Design for Shape Control of Tensegrities

Bram de Jager

Abstract—This paper proposes and demonstrates procedures to optimally design tensegrity structures actuation, based on closed loop shape control requirements, while at the same time a feasible path for realizing a desired shape is synthesized. The procedures are employing different optimization based formulations of a set of requirements needed for shape control. The specific procedure demonstrated is based on a mixed integer linear programming formulation, one of the simplest possible. It is possible to formulate the design problem more generally, but then the computations become more involved, inhibiting a real time implementation of the procedure. The demonstration is for a simple planar tensegrity, but the procedures can be applied to more general structures without modification.

I. INTRODUCTION

Tensegrities are structures that consist of two types of members, tensile ones (tendons) and compressive ones (bars), while simultaneously allowing a state of self-stress. In a class 1 tensegrity structure [1] the bar endpoints, i.e., the nodal points, are only connected to tendons, not to other bars. Tendons are exclusively loaded in tension, otherwise they would buckle because they are normally very slender. Bars are generally loaded in compression only, and not in tension.

Tensegrities are used as art forms [2], architectural entities [3], landmarks [4], biological cell models [5], engineering structures [6], etc.

The main advantage of tensegrity systems is that they allow changes in configuration, or shape, by adjusting the length of the members, without requiring engineered joints, while still being in equilibrium in a state of self-stress. Under the influence of outside forces they can be quite pliable.

Both tendons and bars can play the role of actuator. By controlling the length of all bars and tendons simultaneously, a tensegrity structure is very versatile in adapting its shape, although there are always configurations imaginable for which no equilibrium with the desired sign of the forces is possible. It is not always feasible to equip all tendons or bars as actuators, mainly due to complexity of the structure and cost. It is therefore interesting to know how the possibilities for morphing the structure are limited by not allowing one or more of the members to change length. Some actuator selection results for tensegrities, targeting vibration suppression and load disturbance rejection, are presented in [7]. One of the conclusions in [7] pertaining to the current work was that the selection is highly dependent on the criterion used, so for shape control the selected members might be completely different.

The main goal of the research presented is therefore to obtain preliminary insight in the possibilities to change the shape of a tensegrity, especially in the way this depends on the capability of a member to change its length. Doing this, we also obtain an online implementable path planner.

In the remainder of this paper we first discuss several aspects of modeling tensegrity systems, addressing structural limitations in shape control. This is followed by a discussion of how the shape change requirements are formulated for actuator selection and control design purposes. Finally some computational results are given and discussed.

II. TENSEGRITY STRUCTURE MODEL

Several ways to set up a model for a tensegrity are possible. Initially we follow [8], but later take a diversion to make the model more general.

A tensegrity is a set of \( n \) points \( p_1, \ldots, p_n \in \mathbb{R}^3 \), also called nodes, together with some characterizing functions [8]. One of these is the type of connection \( C \) between pairs of points, which can be a compressive member, or bar, \( b \), a tensile member, or tendon, \( t \), or it can be unconnected, or empty, \( e \),

\[
C: \{1, \ldots, n\}^2 \mapsto \{b, t, e\}. \tag{1}
\]

A connection \( C(i, j) \) has a certain stiffness, characterized by the function \( c \)

\[
c: \{1, \ldots, n\}^2 \mapsto \mathbb{R}^+ \tag{2}
\]

with \( c(i, j) = 0 \) when a connection is empty, so if the pair \( p_i \) and \( p_j \) is not connected or, equivalently, if
$C(i, j) = e$. The function $c$ is symmetric, so

$$c(i, j) = c(j, i) \quad \forall i, j \in \{1, \ldots, n\}. \tag{3}$$

To express the forces due to self-stress of the members a Hookean constitutive law is assumed. The rest-length $r$ of the members is introduced, which will be used later as the control input $u$. Then the stiffness is of the form

$$c(i, j) = \frac{E(i, j)A(i, j)}{r(i, j)}, \tag{3}$$

where $E$ and $A$ are Young’s modulus and the cross sectional area of the member, respectively. Normally, the rest length of bars is larger than the distance between the points the bar connects to, while for tendons it is the reverse, due to pre-stressing the tendons, resulting in a tensile force in tendons and a compressive force in bars.

The members have to be arranged in such a way that the structure has integrity and is not a mechanism. This depends on $p$ and $C$. A necessary condition is expressed by requiring a self-stressed equilibrium for a configuration $p$.

For equilibrium the forces in all $n$ points of the tensegrity should cancel each other

$$\sum_{j=1}^{n} c(i, j) \left(1 - \frac{r(i, j)}{\|p_i - p_j\|}\right) (p_i - p_j) = 0$$

$$\forall i \in \{1, \ldots, n\}, \tag{4}$$

with $\|p_i - p_j\|$ the Euclidean distance between points $p_i$ and $p_j$.

