A negative Bendixson-like criterion for a class of hybrid systems

Alexander Pogromsky, Henk Nijmeijer, and Koos Rooda,

Abstract

A condition which ensures absence of periodic orbits for nonsmooth dynamical systems is presented. The condition is a higher dimensional generalization of Bendixson’s criterion applicable to differential inclusions. The main argument is based on contraction analysis of the $d$-measured volume along the system trajectories. A connection to the methods for estimates of the Hausdorff dimension is emphasized.

1 Introduction

Discontinuous dynamical systems and, particularly, relay systems have attracted considerable attention last decades. While in the studies of nonlinear systems from pure mathematical point of view, the smooth dynamical systems can still be a source of numerous discoveries, in applied disciplines it has been realized that for many applications discontinuities should be taken into account. For example, discontinuities can be used to simplify modelling of friction phenomenon in mechanical systems, to design disturbance tolerant sliding mode controllers, to deal with a switching control strategy in manufacturing lines and so on. One of the hot topics in research in control community now is so called hybrid dynamical systems, which combine continuous and discrete dynamics. Although the existing literature on this subject includes growing number of monographs and papers (see e.g. [1, 2, 3, 4, 5], to mention a few), those systems are far from being completely understood.

Hybrid systems being nonlinear dynamical systems can have very rich phenomenology and one of the main theoretical problems is to predict and to understand it without explicit solution of the equations describing the system. In this paper we are going to study oscillatory behavior in nonlinear hybrid dynamical systems. The class of systems we study is

---

*This work was partially supported by the EU project SICONOS (IST-2001-37172).
†The authors are with the Department of Mechanical Engineering, Eindhoven University of Technology, The Netherlands. a.pogromsky@tue.nl, h.nijmeijer@tue.nl, j.e.rooda@tue.nl
‡A preliminary version of the paper submitted to MTNS’04 mini-symposium “Mathematical theory of oscillations in networks and systems”
described by ordinary nonlinear differential equations with discontinuous right hand side. A particular question which we are going to address is a generalization of Bendixson’s negative divergency test for that class of systems. This simple test gives a sufficient condition for nonexistence of periodic orbits for smooth planar systems. This classical result claims that if in a simply connected domain the divergency of the vector field does not change its sign, this domain does not contain whole periodic orbits. A classical proof of this statement is based on the divergence theorem and cannot be generalized to the higher dimensional case. The main purpose of this paper is to present one of the possible generalization of the Bendixson result for the case of arbitrary dimension taking into account possible discontinuity of the right hand side. There are several higher dimensional generalizations of this criterion, see, e.g. [6, 7, 8, 9]. Muldowney and Li [7, 8, 9] used an approach based on compound matrices to prove a negative Bendixson-like criterion. In this paper we investigate this question by a method which allows to estimate the Hausdorff dimension of invariant compact sets [10, 11, 12, 13, 14]. In doing so, first we present a generalization for the estimate for Hausdorff dimension formula for non smooth systems and then, based on that result we prove a negative criterion for nonexistence of periodic orbits.

From practical point of view a design based on global stability of a system can be too restrictive and conservative. A possible weaker criterion is that all trajectories tend to a set consisting of equilibrium points, that is that the system can not exhibit oscillatory behavior. This fact indicates importance of the Bendixson-like criteria for the design (control) of dynamical systems. A similar motivation can be found, for example in recent paper [15], where for smooth systems a condition which guarantees that almost all trajectories tend to an equilibrium was presented. Another approach to simplify the stability analysis of the discontinuous systems is based on the generalization of the Invariance Principle for differential inclusions, see e.g. [16].

A general idea behind the proof of the nonexistence criterion is relatively simple. If one is able to show that in a simply connected positively invariant domain the flow generated by the system contracts the area of some initial surface, it is sufficient to claim that no periodic orbits can lie entirely in this domain. By reversing time the same holds true for area expanding systems. Together with the Liouville theorem this argument gives another proof of Bendixson’s criterion that can be generalized to arbitrary dimension. To characterize the area of a surface one can use so called Hausdorff 2-measure, so the area contracting systems are those for which Hausdorff 2-measure of any initial measurable set vanishes with time.

