Control and Observer Design for Non-smooth Systems
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Introduction

In this report, the results on the controller and observer design for non-smooth systems attained within WorkPackage 5 of SICONOS are presented. Clearly, control (and observer) design for non-smooth systems is the main objective of WorkPackage 5 and one of the main objectives of SICONOS as a whole. In many engineering systems in which non-smoothness plays a dominant role, we aim at attaining desired behaviour by means of active control. Types of systems addressed by the results in this report are mechanical systems with unilateral contacts and frictional contacts or non-smooth restoring characteristics (such as motion systems, industrial robots, walking robots or structures with flexible one-sided supports). The presence of these non-smooth components or constraints dramatically influence the behaviour of these systems and poses tremendous challenges to the control designs which should ensure desired behaviour. Another issue for such systems is the fact that the feedback control laws to be designed can generally not rely on measurements of the entire state of the system. For example, for mechanical systems commonly only position measurements (and no velocity measurements) are available. Consequently, observer design for non-smooth systems is crucial for the ultimate practical implementation of such controllers. Finally, the joint application of controllers and observers for non-smooth systems is a non-trivial issue which is also addressed by several results in this report. Another application field in which non-smoothness is commonly present is that of power converters, where the non-smoothness is often present in the form of diodes or switches. Also this class of systems is studied in several chapters of this report.

The foregoing clearly motivates the importance of controller and observer design for non-smooth systems. The question remains: "why are dedicated techniques needed for such systems". One of the main reasons lies in the fact that the stability of desired solutions is crucial in control design. The results in the Deliverable D.5.2 on 'Stability of Non-smooth Systems' clearly showed that the approaches taken towards providing such stability results often differ essentially from those for smooth systems. Consequently, the conditions for the stability of certain (desired) solutions, such as equilibrium points or periodic solutions (desired cyclic behaviour), should generally be posed in a dedicated fashion for certain classes of non-smooth systems. One can imagine that the construction of controllers that should enforce such stability (thus satisfying the stability conditions in closed-loop) requires novel design strategies. The latter is exactly the topic of this deliverable, which directly builds upon the results presented in the deliverable on the stability (D.5.2).

It should be noted that due to the fact that different classes of non-smooth systems and different control goals (such as stabilisation, tracking or disturbance attenuation) require dedicated approaches to these problems, a wide variety of results is obtained in WorkPackage 5. For an overview of the publications of WP5 (in the time frame up to March 2005) on this topic, we refer to the intermediate D.5.3 report presented to the Review Committee for the SICONOS Mid-term Review Meeting held in Grenoble, France on April 28-29, 2005. In this deliverable we will present a focussed account of a subset of these results, which represent the focus-areas of WorkPackage 5. Therefore, the current report is structured as follows. It consists of three parts, each of which represents an important focus area of WorkPackage 5:

1. **Part I (Feedback Control Design and Observer Design for Lagrangian Systems)** presents control and observer design approaches tailored to non-smooth Lagrangian systems. Lagrangian systems are a very important class of systems since mechanical systems (one of the main application areas for SICONOS) are generally modelled within a Lagrangian framework. Moreover, Lagrangian systems exhibit a specific structure which can (and should) be exploited when designing controllers for such systems.

2. **Part II (Feedback Control Design and Observer Design for Piece-wise Smooth Systems)** presents control and observer design techniques for other important classes of non-
smooth systems, such as for example piece-wise affine systems and complementarity systems. Obviously not all systems allow to be formulated within the Lagrangian framework and other formalisms for modelling non-smooth systems allow for different control design strategies to be developed for these types of systems.

3. **Part III (Control of Bifurcations in Non-smooth Systems)** takes an entirely different perspective in the sense that the knowledge on bifurcations in non-smooth systems is exploited in designing controllers for these systems. The work presented in this part of the report represents a direct link of the work of WorkPackage 5 to the work of WorkPackage 4.

In figure 1, the connections between the parts of this deliverable report to the deliverable report D.5.2 on 'Stability of Non-smooth Systems', to WP4 on 'Bifurcations Analysis' and to WP6 on 'Applications' is illuminated.

In order to further structure the results discussed in this report, we will categorize non-smooth systems with respect to their degree of non-smoothness or discontinuity (herein, we will already refer to the chapters in this report in which these types of systems are addressed):

- Non-smooth, continuous systems, such as systems described by continuous differential equations with a continuous, non-smooth vector field, which has a discontinuous Jacobian. An example can be a mechanical system with a one-sided flexible support (Chapters 6, 9 and 11).
- Discontinuous systems with a time-continuous state, such as systems described by differential equations with a discontinuous vector field, which can be transformed into differential inclusions with a set-valued right-hand side (often called Filippov systems, referring to the Filippov solution concept). Mechanical systems with Coulomb friction, modelled by a set-valued force law, are a well-known and important engineering example of such systems (Chapters 7, 8, 10 and 12).
- Discontinuous systems with state jumps. Sometimes such systems are called impulsive systems. Mechanical systems with impacts, inducing jumps in the velocity can be formulated within this class of systems (Chapters 2, 3, 4 and 5).

Moreover, we can categorize the results in terms of the type of control problem tackled:

- Stabilisation, i.e. how to design a controller such that (asymptotic) stabilisation of an equilibrium is achieved (Chapters 4, 7, 10 and 11).
- Tracking of reference trajectories (Chapters 3 and 5).
- Observer design and output-feedback design strategies (Chapters 2, 7, 8 and 11).