The self-stress is realized by requiring

$$r(i, j) > \|p_i - p_j\| \text{ for } C(i, j) = b$$

$$r(i, j) < \|p_i - p_j\| \text{ for } C(i, j) = t \tag{5}$$

$$\forall i, j \in \{1, \ldots, n\}.$$

The above can be formulated a little more general, by not already introducing stiffness (2) and a linear constitutive equation (3), but by only requiring force equilibrium at the nodes, irrespective of where the forces stem from. This can be formulated compactly, using force densities [9], as

$$A(p, C)F = 0, \tag{6}$$

where $F$ is a column wise ordered set of force magnitudes in the members and $A$ is a matrix depending on the positions of the nodal points, $p$, and on the connection function, $C$.

Using a particular sign convention, the requirement that tendons are in tension and bars in compression is expressed by

$$F > 0. \tag{7}$$

Now (4) and (5) can be replaced by (6) and (7).

Given $p$ and $C$, the existence of a self-stressed equilibrium is just a linear feasibility problem for $F$, given by (6) and (7), see [10], and can be handled by a linear programming solver. The rest-lengths, $r$, and Hooke’s law do not play a role yet, so a nonlinear constitutive law is possible, that is why this formulation is slightly more general. It is assumed that $r$ can be chosen to realize the desired force magnitude, $F$, and that finally $r$ becomes the control input.

It is also possible to incorporate external forces in the problem formulation, e.g., by using the column wise ordered force components at nodal points, $w(p)$, which leads to

$$A(p, C)F = w(p) \tag{8}$$

for (6), or an equivalent for (4).

There are alternative formulations for the statics of a tensegrity. The main advantages are that (8)

- describes the full nonlinear geometry, so it is useful for large displacements,
- is still independent of the constitutive equations, so it is useful for large deformations,
- explicitly shows the influence of pre-stress via the force density $F$.

Essentially, all results obtained are equivalent with those obtained with a Finite Element Model, the bars and tendons being the elements. Furthermore, the results are exact, assuming that no conditions occur where the members are loaded in a different way than longitudinally.

A simple example of a planar tensegrity is given in Fig. 1. There are three units of 2 bars each. The endpoints of the bars represent the points $p_i$, of which there are 12 in this example, being twice the number of bars. Sixteen tendons are indicated in Fig. 1, but more tendons, connecting different points, can be added.

### III. DESIGN FOR SHAPE CONTROL

When changing the shape of a tensegrity, the rest lengths, $r$, are changed to achieve a desired set of nodal positions, $p$, while the self-stressed equilibrium conditions (4) and (5) or (6) and (7) still hold. When all points $p$ are prescribed, there is only a scaling freedom left in $F$ for non-singular configurations, so the case that is of interest here is when not all nodal points, $p$, are prescribed, so there is some freedom left in choosing $r$ or $F$, besides scaling only.
The particular question of interest is if it is possible to realize the requirements on the desired shape without changing $r$ for some of the members, which means that those members do not need to be actuated. This means that an additional distinction is made in the connection function, which now looks like

$$C: \{1, \ldots, n\} \rightarrow \{b_a, b_u, t_a, t_u, e\}, \quad (9)$$

where the subscript $a$ denotes actuated and $u$ denotes unactuated.

Other problems related to shape control are the computation of a “nice” trajectory for $r$ to go from one shape to another, which is a form of path planning, and the design of feedback control to stabilize the structure during the transition from one shape to another and to suppress vibrations by adding damping. Those two issues are addressed already in [11], so do not need to be addressed here.

The question if some of the members do not need actuation is relatively hard to solve, so some approximations are in order to simplify the problem to be able to get an on-line implementation, while providing some preliminary insights.

From the outset it is stated that we want to formulate the problem in the setting of mixed integer linear programming (MILP). Because the design variables of interest are the nodal positions, $p$, and the member forces, $F$, the equilibrium constraint (6) poses an obstacle, because it is nonlinear in the design variables. For this reason, the existence of a non-trivial $F > 0$ that makes a certain configuration $p$ a self-stressed equilibrium is deferred to an a-posteriori check, it is not enforced directly during the optimization.

Furthermore, we assume that tendons can always be actuated but that bars can be unactuated. The reason for this is that in general compressively loaded members are more expensive to actuate than tensile loaded ones, having to fulfill additional design requirements, namely buckling.

It is assumed that the desired shape can be characterized by a set of linear constraints on the positions,

$$Sp = d, \quad (10)$$

with $S$ a matrix with rank lower than $n$, the number of nodal points.

