Modeling the vibrations of a rotating tyre: a modal approach

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In the past years a considerable effort has been made to develop tyre models for tyre/road noise prediction. Most of these models focus on the high-frequency range (above 400Hz). Existing models for the lower frequency range are either analytical or semi-empirical, which are not suitable for industrial use as part of the design process. In the present paper a modal approach is presented for the modeling of the vibrations of a rotating tyre up to 500Hz, which is the relevant frequency range for interior noise. In this frequency range, the dynamics of a tyre is significantly influenced by the large deformation of the tyre due to the weight of the car. Hence in the approach presented here, first the large stationary deformation of the tyre due to the weight of the car is solved. Next the natural frequencies and modeshapes are calculated of the non-rotating tyre pressed onto the road. These modal data are transformed using an Eulerian approach to obtain the dynamics equations that describe the behavior of the rotating tyre pressed onto the road. The strength of this method is that from a standard FE calculation a fairly simple description of the dynamics of a rotating tyre pressed on the road can be obtained. This approach has been tested on a simplified model of a tyre. It is shown that the model predicts the shift due to rotation in the dispersion curves correctly. The effect of tyre deformation on the dynamics of the tyre is presented. Furthermore, the predicted forced response of the rotating tyre is in agreement with results from models found in the literature.

1 Introduction

Road traffic noise is becoming an increasingly big problem in densely populated areas. One of the main contributions comes from the tyre/road interaction. For the interior noise, to which the occupants of the vehicle are exposed, tyre/road noise is also an important source, especially at lower frequencies. The vibrations of the tyre up to 500 Hz are transmitted via structural transmission paths to the interior effectively, thus causing acoustic noise. In this frequency range tyre/road noise is dominated by vibrational mechanisms [1]. In the literature many models for the vibrational behavior of tyres can be found.

In [2, 3] ring models are used to model a tyre. However, because a ring model is a 2D-model it is unable to describe displacement variations occurring in the axial direction (across the width) of the tread. This means ring models can only be used up to about 300 Hz, the frequency at which waves start to propagate across the tread in axial direction. Another approach is using a plate to model the tyre like in [4, 5, 6]. However these models are only valid above the first ring frequency at about 400 Hz because the curvature is not taken into account. A common drawback of these models is that only a few parameters are available in these models to describe the complete tyre, while in reality the tyre is made out of dozens of materials and consists of many different layers. This makes it very difficult to determine the correct material properties.

Cylindrical shell models have also been used to model the tyre dynamics, but these models mostly consist of only one layer, again making it difficult to determine the correct parameters. In a recent article of Kim and Bolton [7] a cylindrical shell model is used to investigate the influence of the rotation on the dynamics of the tyre. The conclusion is that the main influence of the rotation is a kinematic shift of the dispersion curves which depends on the rotational velocity.

Regarding the application of the finite element method (FEM) to tyres, only works where a modal approach is used and/or the rotation of the tyre is considered will be mentioned here. In [8] Chang and Yang use a modal approach based on a FE model of a tyre to determine the response to a rotating load. They show that resonances occur at frequencies different from the natural frequencies of the non-rotating tyre. However no results regarding the response of the rotating tyre are shown. Furthermore the deformation of the tyre due to ground contact is not taken into account.

In a recent publication Brinkmeier et al. [9] use an Augmented Lagrange Eulerian (ALE) formulation to determine the eigenvalues and -modes of a stationary rolling tyre in ground contact, leading to conclusions similar to those of Kim and Bolton [7]. Unfortunately, only a limited amount of results are shown. The main drawback of this approach is that a special FE code is required for the ALE formulation, which is not included in standard FE
In the present paper a methodology is presented for the modeling of tyres at frequencies up to 500 Hz, which is the most important frequency range for interior noise. Using a detailed FE model of a tyre is a feasible approach in this frequency range and, with a FE model, it possible to relate tyre design parameters to its vibro-acoustic properties. The approach presented in this paper can be summarized as follows:

- A modal approach is used, based on modal information extracted from a FE calculation. In this way it is possible to model the complex build up of the tyre in detail.
- The natural frequencies and modesshapes of a non-rotating tyre are used (standard FE package). This implies that the stiffening of the tyre due to the rotation and the Coriolis effect are not included in this approach. This is not considered a problem as these effects are negligible in comparison to the stiffening due to inflation (7).
- The natural frequencies and modesshapes of the tyre are determined in the deformed state. This way the influence of the ground contact on the dynamics is taken into account.
- The influence of the rotation is taken into account using a coordinate transformation.
- The response of the rotating tyre is determined in non-rotating (Eulerian) coordinates. This way the results can be used directly for a sound radiation analysis.