The main method employed in our study is based on stability/dichotomy-like properties of the solutions with respect to each other rather than with respect to some invariant sets. The first results in this direction were developed by Demidovich [17] and Yoshizava [18]. Methods based on similar ideas are appreciated now in control community [19, 20, 21]. A natural way to investigate those properties is based upon linearization of the dynamical system along any given trajectory which excludes from consideration the class of non
smooth systems. In this paper instead of linearization we investigate behavior of some quadratic forms defined on two trajectories of the system, that allows us to take into account discontinuous systems. The conditions presented in this paper are formulated in terms of inequalities involving two eigenvalues of some matrix pencil.

The paper is organized as follows. In Section II we present necessary background material. Section III contains some result on estimation of Hausdorff dimension of invariant sets. Based on these results in Section IV we present a new version of a generalized Bendixson’s criterion. A particular attention is then drawn to LMI based results for linear systems with relay feedback.

2 Hausdorff dimension

Consider a compact subset $K$ of a compact metric space $X$. Given $d \geq 0$, $\varepsilon > 0$, consider a covering of $K$ by open spheres $B_i$ with radii $r_i \leq \varepsilon$. Denote by

$$
\mu(K, d, \varepsilon) = \inf \sum_i r_i^d
$$

the $d$-measured volume of covering of the set $K$. Here the infimum is calculated over all $\varepsilon$-coverings of $K$. There exists a limit, which may be infinite,

$$
\mu_d(K) := \sup_{\varepsilon > 0} \mu(K, d, \varepsilon).
$$

It can be proved that $\mu_d$ is an outer measure on $X$ (see Proposition 5.3.1, [13]).

**Definition 1** The measure $\mu_d$ is called the Hausdorff $d$-measure.

Some properties of the measure $\mu_d$ can be summarized as follows. There exists a single value $d = d_*$, such that for all $d < d_*$, $\mu_d(K) = +\infty$ and for all $d > d_*$, $\mu_d(K) = 0$, with

$$
d_* = \inf\{d : \mu_d(K) = 0\} = \sup\{d : \mu_d(K) = +\infty\}.
$$

(see Proposition 5.3.2 in [13]).

**Definition 2** The value $d_*$ is called the Hausdorff dimension of the set $K$.

In the sequel, we will use the notation $\dim_H K$ for the Hausdorff dimension of the set $K$. Following Douady and Oesterlé [10], we define the elliptic Hausdorff $d$-measure of a compact set $K \subset \mathbb{R}^n$. Let $E$ be an open ellipsoid in $\mathbb{R}^n$. Let $a_1(E) \geq a_2(E) \geq \ldots \geq a_n(E)$ be the lengths of semiaxes of $E$ numbered in the decreasing order. Represent an arbitrary
number \(d\), \(0 \leq d \leq n\) in the form \(d = d_0 + s\), where \(d_0 \in \mathbb{N}\) and \(s \in [0, 1)\) and introduce the following

\[
\omega_d(E) = \prod_{i=1}^{d_0} a_i(E)(a_{d_0+1}(E))^s. \tag{2}
\]

Fix a certain \(d\) and \(\varepsilon > 0\) and consider all kinds of finite coverings of the compact \(K\) by ellipsoids \(E_i\) for which

\[
[\omega_d(E_i)]^{1/d} \leq \varepsilon
\]

(if \(d = 0\) we put \([\omega_d(E_i)]^{1/d} = a_1(E_i))\). Similar to the definition of Hausdorff \(d\)-measure we denote

\[
\tilde{\mu}_d(K, d, \varepsilon) = \inf \sum_i \omega_d(E_i),
\]

where the minimum is calculated over all coverings.

**Definition 3** The value

\[
\tilde{\mu}_d(K) = \sup_{\varepsilon > 0} \tilde{\mu}(K, d, \varepsilon)
\]

is called the Hausdorff elliptical \(d\)-measure of the compact \(K\).

It was proven in [10, 12] that elliptical and spherical Hausdorff \(d\)-measures are equivalent and therefore, using extremal properties of \(\mu_d\), the values of Hausdorff dimensions determined by means of spherical and elliptic coverings are equal.

For the control community the notions of Hausdorff measure and Hausdorff dimension are not of common use and we would like to clarify the above definitions.