![Fig. 1: Relation of D.5.3 to D.5.2 and the work of WorkPackages 4 and 6.](image-url)
– Disturbance attenuation or suppression of oscillations (*Chapters 6 and 12*).
– Other design issues, such as robustness (*Chapters 4 and 9*) and linearizing controllers (*Chapter 9*).

We like to stress that the results in this report generally consist of the following components:

– a rigorous proof of the fact that the proposed controller (or observer) attains the control goal;
– constructive techniques to actually design controllers that achieve this control goal;
– the validity of the results is generally supported by application of the developed techniques to examples;
– in several cases the examples relate to experimental setups on which the proposed techniques are implemented and tested (*Chapters 4, 6, 7, 8 and 10*). In this scope we would like to refer to Deliverable D.6.3 on 'Experimental Results on Feedback Control' of WorkPackage 6.
Part I: Feedback Control Design and Observer Design for Lagrangian Systems

Part I of this report contains results regarding controller and observer design for non-smooth Lagrangian systems. It comprises the following chapters:

- In Chapter 2, an observer design for linear mechanical systems unobservable in absence of impacts is proposed. Issues such as observability, reconstructibility are addressed and a constructive design technique is presented (Authors: F. Martinelli, L. Menini, A. Tornambé).
- In Chapter 3, the tracking control of Lagrangian systems subject to frictionless unilateral constraints is addressed (Authors: Jean-Matthieu Bourgeot, Bernard Brogliato).
- In Chapter 4, a stability analysis of a control law for walking robots with non-permanent contacts is presented (Authors: Sophie Chareyron, Pierre-Brice Wieber).
- Chapter 5 presents a robustness analysis of passivity-based controllers for complementarity Lagrangian systems (Authors: Jean-Matthieu Bourgeot, Bernard Brogliato).
- In Chapter 6, a control strategy for the disturbance attenuation for a periodically excited piece-wise linear beam system is proposed (Authors: Apostolos Doris, Nathan van de Wouw, Henk Nijmeijer).
- In Chapter 7, a friction compensation strategy for a controlled one-link robot using a reduced-order observer is presented (Authors: Nathan van de Wouw, Niels Mallon, Henk Nijmeijer).
Observer design for linear mechanical systems unobservable in absence of impacts: observability, reconstructibility and a design technique*

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Abstract. This report deals with a class of mechanical systems, constituted by two composite bodies which interact with each other only during collisions: since only one position variable of only one body is measured, the whole system would not be observable without impacts. Each body is possibly constituted by several masses connected by linear springs, so that the internal deformations can be taken into account (the so-called multiple-degrees-of-freedom impacts). For two relevant and different cases, the structural properties of observability and reconstructibility are studied, and observers are proposed in order to estimate all the state variables, including those that would be unobservable in absence of impacts.

1 Introduction

Although the problem of modeling mechanical systems subject to non-smooth impacts is far from being solved in its generality, especially when multiple impacts or the effects of friction have to be taken into account [1], several authors have studied mechanical systems subject to impacts from the control point of view: typical applications are the free-flying space robots [2,3] and the robotic systems where the impacts between the robot and the environment constitute a significant feature of the robot task [4,5].

The need for observers in the control of mechanical systems and, in particular, of robots, is widely recognized [6]-[9], since the velocity measures are often affected by noise, which severely limits the performance achievable by control laws making use of such measures.

State-space methods have been recently used to form systematic approaches to the controller synthesis and design for mechanical systems subject to non-smooth impacts, so to guarantee some stability and tracking requirements (see, e.g., [10] and [11]). A common feature of these approaches is the assumption that all the generalized positions and velocities are available for feedback control purposes. Mechanical systems subject to impacts rarely satisfy this assumption for two main reasons: (1) at each impact time the positions have a corner point, with a consequent jump in the velocity, which renders unreliable or, sometimes, impossible the direct measure or computation of the velocity; (2) since some parts of the mechanical system can interact only at the impact times (e.g., the teeth of two mating gears subject to backlash as in [12]) and not all the generalized coordinates are directly measurable, the impacts could be the only mechanism for the state reconstruction. Hence, the state observation problem, possibly relying on a proper account of the non-smooth impacts, plays a central role on the construction of an output feedback controller also for the considered class of mechanical systems subject to non-smooth impacts. This paper is especially concerned with the problems due to item (2) above.

It is well known that several models can be used for describing the impact phenomena, see, e.g., the book [1]; in this paper, the impacts are assumed to be non-smooth and perfectly elastic.

This renders the proposed results practically useful for those systems in which the parts of the impacting bodies that come in contact during the impacts are quite rigid (e.g., made of glass or steel); nevertheless, internal deformations of the impacting bodies can be taken into account in the proposed class of systems, in view of their structure.

Some results concerned with the state observation of mechanical systems subject to impacts are reported in [13, 14]. In [13], a full order observer, whose state is subject to jumps at the impact times, is proposed for mechanical systems whose dynamics are linear in the absence of impacts, whereas, in [14], a reduced order observer, also switching at the impact times, is proposed for mechanical systems whose dynamics in the absence of impacts are possibly nonlinear. It is stressed that the systems considered in both papers [13, 14] are observable in absence of impacts, since all the position variables are measured (i.e., their continuous-time dynamics are observable); thus, the main contributions of [13, 14] are to show that the impacts do not destroy the observability of the given systems and to show how to design observers that, taking the impacts into account, recover the nice properties of the observers that could be designed (in absence of impacts) for the continuous-time dynamics of the system.