2 Theory

In this section an approach is described which uses the natural frequencies and modesshapes of a loaded, non-rotating (Eulerian) coordinate system that is fixed to the center of the tyre and translates together with the tyre, but does not rotate. This way the natural frequencies of a non-rotating (Eulerian) coordinate system can be modelled as a force which rotates around the non-rotating tyre. This way the natural frequencies in this frequency range and, with a FE model, it possible to relate tyre design parameters to its vibro-acoustic properties.

2.1 Definitions

Two coordinate systems are used: \( e^1 \) and \( e^2 \). \( e^1 \) is the reference coordinate system that is fixed to the center of the tyre and translates together with the tyre, but does not rotate. \( e^2 \) is a body-fixed coordinate system. This means that this system rotates along with the tyre at velocity \( \Omega \). See figure 1. In these systems the angles \( \alpha \) in the body-fixed frame and \( \beta \) in the reference frame are defined, so that for a certain point \( k \) on the tyre \( \beta_k = \alpha_k + \Omega t \).

\[ \beta = \alpha + \Omega t. \]

For the rotation matrix \( A^{21}(t) \) defined as:

\[ A^{21}(t) = \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \] (1)

the following equation holds:

\[ \ddot{e}^2 = A^{21}(t) \dot{e}^1 \] (2)

where \( \Omega \) is the rotational velocity of the rotating reference frame. The derivative of the rotation matrix equals:

\[ \dot{A}^{21}(t) = \hat{\Omega} A^{21}(t) \] (3)

where

\[ \hat{\Omega} = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} \] (4)

And the second derivative equals:

\[ \ddot{A}^{21}(t) = \hat{\Omega}^2 A^{21}(t) \] (5)

For the time being, the excitation is defined as a point force that acts at a fixed position in the reference frame. In the reference frame \( e^1 \) this force can thus be written as:

\[ F^{1T} \dot{e}^1 = \begin{bmatrix} F_x \delta(\beta) \\ F_y \delta(\beta) \end{bmatrix} \] (6)

where \( \delta \) is the Dirac delta function. This leads to a "rotating" force in the body fixed system:

\[ F^{2T} \dot{e}^2 = \begin{bmatrix} F_x \delta(\alpha + \Omega t) \\ F_y \delta(\alpha + \Omega t) \end{bmatrix} A^{12}(t) \dot{e}^2 \] (7)

where

\[ A^{12} = (A^{21})^{-1} = (A^{21})^T \] (8)

2.2 Body fixed modes

As mentioned before for an undeformed tyre the rotation may be modelled as a force which rotates around the non-rotating tyre. This way the natural frequencies
and modeshapes determined by the FE program can be directly used to determine the response by modal superposition. After assembling the mass \( M_{\text{FEM}} \) and stiffness \( K_{\text{FEM}} \) matrices, eq. (9) describes the (undamped) system in the rotating frame \( \hat{e}^2 \):

\[
M_{\text{bf}}^2(\alpha)\ddot{u}^2(t) + K_{\text{bf}}^2(\alpha)u^2(t) = F^2(\alpha + \Omega t, t)
\]

(9)

where \( u^2(t) \) is the displacement column and \( F^2(\alpha + \Omega t, t) \) is the rotating force in \( \hat{e}^2 \).