Suppose we have a two-dimensional bounded surface \(S\) with area \(m(S)\). Let us cover this surface by open spheres as required in definition of Hausdorff measure. Then, for \(d = 1\) and \(d = 3\) we have

\[
\mu_1(S) = \lim_{\varepsilon \to 0} \mu(S, 1, \varepsilon) = +\infty,
\]

\[
\mu_3(S) = \lim_{\varepsilon \to 0} \mu(S, 3, \varepsilon) = 0,
\]

while for \(d = 2\) we have

\[
\mu_2(S) = \frac{m(s)}{\pi}.
\]

This example illustrates the behavior of \(\mu_d(K)\) for a given \(K\) as a function of \(d\). Namely, for the values of \(d\) less than \(\dim_H K\) \(\mu_d(K)\) is infinity and for all values of \(d\) greater than \(\dim_H K\) \(\mu_d(K)\) is zero (see Proposition 5.3.2 in [13]). This situation is schematically presented on Fig. 1.

For the “good” sets such as a piece of arc, a piece of smooth surface, etc. the Hausdorff dimension can be used as a dimension in a normal sense of this word (i.e. in the sense of Brouwer, or Lebesgue). This follows from the result (see, e.g. Proposition 5.3.5 in [13])
which claims that for a set $K$ of positive $n$-dimensional Lebesgue measure the Hausdorff dimension of $K$ is $n$. However for other sets such as Cantor set the value of the Hausdorff dimension can be fractional. The sets of such type are often encountered as invariant sets of “chaotic” systems, that makes the Hausdorff dimension of invariant sets an important characteristic for “chaotic” systems.

3 Upper estimates for the Hausdorff dimension of invariant compact sets

The extremal property of the Hausdorff dimension suggests an idea of how to estimate it for invariant sets of dynamical systems. Namely, if one able to prove that for a given set $K$ its Hausdorff $d$-measure is zero, then it follows that $d$ is an upper estimate of the Hausdorff dimension of $K$. A possible way to show that the Hausdorff $d$-measure is zero is to prove the following inequality

$$\mu_d(\varphi(K)) \leq \nu \mu_d(K)$$

where $\nu < 1$, $\varphi$ is some mapping and $K$ its invariant set, i.e. $\varphi(K) = K$ and hence $\mu_d(\varphi(K)) = \mu_d(K)$. This identity together with (3) implies that $\mu_d(K) = 0$.

When $\varphi$ is a flow generated by a system of differential equations the inequality (3) follows from the fact that $d$-measured volume of an open neighborhood of the invariant set $K$ decreases along the system trajectories. This observation suggests to employ Lyapunov-like technique to estimate the Hausdorff dimension of invariant sets.

As has been mentioned in the Introduction a generalization of Bendixson’s criterion can be derived if one able to show that in some simply connected region there are no invariant sets with Hausdorff dimension greater or equal than 2. This result can be obtained if one takes 2-measured volume as a Lyapunov function candidate. However, in this section we
present a more general result which holds for arbitrary $d$ and in the next section we present a higher dimensional generalization of Bendixson’s criterion.

Consider a system of differential equations

$$\dot{x} = f(x), \quad x \in \Omega \subset \mathbb{R}^n, \quad x_0 \in \Omega$$

(4)

where $f : \Omega \rightarrow \mathbb{R}^n$ is a (possibly) discontinuous vector field defined on some open positively invariant set $\Omega$ which satisfies conditions guaranteeing the existence of solutions $x(t, x_0)$ in $\Omega$ in some reasonable sense, that is, if the function $f$ is discontinuous and satisfies some mild regularity assumptions, one can construct a set-valued function $f$ bounded on any compact set according to numerous possible definitions (e.g., Filippov convex definition, Utkin’s equivalent control, etc.) such that a solution of the differential inclusion

$$\dot{x} \in f(x)$$

is called a solution for system (4). We require that the solution $x(t, x_0)$ is an absolutely continuous function of time and it possesses a continuation property. Additionally we assume that if the solution $x(t, x_0)$ is (right)-unique then continuous dependence on initial conditions in forward time is granted.

The parameterized mapping $x_0 \mapsto x(t, x_0)$, or the flow will be denoted as $\varphi^t : \Omega \rightarrow \Omega$. Consider a differentiable scalar function $V : \Omega \times \Omega \rightarrow \mathbb{R}$, $V(x, x) = 0$.