In this report, the problem is completely different. As a matter of fact, the systems considered in this report are constituted by two separate bodies, with linear dynamics, which interact only at the impact times, colliding with each other. It is assumed that only one of the $n$ position variables describing the configuration of the whole system can be measured, namely the first position variable of the first body. Therefore, it is clear that, in absence of impacts, the dynamics of the second body are unobservable. The first purpose of this report is to show that the considered systems become observable and reconstructible under generically satisfied assumptions if a sufficient number of impacts occur in the considered time interval. The second goal is to design suitable state observers for the same systems, in order to estimate all the state variables, including the ones which would be unobservable in the absence of impacts. Therefore, contrarily to what happens in the papers [13, 14], in this report the impacts play a “positive” role, rendering possible goals that would be not achievable without collisions. In the paper [12], the problem of state estimation has been solved by using a reduced order observer for a simple example (a pair of mating gears subject to backlash) similar to the systems considered here: the interested reader can consider the contribution in [12] as a motivating example for the results presented in this report.

The organization of the report is as follows: the class of systems considered is described in Chapter 2, where it is specified that each of the two bodies is possibly constituted by several masses connected by linear springs, so that the internal deformations that can happen as a consequence of the impacts can be taken into account (these are called multiple-degrees-of-freedom impacts). Among the large variety of systems that fall in the mentioned class, the two extreme cases (which pose completely different problems) are dealt with in this report. Chapter 3 deals with the case when only one of the position variables of the “more complex” body is measured, whereas the state of the simpler one (just one mass) is not measured at all: this case will be referred to as the $(n-1,1)$ case. Chapter 4 deals with the opposite case, when the position of the simpler body (just one mass) is measured, whereas none of the state variables of the more complex body is measured: such a case will be referred to as the $(1,n-1)$ case. For both cases, the structural properties of observability and reconstructibility are studied: it is stressed that, due to the impacts, the considered systems are nonlinear, whence standard results valid for linear systems do not apply in presence of collisions. Furthermore, for the $(n-1,1)$ case, two different observers are proposed: an asymptotic observer and a dead-beat observer, which are designed by two completely different techniques. Both observers are designed assuming that the impacts can be detected in real time, in order to apply suitable jumps to the state estimates (see the subsequent Remark 3). The design of the dead-beat observer is also specified for the $(1,n-1)$ case, since, by noting the analogies and the differences in the design of this type of observer for the two cases, the reader should have no difficulty in extending the result to the general class $(n_a,n_b)$ of considered systems. Moreover, in view of the structure of the system, the design of the dead-beat observer is formally simpler in the $(1,n-1)$ case than in the $(n-1,1)$ case.
2 Preliminaries

The mechanical systems considered in this report are constituted by two composite bodies, each of which is constituted by one or more masses connected by linear springs with zero length when undeformed: all the masses can move along the same line, which is orthogonal to the gravity acceleration vector (the gravity forces are then compensated by such a constraint). Let an inertial frame be defined in the working space.

Let the configuration of the first (respectively, the second) body be completely described (in the whole) by the following vector of generalized coordinates \( q_a := [q_1 \ldots q_{n_a}]^T \in \mathbb{R}^{n_a} \) (respectively, \( q_b := [q_{n_a+1} \ldots q_{n_a+n_b}] \in \mathbb{R}^{n_b} \), with \( n_a \geq 1 \) (respectively, \( n_b \geq 1 \)) being the number of masses constituting the first (respectively, the second) body; finally, let \( v_a = [v_1 \ldots v_{n_a}]^T := [\dot{q}_1 \ldots \dot{q}_{n_a}]^T \) and \( v_b = [v_{n_a+1} \ldots v_{n_a+n_b}]^T := [\dot{q}_{n_a+1} \ldots \dot{q}_{n_a+n_b}]^T \) be the vectors of the generalized velocities. The following compact notations are also used for \( q := [q_a \ q_b] \), \( v := [v_a \ v_b] \) and \( n := n_a + n_b \). The functions \( q_a(t) \) and \( q_b(t) \) are assumed to be continuous functions of time \( t \in \mathbb{R}, t \geq t_0 \), which are, in addition, piece-wise twice differentiable, with \( t_0 \in \mathbb{R} \) being the initial time. Let \( m_b \) be the mass whose position along the motion line has been denoted by \( q_b, h = 1, \ldots, n \). For each pair \( h, j \in \{1, \ldots, n_a \} \) or \( h, j \in \{n_a + 1, \ldots, n_a + n_b \} \), let \( k_{h,j} \in \mathbb{R} \) be the elastic constant of the spring connecting the masses \( h \) and \( j \), whereas let \( k_{h,h} \in \mathbb{R}, h = 1, \ldots, n \), be the elastic constant of the spring connecting the \( h \)-th mass with a fixed point of the working space (which, for simplicity, is assumed to be coincident with the origin of the inertial reference frame). By letting \( k_{h,j} \) and \( k_{h,h} \) to assume any value in \( \mathbb{R} \), exponentially unstable linear mechanical systems are included, as those obtained by linearizing a nonlinear mechanical system about an exponentially unstable equilibrium point. Notice that it is assumed that there is no friction of any kind acting on the mechanical system. For the sake of simplicity, let us assume that the mechanical system is subject to a scalar generalized force \( u(t) \in \mathbb{R} \) at time \( t \geq t_0 \), characterized by the generalized potential: \( U_u = -u(t)E^Tq_a(t) \), where \( E \neq 0 \) is a suitable vector in \( \mathbb{R}^{n_a} \), i.e., the first body is actuated, whereas the second one is in free motion (i.e., no external inputs act on the second body) for all times \( t \) except for the impact times \( t_i \), because the two bodies do not interact with each other except for such impact times. The times \( t_i \in \mathbb{R} \), \( t_i \geq t_0 \), at which the vector function \( q(t) \) is not differentiable (i.e., those times \( t_i \) such that \( \lim_{t \rightarrow t_i^-} E^Tq_a(t) \neq \lim_{t \rightarrow t_i^+} E^Tq_a(t) \), which correspond to corner points) will be referred to as the impact times. The symbols \( \varphi(t_i^-) \) and \( \varphi(t_i^+) \) will be used to denote, respectively, the values taken by the limits \( \lim_{t \rightarrow t_i^-} E^Tq_a(t) \) and \( \lim_{t \rightarrow t_i^+} E^Tq_a(t) \), when they are definite, for any function \( \varphi(t) \). If a function \( \varphi(t) \) is not defined at time \( t = t_i \) (i.e., it is discontinuous at \( t = t_i \)), the symbol \( \varphi(t_i) \) means both \( \varphi(t_i^-) \) and \( \varphi(t_i^+) \).