For the body fixed modes (indicated by the subscript bf) \( M_{\text{bf}}^2(\alpha) = M_{\text{FEM}} \) and \( K_{\text{bf}}^2(\alpha) = K_{\text{FEM}} \). So by using the standard transformation to modal coordinates this can be written as:

\[
\ddot{\eta}(t) + \hat{V}\eta(t) = \Phi_{\text{bf}}^2 T \hat{F}^2(\alpha + \Omega t, t)
\]

(10)

where \( \hat{V} \) is the matrix containing the eigenvalues and \( \Phi_{\text{bf}}^2 = \Phi_{\text{FEM}} \) are the modeshapes determined by the FE program. Clearly this leads to a response in the body fixed frame \( \hat{e}^2 \):

\[
u_{\text{bf}}^2(t) = \Phi_{\text{bf}}^2 \eta(t)
\]

(11)

This is a good approach for degenerated modes, but it is not appropriate to predict the response of a deformed rotating tyre. To be able to predict the response of a deformed rotating tyre another approach has to be used which is described in section 2.3

### 2.3 Reference frame fixed modes

The tyre/road contact has a significant influence on the tyre dynamics. Seen from a modal point of view it leads to the modeshapes being non-axisymmetric. In the undeformed state a tyre has rotation symmetric modeshapes; at each eigenvalue there are two degenerated modeshapes. The degenerated modeshapes are identical eigenmodes, except that they are rotated \( \frac{360}{m} \) degrees, where \( m \) is the number of waves along the circumference (figure 2). When the tyre is excited by a point force two modes of the same frequency combine in such a way that the maximum displacement is in the direction of the excitation force.

![Figure 2: Example of two degenerated modes with n=2.](image)

However, when the tyre is in contact with the ground the modeshapes are not degenerated anymore. They have a clear orientation due to the initial deformation and are not rotation symmetric anymore. The modeshapes are in fact fixed to the reference frame \( \hat{e}^1 \), as are the mass and stiffness matrix.

A mode \( \phi_{\text{bf}} \) which is fixed to the reference frame (indicated by the subscript rf) can be described as follows:

\[
\ddot{\phi}_{\text{rf}} = \Phi_{\text{bf}}^{rf} T \hat{e}^2 = \begin{bmatrix} \phi_x(\beta) \\ \phi_y(\beta) \end{bmatrix}^T \hat{e}^1
\]

(12)

This means that a mode which is constant in the reference frame is time-dependent in the body-fixed frame. Matrices \( \Phi_{\text{bf}}^r \) containing \( m \) modeshapes can now be defined as:

\[
\Phi_{\text{bf}}^r = \begin{bmatrix} \phi_{1,1}^r & \ldots & \phi_{1,m}^r \end{bmatrix} \text{ with } i = 1, 2
\]

(13)

with for the reference frame:

\[
\phi_{1,m}^r = \begin{bmatrix} \phi_{x,m}(\beta) \\ \phi_{y,m}(\beta) \end{bmatrix} = \phi_{\text{FEM},m}
\]

(14)

which are the modeshapes determined in the FE analysis. For the mass matrix this means:

\[
M_{\text{bf}}^1(\beta) = A^{12}(\Omega t)M_{\text{bf}}^1(\alpha + \Omega t)A^{21}(\Omega t)
\]

(15)

And for the stiffness matrix:

\[
K_{\text{bf}}^1(\beta) = A^{12}(\Omega t)K_{\text{bf}}^2(\alpha + \Omega t)A^{21}(\Omega t)
\]

(16)

When using reference frame fixed modes \( M_{\text{bf}}^1(\beta) = M_{\text{FEM}} \) and \( K_{\text{bf}}^1(\beta) = K_{\text{FEM}} \). The equations of motion of the tyre in the body fixed, lagrangian coordinates are given in eq. (17).

\[
M_{\text{bf}}^2(\alpha + \Omega t)\ddot{u}^2(t) + K_{\text{bf}}^2(\alpha + \Omega t)u^2(t) = F^2(\alpha + \Omega t, t)
\]

(17)

Which is the same as eq. (9) except for the mass and stiffness matrices which are now fixed to the reference frame. These equations can be transformed into the reference coordinate system by using the material derivative:

\[
\frac{D}{D\tau} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \beta}
\]

(18)

where the left hand side represents the time derivative in the body fixed (Lagrangian) coordinates, the first term on the right hand side is the time derivative in the reference (Eulerian) coordinates, \( \Omega \) is the rotational speed and \( \beta \) is the circumferential angle in the reference frame. Applying eq. (18) to eq. (17), transforming back to the reference coordinates and pre-multiplying by \( \Phi_{\text{bf}}^r^{Tj}(\beta) \) leads to:

\[
\ddot{\eta}(t) + 2\hat{\Phi}M\hat{\Phi}\dot{\eta}(t) + \hat{\Phi}(\hat{\Phi}^2 + \hat{V})\eta(t) = \Phi_{\text{bf}}^r^{Tj}(\beta)F^1(t)
\]