Define the time derivative of the function $V$ along the two solutions $x_1(t, x_{10}), x_2(t, x_{20})$ of (4) as follows

$$\dot{V} = \frac{\partial V(x_1, x_2)}{\partial x_1} \dot{x}_1(t, x_{10}) + \frac{\partial V(x_1, x_2)}{\partial x_2} \dot{x}_2(t, x_{20}).$$

Since $V$ is Lipschitz continuous and the solutions $x_i(t, x_{i0})$ are absolutely continuous functions of time, the derivative

$$\dot{V}(x_1(t, x_{10}), x_2(t, x_{20}))$$

exists almost everywhere in $[0, \min_i \bar{T}_i)$, where $\bar{T}_i$ is the maximal interval of existence of solution $x_i(t, x_{i0})$ in $\Omega$.

For the function $V$ we can also define its upper derivative as follows

$$\dot{V}^*(x_1, x_2) = \sup_{\xi \in f(x_i)} \left( \frac{\partial V(x_1, x_2)}{\partial x_1} \xi_1 + \frac{\partial V(x_1, x_2)}{\partial x_2} \xi_2 \right).$$

Then for almost all $t \geq 0$ it follows that

$$\dot{V}(x_1(t, x_{10}), x_2(t, x_{20})) \leq \dot{V}^*(x_1(t, x_{10}), x_2(t, x_{20})).$$

We formulate the following hypothesis:
H1. There exists a continuously differentiable $n \times n$ symmetric matrix valued function $P$ defined in the domain $\Omega$, such that the function

$$V(x_1, x_2) = (x_1 - x_2)^\top P(x_1)(x_1 - x_2)$$

(5)

satisfies the following inequalities

$$\alpha_1 ||x_1 - x_2|| \leq V \leq \alpha_2 ||x_1 - x_2||$$

(6)

and

$$\dot{V}^*(x_1, x_2) \leq (x_1 - x_2)^\top Q(x_1)(x_1 - x_2)$$

for all $x_1, x_2 \in \Omega$ and some $\alpha_1, \alpha_2 > 0$ with a symmetric continuous matrix valued function $Q$, bounded on $\Omega$.

Let $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_n(x)$, $x \in \Omega$ be ordered solutions to the following generalized eigenvalue problem

$$\det(Q(x) - \lambda P(x)) = 0$$

which are real since both $Q$ and $P$ are symmetric.

We begin with the following preliminary result:

Lemma 1 Suppose the hypothesis H1 is satisfied. Then any solution $x(t, x_0)$ to (4), $x_0 \in \Omega$ is right-unique and defined on the infinite time interval.

Proof: To prove the first part of the statement it is sufficient to show that two solutions $x(t, x_{10})$ and $x(t, x_{20})$ coincide on the maximal interval of their existence $t \in [0, \hat{T})$. From the integral inequality

$$\delta(t)^\top P(x(t, x_{10}))\delta(t) \leq V(x_{10}, x_{20}) + \int_0^t \delta(s)^\top Q \delta(s)ds$$

where $\delta(t) := x(t, x_{10}) - x(t, x_{20})$ it follows that

$$\delta(t)^\top P(x(t, x_{10}))\delta(t) \leq V(x_{10}, x_{20}) + L \int_0^t \delta(s)^\top P \delta(s)ds$$

where $L = \sup_{x \in \Omega} \lambda_1(x) < \infty$ and $0 \leq t < \min_i \hat{T}_i$ for which the solutions $x(t, x_{10})$ and $x(t, x_{20})$ exist. Then, applying Gronwall’s lemma it follows that $\delta(t)^\top P \delta(t)$ grows at most exponentially with time:

$$\delta(t)^\top P(x(t, x_{10}))\delta(t) \leq V(x_{10}, x_{20})e^{Lt}$$

and nonnegative function $v(t) := \delta(t)^\top P \delta(t)e^{-Lt}$ cannot grow:

$$v(t) \leq V(x_{10}, x_{20})$$
Let \( x_{10} = x_{20} \), i.e. \( v(0) = 0 \). Since \( v \) is nonnegative and \( V(x_{10}, x_{20}) = 0 \) it implies that \( v(t) = 0 \) for all \( 0 \leq t < \min_i T_i \) for which the solutions \( x(t, x_{10}) \) and \( x(t, x_{20}) \) exist. Therefore on a time interval \( 0 \leq t < \min_i T_i \) the solutions \( x(t, x_{10}) \) and \( x(t, x_{20}) \) coincide and all solutions are right-unique and have the same interval of existence.