Assume that a constraint is imposed on the generalized coordinates by the physical nature of the mechanical system under consideration; in particular, assume that (at each time \( t \in \mathbb{R}, t \geq t_0 \)) the vector \( q(t) \) must belong to the following admissible region of \( \mathbb{R}^n \):

\[
\mathcal{R} := \{ q \in \mathbb{R}^n : \ q_{n_a} - q_{n_a+1} \leq 0 \},
\]

that is the \( n_a \)-th mass is constrained to be on the left of the \((n_a + 1)\)-th one, whereas the other masses are not constrained at all.

The subspace of \( \mathbb{R}^n \) identified by \( q_{n_a} - q_{n_a+1} = 0 \) represents the condition of contact between the masses \( n_a \) and \( n_a+1 \). A non-smooth impact can occur at a certain time \( t_i \in \mathbb{R}, t_i \geq t_0 \), only if, at such a time, one has \( q_{n_a}(t_i) - q_{n_a+1}(t_i) = 0 \). Notice the difference between non-smooth impacts and contacts: a non-smooth impact corresponds to a jump of the generalized velocity, \( \dot{q}(t_i^-) - \dot{q}(t_i^+) \neq 0 \), whereas a contact corresponds to the fulfillment of the constraint \( q_{n_a}(t) - q_{n_a+1}(t) \leq 0 \) with the equality sign, \( q_{n_a}(t) - q_{n_a+1}(t) = 0 \); a non-smooth impact can occur only in presence of a contact, whereas the nice versa is not necessarily true.

A time \( t_i \in \mathbb{R}, t_i \geq t_0 \), is an impact time if \( q_{n_a}(t_i) - q_{n_a+1}(t_i) = 0 \) and \( v_{n_a}(t^-) - v_{n_a+1}(t^-) > 0 \) (i.e., the inequality constraint would be violated in absence of impulsive forces, as those generated as a reaction to the non-smooth impact). However, in the following, when only the position \( q_i(t) \) of the first mass will be assumed to be measured, it will be difficult to distinguish (in practice) an
impact time from a **degenerate impact time**, i.e., a time $t_i \in \mathbb{R}$, $t_i \geq t_0$, such that $q_{n_a}(t_i) = q_{n_a+1}(t_i) = 0$ and $v_{n_a}(t_i^-) - v_{n_a+1}(t_i^-) = 0$ and there exists $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, such that $q_{n_a}(t_i - \varepsilon) - q_{n_a+1}(t_i - \varepsilon) < 0$ for all $\varepsilon \in (0, \varepsilon)$. Hence, the notation $t_i, i \in \mathbb{N}$, will denote the impact times (degenerate or not) ordered so that $t_{i+1} > t_i$, for all $i \in \mathbb{N}$.

For simplicity, the following assumption will be assumed to be true throughout the report, without an explicit mention to it.

**Assumption 1** The initial conditions and the input function $u(\cdot)$ are such that there are no intervals during which the mechanical system is in a situation of sustained contact.

Between the initial time and the first impact time, and between two adjacent impact times (i.e., for $t \in (t_i, t_{i+1})$, $i \in \mathbb{Z}^+$), the system is described by the Euler-Lagrange equations:

$$
H_a \ddot{q}_a(t) + K_a q_a(t) = E u(t), \quad (1a)
$$

$$
H_b \ddot{q}_b(t) + K_b q_b(t) = 0, \quad (1b)
$$

where $H_a := \text{diag}(m_1, \ldots, m_n)$, $H_b := \text{diag}(m_{n_a+1}, \ldots, m_{n_a+n_b})$, and $K_a, K_b$ are suitable symmetric matrices whose entries depend on the elastic constants $k_{h,j}$ and $k_{h,h}$.

