the angle between the magnetic field and the current through the resistor. By combining four AMR resistors in a bridge of Wheatstone a change in resistance is converted to a voltage difference. Two of these bridges of Wheatstone are located in the sensor, but are rotated 45° degrees with respect to each other. As a result the voltage difference of bridges A and B can be written as

$$\Delta V_A = V_s S \sin 2\theta\,,\ \Delta V_B = V_s S \cos 2\theta \tag{4}$$

where V_s is the voltage supplied to the bridges and S is the AMR material constant. The angle θ can be calculated from these signals by

$$\theta = \tfrac{1}{2}\arctan(\Delta V_A/\Delta V_B) \tag{5}$$

regardless of the value of voltage V_s and constant S. Due to manufacturing tolerances the bridges will show an offset when no magnetic field is applied. This offset can be corrected in software when both signals are recorded.

The velocity of the platform is measured using a Polytec Vibrometer, type OFV 3000 with a OFV 302 sensorhead. The measurement is based on the Doppler shift of a laser beam reflected on the platform.

3. MATHEMATICAL MODEL OF THE SETUP

Assuming the setup consists of rigid bodies the equations of motion can be derived using Lagrangian mechanics. The generalized coordinates are chosen as

$$\underline{q}^T = [\theta_1, \theta_2, x] \tag{6}$$

which are the angles of the pendulums from the vertical and the translation of the platform.

The kinetic energy $T(\underline{q}, \dot{\underline{q}})$ of the system can be expressed as

$$T(\underline{q}, \dot{\underline{q}}) = \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1 \cdot \dot{\boldsymbol{r}}_1 + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2 \cdot \dot{\boldsymbol{r}}_2 + \frac{1}{2} M \dot{\boldsymbol{r}}_3 \cdot \dot{\boldsymbol{r}}_3 \tag{7}$$

where r_1, r_2 and r_3 are respectively the translation of the center of mass of pendulum 1, 2 and the platform, m_1 and m_2 are the mass of pendulum 1 and 2, l_1 and l_2 the lengths of the center of mass to the pivot point of pendulum 1 and 2, M is the mass of the platform and

$$\boldsymbol{r}_1 = (x + l_1 \sin\theta_1) \cdot \boldsymbol{e}_1 - l_1 \cos\theta_1 \cdot \boldsymbol{e}_2 \tag{8a}$$
$$\boldsymbol{r}_2 = (x + l_2 \sin\theta_2) \cdot \boldsymbol{e}_1 - l_2 \cos\theta_2 \cdot \boldsymbol{e}_2 \tag{8b}$$
$$\boldsymbol{r}_3 = x \cdot \boldsymbol{e}_1 \tag{8c}$$

The potential energy $V(\underline{q})$ of the system consists of the force of gravity acting on the pendulums and the energy stored in the spring,

$$V(\underline{q}) = m_1 g l_1 (1 - \cos\theta_1) + m_2 g l_2 (1 - \cos\theta_2) + \tfrac{1}{2} k x \tag{9}$$

where g is the constant of gravity and k is the spring stiffness of the platform. The generalized forces $\underline{Q}^{nc}$ include viscous damping in the hinges of the pendulums and the platform and the torque $\mathrm{f}_i(\theta_i, \dot{\theta}_i)$ exerted by the escapement on the pendulums and can be written as

$$\underline{Q}^{nc} = \begin{bmatrix} \mathrm{f}_1(\theta_1, \dot{\theta}_1) - d_1 \dot{\theta}_1 \\ \mathrm{f}_2(\theta_1, \dot{\theta}_1) - d_2 \dot{\theta}_2 \\ -d_3 \dot{x} \end{bmatrix} \tag{10}$$

where d_i are the viscous damping constants of respectively the two pendulums and the platform. With Lagrange's equations of motions

$$\frac{d}{dt}\left(T_{,\dot{\underline{q}}}\right) - T_{,\underline{q}} + V_{,\underline{q}} = \left(\underline{Q}^{nc}\right)^T \tag{11}$$

the equations of motion for the system become

$$\begin{aligned} m_1 {l_1}^2 \ddot{\theta}_1 + m_1 l_1 g \sin\theta_1 + m_1 l_1 \cos(\theta_1)\ddot{x} \\ + d_1 \dot{\theta}_1 = \mathrm{f}_1(\theta_1, \dot{\theta}_1) \\ m_2 {l_2}^2 \ddot{\theta}_2 + m_2 l_2 g \sin\theta_2 + m_2 l_2 \cos(\theta_2)\ddot{x} \\ + d_2 \dot{\theta}_2 = \mathrm{f}_2(\theta_2, \dot{\theta}_2) \end{aligned} \tag{12}$$

$$\begin{aligned} M\ddot{x} + d_3 \dot{x} + kx \\ + \sum_{i=1}^{n} m_i l_i \left(\ddot{\theta}_i \cos\theta_i - \dot{\theta}_i^2 \sin\theta_i\right) = 0. \end{aligned}$$