Now let us show that all solutions are defined on the infinite time interval. Suppose there is a solution \( x(t, x_0) \) which is defined only on a finite interval \([0, \bar{T})\). Let \( z = x(\bar{T} - \varepsilon, x_0) \) for some \( 0 < \varepsilon < \bar{T} \). Then due to time-invariance of the system the solution \( x(t, z) \) is well defined at most on the interval \([0, \varepsilon)\) that contradicts to the fact that all solutions have the same interval of their existence.

The previous lemma shows that the Cauchy problem (4) is well-posed and continuous dependence on initial conditions follows.

Consider a compact set \( S \) of finite Hausdorff \( d \)-measure for some \( d = d_0 + s \), \( d \leq n \), where \( d_0 \in \mathbb{N} \) and \( s \in [0, 1) \). Suppose that \( S \in \Omega \), then \( \varphi^t(S) \in \Omega \) for all positive \( t \).

**Theorem 1** Suppose hypothesis H1 is satisfied. If
\[
\sup_{x \in \Omega} (\lambda_1(x) + \ldots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x)) < 0
\] (7)
for any \( x \in \Omega \). Then
\[
\lim_{t \to \infty} \mu_d(\varphi^t(S)) = 0.
\]

The proof of this result will be presented in the full version of this paper.

It is worth mentioning that the hypothesis H1 can be relaxed. Particularly one can assume that the following two assumptions are satisfied:

**H1a.** There exists a continuously differentiable \( n \times n \) symmetric matrix valued function \( P \) defined in the domain \( \Omega \), such that the function (5) satisfies (6) and the following inequality
\[
\dot{V}^*(x_1, x_2) \leq (x_1 - x_2)^\top Q(x_1)(x_1 - x_2) + o(||x_1 - x_2||^2)
\]
for all \( x_1, x_2 \in \Omega \) with a symmetric continuous matrix valued function \( Q \), bounded on \( \Omega \).

**H2.** All solutions starting in \( \Omega \) are defined for all \( t \geq 0 \).

As one can easily prove similarly to the proof of Lemma 1, the hypothesis H1a implies right uniqueness of the solutions to (4). Now we formulate the following result.

**Theorem 2** Suppose hypotheses H1a and H2 are satisfied. If for some \( d = d_0 + s \), \( 0 < d_0 \leq n \), \( 0 \leq s < 1 \) it follows that
\[
\sup_{x \in \Omega} (\lambda_1(x) + \ldots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x)) < 0
\] (8)
for any \( x \in \Omega \). Then
\[
\lim_{t \to \infty} \mu_d(\varphi^t(S)) = 0.
\]
We are going to present only sketch of the proof. The previous proof is based on an estimation of the \( d \)-measured volume of an ellipsoid that covers \( \varphi'(B_x) \) of radius \( r \). In the proof of this result we have to find a new estimate that takes into account the term \( o(r^2) \) from the assumption H1a. This can be made by Douady-Oesterlé lemma [10] (see also Lemma 5.4.1 in [13]). To complete the proof one can follow the same arguments as in the final part of Theorem 5.4.1 in [13].

The main result of this section is the following theorem

**Theorem 3** Suppose hypotheses H1a and H2 are satisfied, and there exist positive integer \( d_0 \) and real \( s \in [0, 1) \) such that

\[
\lambda_1(x) + \ldots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x) < 0
\]  

(9)

for any \( x \in \Omega \). Suppose that there is an invariant compact set \( K \in \Omega \). Then \( \dim_H K \leq d_0 + s \).

**Proof:** From the previous result it follows that \( \mu_d(\varphi^t(K)) \to 0 \) as \( t \to \infty \). Since \( K \) is invariant \( \mu_d(K) = \mu_d(\varphi^t(K)) \). Therefore \( \mu_d(K) = 0 \).

For continuously differentiable right-hand side of the system (4) the previous theorem can be formulated in the infinitesimal form.

**Corollary 1** [14] Suppose that hypothesis H2 is satisfied and there is an invariant compact set \( K \in \Omega \). Assume also that there is a continuously differentiable symmetric uniformly positive definite \( n \times n \) matrix function \( P \) defined and bounded in \( \Omega \) such that the solutions of the following generalized eigenvalue problem

\[
\det(Q(x) - \lambda P(x)) = 0
\]

satisfy the inequality (9), with the matrix \( Q \) defined as

\[
Q = P(x) \frac{\partial f(x)}{\partial x} + \frac{\partial^\top f(x)}{\partial x} P(x) + \dot{P}(x).
\]

Then \( \dim_H K \leq d_0 + s \).