(19)

Where:

\[
\hat{\Phi}M\hat{\Phi} = \Phi_{\text{bf}}^{rf} (\beta)M_{\text{bf}}^1 \left( \hat{\Phi}_{\text{bf}}^r(\beta) + \Omega \frac{\partial \Phi_{\text{bf}}^r(\beta)}{\partial \beta} \right)
\]

(20)
\[ \Phi \dot{M} \ddot{\Phi} = \Phi_{rf}^{T}(\beta)M_{rf}^{1}\left(\hat{\Omega}^{2}\Phi_{rf}^{1}(\beta) + 2\Omega \hat{\Omega} \frac{\partial \Phi_{rf}^{1}(\beta)}{\partial \beta} + \Omega^{2} \frac{\partial^{2} \Phi_{rf}^{1}(\beta)}{\partial \beta^{2}}\right) \]

(21)

and \( \Phi_{rf}^{1}(\beta) = \Phi_{\text{FEM}} \) are the modeshapes determined in the FE analysis. The response in the reference frame can now directly be determined from:

\[ \Phi_{rf}^{1}(\omega) = \Phi_{rf}^{1} \cdot \Phi_{rf}^{T}(\beta) \eta(t) \]

(22)

Although not shown here, equation (19) can be easily extended to include proportional damping [10]. While the equations are uncoupled in eq. (10), this is no longer the case in eq. (19). The matrices \( \Phi \dot{M} \dot{\Phi} \) and \( \Phi \dot{M} \ddot{\Phi} \) are not diagonal, which means that eq. (19) is a system of coupled equations. If necessary a second eigenvalue analysis can be performed on eq. (19) to determine the natural frequencies and modeshapes of the rotating tyre and the equations can be uncoupled. If eq. (19) and (22) are transformed to the frequency domain and combined, the receptance matrix of the rotating tyre can be found:

\[ \mathbf{r}(\omega, \Omega) = \mathbf{H}_{rf}^{1}(\omega, \Omega) \mathbf{F}^{1}(\omega) \]

(23)

so that

\[ \Phi_{rf}^{1}(\omega, \Omega) = \mathbf{H}_{rf}^{1}(\omega, \Omega) \mathbf{F}^{1}(\omega) \]

(24)

Eq. (24) relates a force acting at any point on the tyre to the displacement at every point on the tyre. Since the forces outside the contact area are zero, it is possible to use only a small sub-matrix of \( \mathbf{H}_{rf}^{1}(\omega, \Omega) \) defined by those points of the tyre that are likely to come into contact with the road. Since the contact between the tyre and the road is of non-linear nature the description of the vibrational properties of the tyre has to be formulated in the time-domain. An inverse FFT has to be performed on the sub-matrix of \( \mathbf{H}_{rf}^{1}(\omega, \Omega) \) to determine the Green’s functions (impulse response) of the tyre. Then only this sub-matrix is needed to calculate the contact forces and the response of the whole tyre can be determined afterwards using the full matrix.

### 3 Results

To test the approach described in the previous section two models are used. A 2D ring model and a slightly more complex 3D model, which has a geometry that looks more like a real tyre. The results will be compared to literature (mainly [7]). For the ring models 200 beam elements have been used in the FE calculation which resulted in a 600 DOF model. In total 80 modes have been used with frequencies up to 2000 Hz. The parameters of the ring model can be found in table 1. For the 3D model (figure 3) 3700 brick elements have been used resulting in 7600 DOFs. An inflation pressure of 10⁶ Pa has been used and 100 modes have been determined with frequencies up to ±700 Hz. The parameters of the 3D model can be found in table 2. The parameters of these models are not realistic for tyres.