These equations for the metronomes can be simplified by dividing all terms by $m_i {l_i}^2$, which give for $i = 1, 2$

$$\begin{aligned} &\ddot{\theta}_i + {\omega_i}^2 \sin\theta_i + \frac{1}{l_i} \cos(\theta_i)\ddot{x} + \frac{d_i}{m_i {l_i}^2} \dot{\theta}_i \\ &= \frac{1}{m_i {l_i}^2} \mathrm{f}_i(\theta_i, \dot{\theta}_i) \end{aligned}$$

where $\omega_i = \sqrt{g/l_i}$.

The equations of motion can be written in dimensionless form using the following transformations. The dimensionless time is defined as $\tau = \omega t$ and the position of the platform as $y = x/l = x\omega^2/g$, where ω is the mean angular frequency of both pendulums. The derivatives of the angles with respect to the dimensionless time are written as

$$\frac{d\theta}{dt} = \frac{d\theta}{d\tau}\frac{d\tau}{dt} = \omega\theta', \quad \frac{d^2\theta}{dt^2} = \omega^2\theta''.$$

The equations of motion now become

$$\begin{aligned} &\theta_i'' + \gamma_i^2 \cos\theta_i y'' + \gamma_i^2 \sin\theta_i \\ &+ \delta_i \theta_i' = \epsilon_i \, \mathrm{f}(\theta_i, \theta_i'), \\ &y'' + 2\Omega\xi y' + \Omega^2 y + \\ &\sum_{i=1}^{2} \beta_i \gamma_i^{-2} \left(\cos\theta_i \theta_i'' - \sin\theta_i {\theta_i'}^2\right) = 0, \end{aligned} \tag{13}$$

with coupling parameter $\beta_i = \frac{m_i}{M}$, scaled eigenfrequency of the metronomes $\gamma_i = \omega_i/\omega$, damping factor $\delta_i = \frac{d{\omega_i}^2}{m_i g}$, eigenfrequency of the platform $\Omega^2 = \frac{k}{M\omega^2}$ and damping ratio of the platform $\xi = \frac{d_3}{2\sqrt{kM}}$. The factor $\epsilon_i = \frac{{\omega_i}^4}{m_i g^2}$ will be set to 1 in further equations, since this factor can be taken into account in the model of the escapement.

3.1 Escapement

So far the escapement has been indicated by the function $\mathrm{f}(\theta, \theta')$. A close inspection of the metronomes shows that the escapement gives the pendulum a push when going upward. Without deriving an accurate mechanical model of the escapement mechanism this torque is approximated by the following normalized expression:

$$\begin{aligned} &\mathrm{f}(\theta, \theta') = 0, \quad \text{if} \quad \theta < \phi \vee \theta > \phi + \Delta\phi \\ &\mathrm{f}(\theta, \theta') = \frac{1 - \cos(2\pi\frac{\theta - \phi}{\Delta\phi})}{2\Delta\phi}, \\ &\text{if} \quad \phi \le \theta \le \phi + \Delta\phi \wedge \theta' > 0 \end{aligned} \tag{14}$$

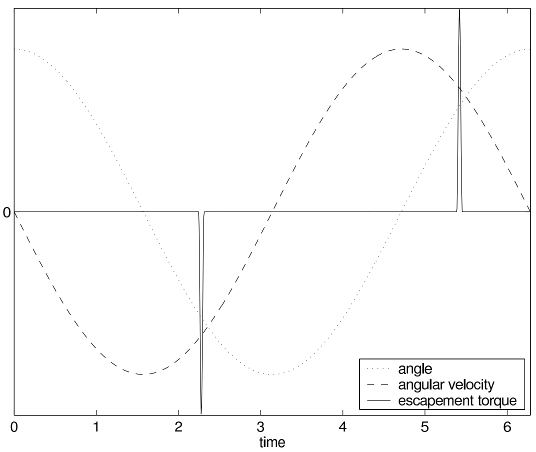

Fig. 3. The torque exerted by the escapement on the pendulum is plotted versus time together with the angle and velocity of the pendulum. The vertical axis is scaled in order to illustrate the torque more clearly.

where θ_1 and θ_2 are angles between which the mechanism works. In figure 3 the torque of the escapement is plotted versus time when the pendulum would follow a sinusoidal trajectory.