For $t \in (t_i, t_{i+1})$, $i \in \mathbb{Z}^+$, the equations of motion of the mechanical system can be rewritten in the following first-order normal form:

$$
\dot{x}_a(t) = A_a x_a(t) + B_a u(t), \quad (2a)
$$

$$
\dot{x}_b(t) = A_b x_b(t), \quad (2b)
$$

where

$$
x_a := \begin{bmatrix} q_a \\ v_a \end{bmatrix}, \quad x_b := \begin{bmatrix} q_b \\ v_b \end{bmatrix},
$$

$$
A_a := \begin{bmatrix} 0 & I \\ -H_a^{-1}K_a & 0 \end{bmatrix}, \quad B_a := \begin{bmatrix} 0 & I \\ H_a^{-1}E \end{bmatrix}, \quad A_b := \begin{bmatrix} 0 & I \\ -H_b^{-1}K_b & 0 \end{bmatrix}.
$$

Now, assume that only the position of the first mass is measurable

$$
y(t) := C_a x_a(t), \quad (3)
$$

where $C_a \in \mathbb{R}^{1 \times n_a}$, $C_a := [1 \ 0 \ldots \ 0]$.

Notice that equations (2a) and (2b) are decoupled one from the other for all $t \in (t_i, t_{i+1})$ and that (in absence of impacts) the dynamics (the free modes) of the second body (i.e., the dynamics of the masses $m_{n_a+1}, \ldots, m_{n_a+n_b}$) are unobservable (in the usual sense of linear systems) from the output $y(t)$.

We assume that the non-smooth impacts are **elastic**, namely that the coefficient of restitution characterizing the impacts is $e = 1$ (i.e., there is no loss of kinetic energy due to the impacts).

The **restitution** rule for computing the post-impact velocities is given by:

$$
v_h(t_i^+) = v_h(t_i^-), \quad h \neq n_a, h \neq n_a + 1, \quad (4a)
$$

$$
v_{n_a}(t_i^+) = M_{1,1} v_{n_a}(t_i^-) + M_{1,2} v_{n_a+1}(t_i^-), \quad (4b)
$$

$$
v_{n_a+1}(t_i^+) = M_{2,1} v_{n_a}(t_i^-) + M_{2,2} v_{n_a+1}(t_i^-); \quad (4c)
$$

or, in compact form, by

$$
\begin{bmatrix} v_{n_a}(t_i^+) \\ v_{n_a+1}(t_i^+) \end{bmatrix} = \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \begin{bmatrix} v_{n_a}(t_i^-) \\ v_{n_a+1}(t_i^-) \end{bmatrix},
$$

where $M_1 := [M_{1,1} \ M_{1,2}]$ and $M_2 := [M_{2,1} \ M_{2,2}]$, with

$$
M_{1,1} := \frac{m_{n_a} - m_{n_a+1}}{m_{n_a} + m_{n_a+1}}, \quad M_{1,2} := \frac{2m_{n_a+1}}{m_{n_a} + m_{n_a+1}}, \quad M_{2,1} := \frac{2m_{n_a}}{m_{n_a} + m_{n_a+1}}, \quad M_{2,2} := \frac{m_{n_a+1} - m_{n_a}}{m_{n_a} + m_{n_a+1}}.
$$

The whole mechanical system is described by the state equations (2), by the output equation (3) and by the restitution rule (4), and has as state vector $x := [x_a^T \ x_b^T]^T$. 


Remark 1. Observe that elasticity of impacts (i.e. $e = 1$) is just a simplifying assumption. If the impacts are not elastic, i.e. $e \in (0, 1)$, it is sufficient to replace $M_{i,j}$ in (4b) and (4c) with $\tilde{M}_{i,j}$ as follows:

$$\tilde{M}_{1,1} := \frac{m_{n_a} - e m_{n_a + 1}}{m_{n_a} + m_{n_a + 1}}, \quad \tilde{M}_{1,2} := \frac{(1 + e) m_{n_a + 1}}{m_{n_a} + m_{n_a + 1}}, \quad \tilde{M}_{2,1} := \frac{(1 + e) m_{n_a}}{m_{n_a} + m_{n_a + 1}}, \quad \tilde{M}_{2,2} := \frac{m_{n_a + 1} - e m_{n_a}}{m_{n_a} + m_{n_a + 1}},$$

and modify the rest of the report accordingly.  

Now, define the following set of admissible initial conditions:

$$\mathcal{A} := \{ x \in \mathbb{R}^{2n} : \ (q_{n_a} - q_{n_a + 1} \leq 0) \text{ and } (v_{n_a} - v_{n_a + 1} \leq 0 \text{ if } q_{n_a} - q_{n_a + 1} = 0) \};$$

so that for each initial condition belonging to $\mathcal{A}$, the inequality constraint is not violated at the initial time and cannot be violated at times immediately after the initial one.

Remark 2. Notice that, although the dynamics of the system considered are linear in the absence of impacts (as given by (2)), the presence of the inequality constraint defining the admissible region renders the whole system nonlinear. An elegant model of such a class of systems could be obtained through the complementarity framework (for a survey on such a general approach see [15]).

Let $y(t, x_0, u(\cdot))$ denote the position $q_1(t)$ at time $t \geq t_0$ of the first mass (i.e., the measurable position), when the system starts from the initial state $x(t_0) = x_0$ and is subject to the input function $u(\cdot)$. The following definitions are taken from [16].

Definition 1. Two states $x_1, x_2 \in \mathcal{A}$ of system (2), (3) and (4), $x_1 \neq x_2$ are distinguishable in the interval $(T_1, T_2)$ if there exists an input function $u(\cdot)$ such that $y(t, x_1, u(\cdot)) \neq y(t, x_2, u(\cdot))$ for at least a time $t \in (T_1, T_2)$, whereas they are called undistinguishable in the interval $(T_1, T_2)$ in the opposite case.

Definition 2. System (2), (3) and (4) is observable if there is no pair $(x_1, x_2) \in \mathcal{A} \times \mathcal{A}$ of undistinguishable states in the interval $(t_0, +\infty)$.