In [14] this result was derived using the linearization of the flow \( \varphi^t \) with an approach close to that due to Douady-Oesterlé and Leonov. It is seen now that this result can be also obtained from a more general argument which is applicable to discontinuous systems. We have presented only local conditions that are easier to verify analytically for particular examples. One can derive a further generalization via integral conditions of the form

\[
\sup_{x_{i0}} \int_0^r (\lambda_1(x(t, x_{i0})) + \ldots + s\lambda_{d_0+1}(x(t, x_{i0}))) dt < 0
\]

which can be useful for numerical methods.
3.1 Example: the Lorenz system

It is well known that all trajectories of the Lorenz system

\[
\begin{align*}
\dot{x} &= \sigma(y-x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= -bz + xy
\end{align*}
\]

(10)

are ultimately bounded for arbitrary positive \(\sigma,r,b\), that is, there is an invariant compact set \(K\). Let us estimate its Hausdorff dimension.

**Proposition 1** Suppose the parameters of the system are such that the following inequality

\[
\limsup_{t \to \infty} \left[ \frac{y(t)^2}{b} + (z(t) - 2r)^2 \right] \leq 4r^2.
\]

(11)

is satisfied for all initial conditions. Then

\[
\dim_H K \leq 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4r\sigma}}.
\]

(12)

**Proof:** The Jacobian of the Lorenz system is

\[
J = \begin{pmatrix}
-\sigma & \sigma & 0 \\
 r - z & -1 & -x \\
 y & x & -b
\end{pmatrix}
\]

and let \(P\) be

\[
P = \begin{pmatrix}
 r/\sigma & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
\end{pmatrix}
\]

then

\[
PJ + J^T P = \begin{pmatrix}
-2r & 2r - z & y \\
2r - z & -2 & 0 \\
y & 0 & -2b
\end{pmatrix}.
\]

We know that \(\lambda_1 + \lambda_2 + \lambda_3 = -2(\sigma + 1 + b)\), and from \(-2(\sigma + 1 + b) < 0\) it follows that \(\lambda_3 < 0\) for all \(x,y,z\). Hence, by the Corollary 1, to find an upper estimate of \(s\) it is sufficient to find a lower estimate of \(\bar{\lambda}_3\). Given this estimate, we then evaluate the upper bound on the Hausdorff dimension as

\[
\dim_H K \leq 3 - \frac{2(\sigma + b + 1)}{|\lambda_3|}.
\]
The smallest negative eigenvalue of some symmetric matrix $Q$ is such a number $\bar{\lambda}$ that the matrix $Q - \lambda I$ is positive definite as long as $\lambda < \bar{\lambda}$. For all nonpositive $\lambda$ the following matrix inequality

$$PJ + J^T P - \lambda P \geq PJ + J^T P - \lambda P_1$$

is satisfied, where $P_1 = \text{diag}\{r/\sigma, 1, 0\}$. Therefore, to find a lower estimate for $\lambda_3$ it is sufficient to find a lower estimate of the smallest nonpositive solution of the following equation

$$\det(PJ + J^T P - \lambda P_1) = 0$$

or,

$$r \left( 2 + \frac{\lambda}{\sigma} \right) (2 + \lambda) = \frac{y^2}{b} + (2r - z)^2 + \lambda \frac{y^2}{2b}.$$  

Using the inequality (11) and neglecting the term $\lambda y^2/2b$ (since we are looking for a negative lower estimate of the smallest solution for $\lambda$) it is then straightforward to complete the proof.

In [22] it has been proved that for the standard values of parameters of the Lorenz system $\sigma = 10, r = 28, b = 8/3$, the inequality (11) is satisfied. Thus in this case, $\dim H K \leq 2.4013$. A typical trajectory of the Lorenz system for those values of the parameters is presented in Fig. 2.

![Figure 2: A typical trajectory of the Lorenz system.](image-url)
dimension of the Lorenz attractor equals to the local Lyapunov dimension of the origin [25]. Our result can be proved with a much simpler derivation.

4 A higher-dimensional generalization of Bendixson’s criterion

We begin with some definitions.

**Definition 4** [28] A set $S \subset \mathbb{R}^n$ is called $d$-dimensional rectifiable set, $d \in \mathbb{N}$ if $\mu_d(S) < \infty$ and $\mu_d$ almost all of $S$ is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^d$ to $\mathbb{R}^n$.