4. EXPERIMENTAL RESULTS

Several experiments are performed in order to gain experience with the dynamics of the system. Parameters which can be varied in the experiment are the mass of the platform, the mass and frequency of the metronomes and the amount of damping in the system. Converted to the dimensionless parameters the influence of the physical parameters is:

$$\Delta = (\omega_1 - \omega_2)/\omega\,, \quad \omega = (\omega_1 + \omega_2)/2$$
$$\beta = m/M$$
$$\Omega = \sqrt{k/M}/\omega$$
$$\xi = d/2/\sqrt{kM}$$

The experiments show different phenomena, first of all when the damping of the platform is too small (< 2.0 kg/s) the pendulums hit the frame. These experiments are discarded since we are want to avoid collisions in the experiments. Apparently, without enough damping the platform and consequently the metronomes are excited too much. For larger damping synchronization with different phase differences

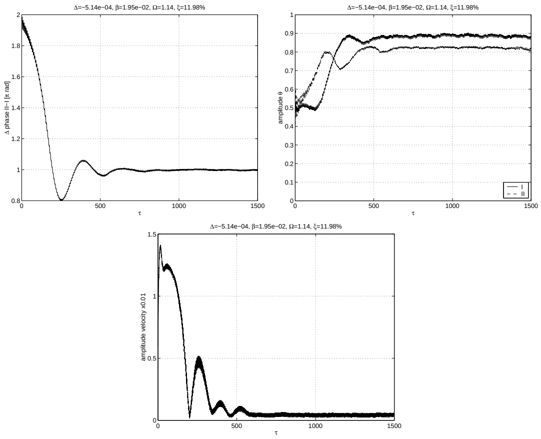

Fig. 4. Experiment which shows anti phase synchronization.

is observed. Four different types of responses can be identified, anti phase synchronization with damped oscillation of the phase difference, fast anti phase synchronization without oscillations of the phase difference, intermediate (neither anti nor in) phase synchronization with a large amplitude difference of the angles of the metronomes and finally in phase synchronization.

All experiments with small Δ show anti phase synchronization with damped oscillation of the phase difference. And for these experiments Ω is larger than 1. The experiments with fast synchronization without oscillation of the phase difference all have $\Omega \approx 1$. All experiments with other than anti phase synchronization have $\Omega < 1$.

Three different synchronization phenomena can be observed in the experiments. In figures 4, 5 and 6 typical experiments are plotted.

5. CONCLUSIONS

This paper presents some experimental results on synchronization of two metronomes attached to a common beam that can move in horizontal direction. From those experiments it becomes evident that different synchronization regimes can (co-) exist depending on the system parameters. It is worth mentioning that we have observed from those experiments some intermediate seemingly chaotic regimes of oscillations. Further research will be devoted towards theoretical studies of those oscillations.

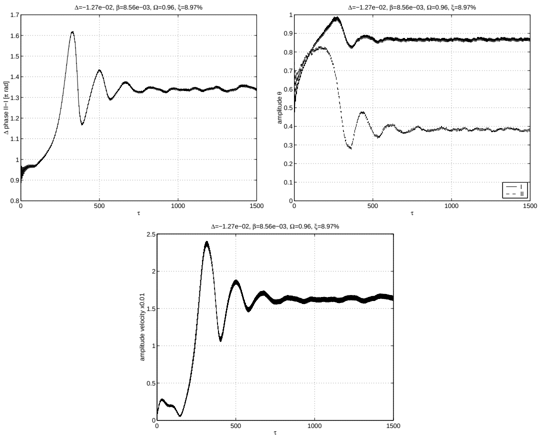

Fig. 5. Experiment which shows in-between phase synchronization.

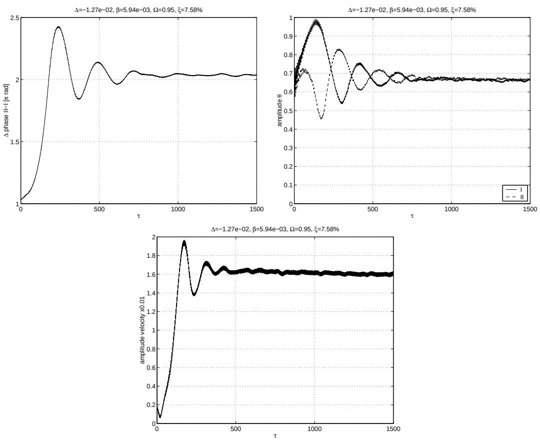

Fig. 6. Experiment which shows in phase synchronization.

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