Definition 3. System (2), (3) and (4) is reconstructible with respect to the input function $u(\cdot)$ in the interval $[t_0, T_f]$ if the knowledge of the functions $u(\cdot)|_{[t_0, T_f]}$ and $y(\cdot)|_{[t_0, T_f]}$ allows the final state $x(T_f)$ to be uniquely determined.

3 A multi-degrees-of-freedom body impacting against a mass in free motion (the case $n_a = n - 1$ and $n_b = 1$)

Consider the mechanical system of Chapter 2, with $n_b = 1$, that is the case of an actuated multi-degrees-of-freedom body impacting against a mass in free motion (i.e., no external forces act on the mass). The mechanical system considered in this section is then constituted by $n = n_a + 1$ masses, that is the $(n - 1)$-th mass is constrained to be on the left of the $n$-th one, whereas the other masses are not constrained at all.

Equation (1b) takes the following form:

$$\ddot{q}_{n_a}(t) + k_{n,n} q_{n_a}(t) = 0,$$  \hspace{1cm} (5)

whence $A_b := \begin{bmatrix} 0 & 1 \\ -\frac{k_{n,n}}{m_n} & m_n \end{bmatrix}$, whereas equations (4b)-(4c) take the following form:

$$\begin{bmatrix} v_{n-1}(t_i^+) \\ v_n(t_i^+) \end{bmatrix} = \begin{bmatrix} M_1 & v_{n-1}(t_i^-) \\ M_2 & v_n(t_i^-) \end{bmatrix}.$$  \hspace{1cm} (6)

For later use, define the following transfer function from $u(s)$ to $q_{n-1}(s)$:

$$W_{n-1}(s) := C_{n-1} (s I - A_n)^{-1} B_n,$$

where $C_{n-1} := [0_{1,n-2} \ 0_{1,n-1}]$. 

Observer design for linear mechanical systems unobservable in absence of impacts 13
3.1 Observability

The following theorem gives a sufficient condition for the observability of the mechanical system subject to non-smooth impacts in the \((n-1,1)\) case.

**Theorem 1.** The mechanical system \( (2), (3) \) and \((6) \) is observable if

1. the transfer function \( W_{n-1}(s) \) is non-null;
2. the pair \((C_a, A_a)\) is observable.

**Proof.** It is now shown that condition \( (1) \) of the theorem implies that for each initial state \( x(t_0) \in A \) there exists an input function \( \bar{u}(\cdot) \) such that there exists at least a non-degenerate impact time \( t_1 > t_0 \). First, it is stressed that if the mechanical system starts from \( A \), then it has a linear behavior up to the first impact time (the only non-linearities are those introduced by the inequality constraint). For the fixed initial condition in \( A \), denote by \( q_{h,f}(t), h = 1,2,\ldots,n \), and \( q_{h,\ell}(t), h = 1,2,\ldots,n-1 \), the free and the forced responses, respectively, in the position variables obtained by integrating equations \( (2) \) (notice that such functions, which are defined for all \( t \geq t_0 \), coincide with the actual responses only up to the first impact time). Since equations \( (2) \) are time-invariant, without loss of generality assume in this part of the proof that \( t_0 = 0 \).

Let \( \hat{t} > t_0 \) and \( \Delta > 0 \) be fixed and let \( k_d := \frac{q_{n,\ell}(\hat{t}) + \Delta - q_{n-1,\ell}(\hat{t})}{\hat{t}} \).

Now, define the ramp function \( q_{n-1,d}(t) = k_d t \), having Laplace transform \( \mathcal{L}[q_{n-1,d}(s)] = \frac{k_d}{s^2} \).

Then, the corresponding input function is given by \( u_d(s) = \frac{k_d}{s^2} W_{n-1}^{-1}(s) \); as this function could contain impulsive functions (when it is not strictly proper), the function \( q_{n-1,d}(s) = \frac{k_d}{s^2} \) is modified as \( q_{n-1,d}(s) = \frac{k_d}{s^2(1+\tau s)^\delta} \), with \( \tau > 0 \) sufficiently small and the positive integer \( \delta \) sufficiently high, thus rendering \( u_d(s) = \frac{k_d}{s^2(1+\tau s)^\delta} W_{n-1}^{-1}(s) \) strictly proper. Taking \( \bar{u}(t) = \mathcal{L}^{-1}[u_d(s)] \) and defining \( q_{n-1,f}(t) = \mathcal{L}^{-1}[\frac{k_d}{s^2(1+\tau s)^\delta}] \), it is easy to see that by choosing \( \tau \) sufficiently small (since the impacts have been ignored in the computation of \( q_{h,\ell}(t), h = 1,2,\ldots,n \), and \( q_{h,\ell}(t), h = 1,2,\ldots,n-1 \), the \((n-1)\)-th mass is not constrained to be on the left of the \( n \)-th mass at time \( \hat{t} \), one has

\[ q_{n-1,\ell}(\hat{t}) + q_{n-1,f}(\hat{t}) > q_{n,\ell}(\hat{t}) \]