The rectifiable sets are the generalized surfaces of geometric measure theory. Any 1-dimensional closed rectifiable contour $\gamma$ bounds some two-dimensional rectifiable set, for example the cone over $\gamma$.

A set is said to be simply connected if any simple closed curve can be shrunk to a point continuously in the set.

**Theorem 4** Suppose that assumptions $H1a$ and $H2$ are satisfied, let $\Omega$ be a simply connected set. Suppose that

$$\lambda_1(x) + \lambda_2(x) < 0 \quad (13)$$

for any $x \in \Omega$. Then no periodic orbit can lie entirely in $\Omega$.

**Proof:** The proof of Theorem 4 follows an idea used in the proof of the Leonov theorem ([27], see also Theorem 8.3.1 in [26]). Suppose (13) holds but there is a periodic orbit $\gamma$ passing through a point $x_0 \gamma := \{x \mid \exists t \geq 0, x = x(t, x_0)\}$ which lies entirely in $\Omega$.

Since as assumed the function $f$ is bounded on any compact set it follows that there is a positive constant $L > 0$ such that for all $x_0 \in \gamma$

$$||x(t_1, x_0) - x(t_2, x_0)|| \leq L|t_1 - t_2|$$

and thus the set $\gamma$ is an image of Lipschitz continuous function. Therefore the set $\gamma$ is a rectifiable one-dimensional set. From the theorem on existence of area-minimizing surfaces (see Theorem 5.6 in [29]) it follows that there exist a 2-dimensional rectifiable set $\bar{S} \in \mathbb{R}^n$ such that its boundary is $\gamma$ and it has minimal Hausdorff 2-measure.

Let $S$ be a rectifiable two-dimensional set $S \subset \Omega$ with boundary $\gamma$. The existence of such set follows from the fact that $\Omega$ is simply connected. As before, we denote by $\varphi^t$ the flow of system (4). Let $\mu(S)$ be the Hausdorff 2-measure of a 2-dimensional surface $S$. 12
Since $\gamma$ is invariant under $\varphi^t$ and $\varphi^t(S) \subset \Omega$ for any $t \geq 0$ ($\Omega$ is positively invariant) we have
\[
\inf_{t \geq 0} \mu(\varphi^t(S)) \geq \mu(S) > 0.
\] (14)

At the same time, using (13) from Theorem 2 it follows that
\[
\lim_{t \to \infty} \mu(\varphi^t(S)) = 0,
\] (15)
which contradicts (14). Therefore, (13) ensures the absence of periodic trajectories lying in $\Omega$.

It is worth saying that this theorem being applied to smooth systems together with its time reversed version (for the smooth systems we have local right and left uniqueness) gives the classical Bendixson divergency condition.

The main idea of the proof (see [27]) is based on the existence of a surface with minimal area given its boundary. Although the mathematical problem of proving existence of a surface that has minimal area and is bounded by a prescribed curve, has long defied mathematical analysis, an experimental solution is easily obtained by a simple physical device. Plateau, a Belgian physicist, studied the problem by dipping an arbitrarily shaped wire frame into a soap solution. The resulting soap film corresponds to a relative minimum of area and thus produces a minimal surface spanned by that wire contour. A classical solution to Plateau’s problem can be found, for example, in [30] with some regularity assumptions on the contour $\gamma$ that can be violated if $\gamma$ is a closed orbit corresponding to a periodic solution of a system of differential equations with discontinuous right hand sides. Fortunately, the argument based on geometric measure theory allows to overcome this difficulty.

### 4.1 Example

Consider the following system:
\[
\dot{x} = Ax + Bu, \quad u = -\text{sign}(y), \quad y =Cx
\]
where $x \in \mathbb{R}^3$, $u, y \in \mathbb{R}^1$ and the matrices $A, B, C$ are given as follows
\[
A = \begin{pmatrix}
\alpha & 1 & 1 \\
-1 & \beta & -1 \\
-1 & 1 & -1
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
b
\end{pmatrix}, \quad C = (0 \ 0 \ 1)
\]
with positive $b$. Consider the smooth function (5) in the form
\[
V = (x_1 - x_2)^\top(x_1 - x_2)
\]
For this system the corresponding solution according to Filippov convex definition coincides with the Utkin solution. At the discontinuity points of the right hand side, the corresponding set valued function in the differential inclusion is obtained by closure of the graph of the right hand side and by passing over to a convex hull. As shown in [1], p.155, these procedures do not increase the upper value of $\dot{V}^*$ and hence it is sufficient to compute the derivative of $V$ only in the area of continuity of the right hand side. The derivative of $V$ in this area satisfies

$$\dot{V} \leq 2(x_1 - x_2)^T \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -1 \end{pmatrix} (x_1 - x_2)$$