The requirement that the rest length of a bar does not change can be enforced by requiring the corresponding force magnitude component $F_k$ to be fixed and the actual length $\|p_i - p_j\|$ to be fixed also, or, more generally, to enforce a linear equation between $F_k$ and $\|p_i - p_j\|$ when a Hookean law is used. Requiring a length to be constant in a linear setting is not easy, so we simplify this by a linear approximation. Because it is desired for this approximation to always hold, also for large displacements, is has been formulated in the sense that rotations of bars are not allowed, but only translations and elongations $\Delta l$. For an unactuated bar then holds that $\Delta l = 0$, assuming the effect of changes in $F$ on length changes $\Delta l$ is of second order, while changes in rest-length $r$ are the main influence.

It is possible to allow for the bars to rotate. A way to achieve this is to allow a bar to have different angles, by gridding an angular sector, and requiring a bar to occupy exactly one grid point, introducing additional binary variables, which is not appealing, so this possibility is not used (yet).

The constraint on bar length change can be incorporated in a mixed integer set-up using a binary variable $s$ (a switch) as

$$\Delta l - s\Delta \bar{l} \leq 0 \quad (11)$$

$$\Delta l + s\Delta \bar{l} \geq 0$$

where $\Delta \bar{l}$ is the maximal allowable change in length. When $s = 0$, $\Delta l = 0$ is enforced, so no change in length is allowed, when $s = 1$ the length change $\Delta l$ can vary between $-\Delta \bar{l}$ and $\Delta \bar{l}$. The upper and lower bound on $\Delta l$ are based on physical considerations, and are chosen here symmetric for ease of notation.

To complete the formulation, besides the constraints (10) and (11) an objective is needed to formulate the problem as a standard MILP, although a formulation purely as a constraint satisfaction problem would also be possible, but will in general not deliver a unique solution for $p$, although this is also not guaranteed with an objective. A possible objective is to minimize the changes in length $\Delta l$ of the bars that are actuated, but $\Delta l$ can be both positive and negative, and a quadratic objective is not wanted. To get a linear objective function, $|\Delta l|$ is essentially used as objective. Minimizing $|\Delta l|$ can be cast in the form of a linear

![Fig. 1. Planar tensegrity structure build-up from 3 units. Bars: —, tendons: - -](image-url)
program (LP), due to the similarity between minimizing $|x|$ and minimizing the following LP

$$\begin{align*}
\min & \quad y \\
\text{sub} & \quad y \geq x \\
& \quad y \geq -x
\end{align*}$$

whose solution is $y = |x|$, see [12]. An additional design variable needs to be introduced to achieve this.

To summarize, the following MILP is formulated to optimize a tensegrity with shape constraints and to check if some bars can be unactuated

$$\begin{align*}
\min & \quad p, \Delta l, \bar{l}, y \\
\text{sub} & \quad \Delta l \bar{l} = d_t \\
& \quad S_p = d \\
& \quad y \geq \Delta l \\
& \quad y \geq -\Delta l \\
& \quad \Delta l - s \Delta \bar{l} \leq 0 \\
& \quad \Delta l + s \Delta \bar{l} \geq 0 \\
& \quad s \in \{0, 1\}
\end{align*}$$

with $h$ a column with positive coefficients. Note that with the assumptions that bars are not connected to other bars and all points $p$ connect to at least one bar, nodal points belong to exactly one bar, so all constraints need to hold for bars only. The nodal positions of the bars is restrained to be determined by a translation and an elongation $\Delta l$. The translation is essentially achieved by restricting the positions $p$ by the constraint

$$\begin{align*}
\Delta l \bar{l} = d_t.
\end{align*}$$

IV. DESIGN EXAMPLES

The computation is demonstrated for a planar tensegrity composed of 3 basic units, see Fig. 1. The number of nodes is $n = 12$. This is the same system as discussed in [11], but here another problem is tackled. The problem formulated in the previous section is looked at, and enhancements to the formulation are devised to get results that have some practical significance. The configuration and nominal shape are given in Fig. 2. The bars are indicated, and the ends of the bars are the points of the tensegrity structure, lying at $y = 0$ and $y = -1$. Tendons are omitted from the figures to follow, to avoid a cluttered appearance.

The connection $C$ is fixed, except for the freedom to assign a bar to the actuated or unactuated category, but the number of bars and tendons, and their connection topology, stays the same. It is not possible to eliminate bars or tendons, because the structure is already minimal, i.e., if one of the members is removed, no equilibrium with pre-stress in all members is possible, the structure collapses.

Two cases for the desired shape are discussed.