then, there must exist at least a non-degenerate impact time in the interval \( (t_0, \hat{t}) \). Now, let \( x_1, x_2 \in A \), with \( x_1 \neq x_2 \), be two arbitrary states. Let the input function \( \bar{u}(\cdot) \) be computed as described above, taking \( x_1 \) as initial state. Let \( t_{1,1} \) be the first impact time corresponding to \( \bar{u}(\cdot) \) when the initial state is \( x_1 \) (such an impact time certainly exists by construction), and let \( t_{1,2} \) be the first impact time corresponding to \( \bar{u}(\cdot) \) when the initial state is \( x_2 \) when it exists, or otherwise let \( t_{1,2} = +\infty \). Let \( t_1 := \min(t_{1,1}, t_{1,2}) \). Denoting by \( T > t_1 \) a time such that there are no impacts in \((t_1, T)\) corresponding to \( \bar{u}(\cdot) \) starting from both \( x_1 \) and \( x_2 \), it is now shown that the two states are distinguishable in the interval \( [t_0, T] \), that is

\[ y(\cdot, x_1, \bar{u}(\cdot))|_{[t_0,T]} \neq y(\cdot, x_2, \bar{u}(\cdot))|_{[t_0,T]} \]

where the symbol \( \varphi(\cdot)|_{[T_i,T_f]} \) denotes the function \( \varphi(\cdot) \) restricted to the interval \( [T_i, T_f] \) at subscript. Let \( x_1 = \begin{bmatrix} x_{a,1} \\ x_{b,1} \end{bmatrix} \) and \( x_2 = \begin{bmatrix} x_{a,2} \\ x_{b,2} \end{bmatrix} \) and denote by \( q_{h,j}(t), v_{h,j}(t), h = 1,2,\ldots,n \), the time behavior of the mechanical system when the initial condition is \( x_j, j = 1,2 \); if \( x_{a,1} \neq x_{a,2} \), then by condition \( (2) \) of the theorem, taking into account that there are no impacts in the interval \( [t_0, t_1] \), it follows that the two states \( x_1 \) and \( x_2 \) are distinguishable in the interval \( [t_0, t_1] \). Hence, only the case \( x_{a,1} = x_{a,2}, x_{b,1} \neq x_{b,2} \) needs further attention.
First, suppose that \( t_1 = t_{1,1} \) and \( t_{1,2} > t_{1,1} \). From the restitution rule at time \( t_1 \), if the initial state is \( x_1 \) (whence, \( t_1 \) is a non degenerate impact time), it follows

\[
\begin{bmatrix}
q_{n-1,1}(t_1^-) \\
v_{n-1,1}(t_1^-)
\end{bmatrix} = M_1 \begin{bmatrix}
q_{n-1,1}(t_1^-) \\
v_{n-1,1}(t_1^-)
\end{bmatrix},
\]

whereas, if the initial state is \( x_2 \) (whence, \( t_1 \) is not an impact time), it follows

\[
\begin{bmatrix}
q_{n-1,2}(t_1^-) \\
v_{n-1,2}(t_1^-)
\end{bmatrix} = \begin{bmatrix}
q_{n-1,2}(t_1^-) \\
v_{n-1,2}(t_1^-)
\end{bmatrix} = \begin{bmatrix}
q_{n-1,1}(t_1^-) \\
v_{n-1,1}(t_1^-)
\end{bmatrix}.
\]

Since \( t_1 \) is a non degenerate impact time when the initial state is \( x_1 \), then \( v_{n-1,1}(t_1^-) \neq v_{n-1,1}(t_1^+) \), which implies \( v_{n-1,1}(t_1^-) \neq v_{n-1,2}(t_1^+) \). This, in view of condition (2) of the theorem, implies

\[
y(\cdot, x_1, \bar{u}(\cdot))|_{(t_1, T)} \neq y(\cdot, x_2, \bar{u}(\cdot))|_{(t_1, T)}.
\]

Secondly, if \( t_{1,1} = t_{1,2} \) and \( x_1 \neq x_2 \), then \( q_{n,1}(t_1) = q_{n,2}(t_1) \), because \( t_1 \) is an impact time and \( x_{a,1} = x_{a,2} \). Moreover, \( v_{n,1}(t_1^-) \neq v_{n,2}(t_1^-) \), because (otherwise) by a backward integration it would be \( x_1 = x_2 \), which is a contradiction. Then, it can be seen that

\[
\begin{bmatrix}
q_{n-1,j}(t_1^-) \\
v_{n-1,j}(t_1^-)
\end{bmatrix} = M_1 \begin{bmatrix}
q_{n-1,j}(t_1^-) \\
v_{n-1,j}(t_1^-)
\end{bmatrix}, \quad j = 1, 2,
\]

\[
\begin{bmatrix}
q_{n-1,1}(t_1^-) \\
v_{n-1,1}(t_1^-)
\end{bmatrix} \neq \begin{bmatrix}
q_{n-1,2}(t_1^-) \\
v_{n-1,2}(t_1^-)
\end{bmatrix}.
\]

This, together with condition (2) of the theorem, implies

\[
y(\cdot, x_1, \bar{u}(\cdot))|_{(t_1, T)} \neq y(\cdot, x_2, \bar{u}(\cdot))|_{(t_1, T)}.
\]

Finally, if \( t_1 = t_{1,1} \) and \( t_{1,2} > t_1 \), a reasoning similar to the one carried out in the case \( t_1 = t_{1,1} \) and \( t_{1,2} > t_1 \) can be repeated. ■

### 3.2 Reconstructibility

The following theorem gives a sufficient condition for the mechanical system under consideration to be reconstructible in the \((n - 1,1)\) case.