#### 3.1 2D Ring-model

To test the theory with the reference frame fixed modes from section 2.3 an analytical description of the 2D ring is used. The radial displacements of the modeshapes of a ring can be written as \( u = \cos(na) \) and \( v = \sin(na) \). Here \( n \) is the number of waves along the circumference and the corresponding eigenfrequency is \( \omega_{n} \). These analytical modeshapes are used to calculate the eigenvalues of the rotating ring.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>0.3 m</td>
</tr>
<tr>
<td>( E )</td>
<td>2 \times 10^{9} \text{N/m}^{2}</td>
</tr>
<tr>
<td>( \nu )</td>
<td>0.3</td>
</tr>
<tr>
<td>( \rho )</td>
<td>7800 \text{kg/m}^{3}</td>
</tr>
<tr>
<td>( r )</td>
<td>0.005 m</td>
</tr>
</tbody>
</table>

It can be shown that, when only two modes are used, equation (19) becomes:

\[ \dot{\eta}(t) + 2\Omega \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix} \eta(t) + (V - n^{2} \Omega I) \eta(t) = \Phi_{rf}^{1} \mathbf{F}^{1} \]

(25)

And the eigenvalues in of the rotating ring can be determined by:

\[ \det(\lambda I - B_{rf}^{1}) = 0 \]

(26)
where

$$B^1 = \begin{bmatrix} 0 & 1 \\ n^2 \Omega^2 I - V & -2\Omega \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix} \end{bmatrix}$$

(27)

Leading to:

$$\lambda = \left[ i(\omega_n \pm n\Omega) - i(\omega_n \pm n\Omega) \right]$$

(28)

Which clearly shows the splitting of the eigenvalues.

As discussed in section 2 the rotation of the tyre leads to a shift of the dispersion curves. This effect is important and therefore should also be properly reproduced. To show the splitting of the eigenvalues on the dispersion curves, the FRF’s of all nodes on the ring are determined using eq. (24) for an excitation at $\beta = 0$. This is done for the frequency range 100-1000 Hz. Then for each frequency for which the steady state response is calculated, a spatial FFT is performed for the “upstream” half and the “downstream” half of the ring. This leads to the continuous dispersion plots shown in figures 4 and 5. The triangular symbols in the figures indicate the eigenfrequencies determined in the eigenvalue analysis for $\Omega = 0$. The square symbols are the corrected eigenvalues using equation (32) from [7]. In figure 5 it can be seen that eigenfrequency shifts are predicted correctly.

3.2 Simplified 3D tyre-model

In the previous sections the theory from section 2.3 has been verified with use of a 2D ring model. In this section it is shown that for the 3D model the splitting of frequencies is also correctly modeled.

Figure 6 shows the point admittance in the radial direction at a point on the middle circle of the 3D model for a rotational velocity of 0 rad/s and 100 rad/s. In the figure the 1 indicates a mode with $n=1$. The 1* indicates the two modes which appear when the tyre is rotating. These modes are shifted $\pm \frac{20\pi}{2\pi} \approx \pm 20$ Hz with respect to the situation where $\Omega = 0$ rad/s. 2 and 2* indicate mode with $n=2$ for the non-rotating and rotating tyre respectively. The latter are shifted $\pm \frac{20\pi}{2\pi} \approx \pm 20$ Hz. 3 indicates a modeshape with $n=0$. The frequency of this mode does not shift due to the rotation as there is no variation of this modeshape along the circumference. This effect is also shown in figure 6, where the eigenvalues corresponding with the 10 first eigenmodes (in the non-rotating situation) are shown. Again the splitting of the eigenvalues can be seen. The slope of the lines is determined by the number of wavelengths along the circumference. These results are the same as predicted in [7].

4 Conclusions

This paper examines an approach to model the tyre vibrations in the 0-500 Hz region. Determining the eigenvalues and eigenmodes of a detailed FE-model of the tyre and then using these to construct a modal base of the tyre seems a computationally cheap way of calculating the dynamic response of the tyre without neglecting its complex build-up.

It has been shown that it is necessary to use reference frame fixed modes to model the behavior of the tyre correctly when an asymmetry in the modeshapes is present (e.g. caused by initial deformation). These modes are constant in the reference frame, but rotate along with the excitation in the body fixed frame. The approach pre-
The modelling approach presented in this paper has some drawbacks:

- In eq. (20) and (21) the full mass matrix of the FE model is used. For a large model, exporting the mass matrix and computing $\Phi M \Phi$ and $\Phi M^{\phi} \Phi$ from eq. (19) could be a serious problem. However, it should be kept in mind that these matrices only have to be computed once and that their final size is equal to the number of modes chosen.

The proposed methodology has been applied to two relatively simple models showing that the effects of rotation are modeled correctly and are in accordance with results from literature.

References