A. Case 1

The desired shape is such that the Cartesian coordinates of the upper layer of points of the tensegrity are related by the equality

$$\alpha x - y = 0$$

with $0 < \alpha < 3/14$. By varying $\alpha$ from 0 to $3/14$ a path is generated for the initial flat shape to the final tilted shape. Note that this constraint does not fix the points, but only requires the points to sit on a line with specified slope. This constraint is cast in the form $S_p = d$, together with boundary conditions for the points at the left side of the structure and for the top right node. To be specific, the top left node cannot move in both $x$ and $y$ direction while the bottom left node’s $x$-position is fixed. This removes the 3 degrees-of-freedom of a planar rigid body. The $x$ position of the top right node is also fixed, being related to a load application point.

Without additional constraints the results appeared to be bogus, implying that a tensegrity equilibrium cannot be guaranteed and is even unlikely. There is just too much freedom in (13) to deliver practical results, due to omitting the nonlinear self-stressed equilibrium constraint.

Therefore, compared to (13), the following constraints are added to get reasonable results, for which it can be expected that the existence of a self-stress equilibrium is possible, and even is likely:

- the sequence of nodal points in the lower and upper layer in $x$-direction is kept, by requiring the $x$ positions of nodes to correspond to the same order as in the nominal shape, with some slack; this involves a sequence of inequality constraints,
- the bars cross each other in the nominal shape, and these crossings of the bars are required to be preserved in the new shape, with some slack; this leads to a set of inequality constraints, using the known slope of the bars.
In [10] these situations were signaled as triggers for leaving the set of feasible configurations or shapes. Essentially, we replace the nonlinear self-stressable equilibrium constraint with a, possibly larger, set of linear constraints that approximate the nonlinear one, based on physical insight.

With these additional constraints, that can be cast in the form

\[ T_a p \leq d_a, \]

the problem to be solved is characterized by

- 42 design variable, of which 6 integer (binary) ones, corresponding to the 6 bars
- 22 equality and 43 inequality constraints
- all design variables having upper and lower bounds set, to avoid stray solutions.

Results of the optimization are given in Figs. 3–6, showing the shape of the structure for some values of \( \alpha \), so different slopes for the line that connects the points on the upper layer. For path planning purposes a finer resolution for \( \alpha \) has been used, the figures are only a sample.

Note that

- no initial conditions were used for the design variables
- all constraints are satisfied
- for \( \alpha = 1/14 \) no bars need to be actuated, only 2 of the bars are actuated for \( \alpha = 2/14 \), while 3 bars are actuated for \( \alpha = 3/14 \). The actuated bars are the ones that are longer or shorter than \( \sqrt{10} \) [m].
- the MILP is solved with a branch and bound (BB) method implemented in an interpreted language
- only 21 BB iterations were needed to find the optimal mixed integer solution for \( \alpha = 3/14 \)
- the computation time for the optimization is less than 1/2 second per BB iteration with the software used, but can be improved a lot by using compiled software
- the optimal solutions for \( p \) enables the computation of \( F \) as a simple linear program, generating all information needed to compute \( r = u \), if the constitutive equations are known; so the method enables the generation of a feasible path, for both the variables to be tracked, \( p \) and \( F \), and the control input, \( u \)
- the last two points imply that we have constructed a path planner that can be used on-line, when the sampling period for path generation is compatible with the computation time of the algorithm, which will be so for some applications, but not for all.

B. Case 2

In this case the desired shape is characterized only by constraining the right boundary, effectively changing its length and using the structure as a deployable system. The additional constraints introduced in the previous section are also employed here. The only difference is in the shape constraint and in some of the bounds on the design variables.

By varying the length between 5 [m] and 11 [m], Figs. 7–10 were generated.

Note that the solution for the 5 [m] length situation is highly singular, so not acceptable, while the solution...
for the nominal length of 7 [m], Fig. 8, differs from the equivalent structure found in the previous case, Fig. 3, due to the different shape constraints and bounds. It illustrates the non-uniqueness of the solution for this length, because the required Δl for these solutions are in both cases equal to 0, and no actuated bars are needed. The solutions for 7 [m], 9 [m], and 11 [m] are not symmetric, due to non-uniqueness, due to the minimization of Δ_l, and/or due to an uneven number of actuated members.

When symmetric solutions are required, additional constraints can be added – a flexible approach, or, by solving the constraints, some design variables can be eliminated – an efficient approach. Both approaches can be accommodated easily.

V. DISCUSSION

The results obtained indicate that a simple mixed integer linear programming formulation is often sufficient to compute configurations realizing different shapes for a tensegrity that needs a limited number of actuators only. This provides preliminary insight in design issues, which can be helpful in the conceptual design phase and in setting up the final design specifications. For a rigorous treatment, however, it seems that one still has to resort to a design using non-linear optimization, to enforce a self-stressed equilibrium, which, with present technology, is not a suitable approach for online implementation.

REFERENCES