The previous theorem suggests that if $\min\{\alpha, \beta\} \geq -1$, a sufficient condition for the absence of periodic solutions is

$$\alpha + \beta < 0 \quad (16)$$

To demonstrate that the violation of the condition (16) can result in oscillatory behavior we performed a computer simulation for the following parameter values: $\alpha = 1, \beta = -1/2, b = 1$. The results of the simulation are presented in Figure 3. It is seen that the system possesses orbitally stable limit cycle.

![Figure 3: Oscillatory behavior for $\alpha + \beta > 0$.](image)

### 4.2 An LMI based criterion for Lur’e systems with discontinuous right hand side

In the previous example the matrix $A$ was chosen as a sum of a diagonal and skew-symmetric matrix that made all necessary calculations trivial. Next we present an LMI based criterion which ensures the absence of periodic solutions for the following system:

$$\dot{x} = Ax + Bu, \quad u = -b\text{sign}(y), \quad y = Cx \quad (17)$$
where \( x \in \mathbb{R}^n, \ n \geq 2, \ u, y \in \mathbb{R}^1, \ b > 0 \) and the matrices \( A, B, C \) are of corresponding dimensions.

**Theorem 5** Suppose that there exists \( \mu \) and positive definite matrix \( P \) such that the following inequality
\[
\begin{bmatrix}
P(A - \mu I_n) + (A - \mu I_n)^T P & PB - C^T \\
B^T P - C & 0
\end{bmatrix} \geq 0
\]

is satisfied. Then if
\[
\text{tr}A - (n - 2)\mu < 0
\]
the system (17) does not have periodic solutions.

**Proof:** According to theorem hypothesis the matrix \( P \) satisfies the following equation \( PB = C^T \). Thus taking the derivative of the following function
\[
V = (x_1 - x_2)^T P(x_1 - x_2)
\]
yields (as in the previous example it is sufficient to compute the derivative in the area of continuity of the right hand side)
\[
\dot{V} = (x_1 - x_2)^T (PA + A^T P)(x_1 - x_2)
\]
\[
-2b(Cx_1 - Cx_2)(\text{sign}Cx_1 - \text{sign}Cx_2)
\]
\[
\leq (x_1 - x_2)^T (PA + A^T P)(x_1 - x_2) \quad (18)
\]

Now consider the smallest solution \( \lambda_n \) of the following equation
\[
\det(PA + A^T P - \lambda P) = 0 \quad (19)
\]
From the hypothesis it follows that \( \lambda_n \geq 2\mu \). On the other hand if \( \lambda_i, \ i = 1, \ldots, n \) are the solutions of (19) then
\[
\lambda_1 + \ldots + \lambda_n = 2\text{tr}A
\]
Since
\[
\lambda_i \geq \lambda_n \quad (20)
\]
it follows that
\[
\lambda_1 + \lambda_2 \leq 2(\text{tr}A - (n - 2)\mu) < 0
\]
and according to Theorem 4 the system (17) has no periodic solutions. \( \blacksquare \)
5 Conclusions

In this paper we presented a new discontinuous version of Bendixson’s like criterion. The criterion is based on a new result on estimation of the Hausdorff dimension of invariant sets for (possibly) discontinuous systems. The new criterion can be applied for design (control) of discontinuous systems when the requirement of global stability is too restrictive. Our study is based on dichotomy-like properties of solutions of dynamical systems with respect to each other rather than with respect to some invariant sets. We hope that a further development of this approach will allow to better understand and classify bifurcations in nonsmooth dynamical systems.

Acknowledgment

The authors thank Prof. G.A. Leonov for fruitful discussions during his visit to Eindhoven University of Technology.

References


[23] Eden A. “Local Lyapunov exponents and a local estimate of Hausdorff dimension,”

[24] Ljashko A S 1994 Estimates for fractal and Hausdorff dimensions of invariant sets of
State University, Russia.


edition.


York), 1959.