**Theorem 2.** Let \( T_f > t_0 \). Under the assumption that pair \((C_a, A_a)\) is observable, system (2), (3) and (6) is reconstructible in the interval \([t_0, T_f]\) with respect to all the input functions \( \bar{u}(\cdot) \) satisfying the following two conditions:

(a) there exists at least an impact time in the interval \([t_0, T_f]\),

(b) the number of the impact times in the interval \([t_0, T_f]\) is finite.

**Proof.** First, notice that for reconstructing the state of the system in the interval \([t_0, T_f]\) it is sufficient to reconstruct the state of the system in any subinterval \([t_0, T]\), with \( T < T_f \), because the state of the system along the interval \([T, T_f]\) can be directly computed by integrating the Euler-Lagrange equations (2) in such an interval, taking into account that the restitution rule is to be used for the computation of the post-impact velocities as functions of the pre-impact velocities at each impact time. The following proof is constructive. Let us now show how to compute the state of the mechanical system at time \( T > t_1 \), with \( T < \min\{t_2, T_f\} \), with \( t_1 \) and \( t_2 \) being the first and the possible second impact time, respectively. By well known results from the linear system theory (see, e.g., [17, Chapters 6 and 9]), in view of the observability of pair \((C_a, A_a)\) and taking into
account that in the interval \((t_0, t_1)\) there are no impact times, from the knowledge of the functions \(u(\cdot)|_{[t_0,t_1]}\) and \(y(\cdot)|_{[t_0,t_1]}\) it is possible to compute \(x_n(t_1^-)\) in a unique way, and therefore \(q_{n-1}(t_1^-)\), \(v_{n-1}(t_1^-)\). In a similar way, by using the knowledge of the functions \(u(\cdot)|_{[t_1,T]}\) and \(y(\cdot)|_{[t_1,T]}\) it is possible to compute \(x_n(t_1^-)\), and therefore \(v_{n-1}(t_1^+)\). Since \(t_1\) is an impact time \(q_n(t_1^+) = q_{n-1}(t_1^-)\).

From the restitution rule (6), \(v_n(t_1^+)\) can be computed as a function of the velocities of the \((n-1)\)-th mass immediately before and immediately after the impact time \(t_1\):

\[
v_n(t_1^+) = \frac{m_n - m_{n-1}}{2m_n} v_{n-1}(t_1^-) + \frac{m_{n-1} + m_n}{2m_n} v_{n-1}(t_1^-).
\]

By a direct integration of the equation of motion (5) of the \(n\)-th mass in the interval \((t_1, T)\) from the initial state \(x_n(t_1^+) = \begin{bmatrix} q_n(t_1^+) \\ v_n(t_1^+) \end{bmatrix}\) just now obtained, it is possible to compute the value of the final state \(x_n(T)\). Finally, in view of the observability of pair \((C_a, A_a)\) and taking into account that in the interval \([t_1, T]\) there are no impact times, from the knowledge of the functions \(u(\cdot)|_{[t_1,T]}\) and \(y(\cdot)|_{[t_1,T]}\) it is possible to compute \(x_n(T)\) in a unique way.

3.3 Asymptotic observer design

Assume that pair \((C_a, A_a)\) is observable. Let \(\tilde{q}_a, \tilde{v}_a \in \mathbb{R}^{n-1}\) and \(\hat{q}_n, \hat{v}_n \in \mathbb{R}\) be the estimates of \(q_a, v_a\) and \(q_n, v_n\), respectively, denote by \(\hat{q}_h\) and \(\hat{v}_h\), \(h = 1, \ldots, n-1\), the \(h\)-th components of \(\hat{q}_a\) and \(\hat{v}_a\), respectively, and let

\[
\begin{align*}
\dot{x}_a := \begin{bmatrix} \hat{q}_a \\ \hat{v}_a \end{bmatrix}, & \quad \dot{x}_b := \begin{bmatrix} \hat{q}_n \\ \hat{v}_n \end{bmatrix}, & \quad \dot{x} := \begin{bmatrix} \hat{x}_a \\ \hat{x}_b \end{bmatrix}.
\end{align*}
\]

The observer proposed for system (2), (3) and (6) is given by:

\[
\begin{align*}
\dot{x}_a(t) &= A_a x_a(t) + B_a u(t) + G_c (y(t) - C_a x_a(t)), & t \in (t_i, t_{i+1}), \\
\dot{x}_b(t) &= A_b \dot{x}_b(t), & t \in (t_i, t_{i+1}), \\
\dot{q}_h(t_i^+) &= \hat{q}_h(t_i^-), & h = 2, \ldots, n-1, \ i \in \mathbb{N}, \\
\dot{v}_h(t_i^+) &= \hat{v}_h(t_i^-), & h = 1, \ldots, n-2, \ i \in \mathbb{N}, \\
\dot{q}_1(t_i^+) &= y(t_i), & i \in \mathbb{N}, \\
\dot{q}_n(t_i^+) &= \hat{q}_{n-1}(t_i^-), & i \in \mathbb{N}, \\
\hat{v}_{n-1}(t_i^+) &= M_1 \hat{v}_{n-1}(t_i^-), \\
\hat{v}_n(t_i^+) &= M_2 \hat{v}_n(t_i^-),
\end{align*}
\]

where \(G_c\) is a suitable column vector, depending on the scalar parameter \(\varepsilon \in (0, 1]\), whose choice will be specified and motivated in the following.

Remark 3. Notice that, in view of the jumps (7e)-(7g) the applicability of the observer given in (7) relies on the possibility of detecting the impacts in real time. The behaviour of such an observer in presence of small delays in the detection of the impacts can be investigated by extending the technique proposed in [13], where a similar study was carried out for a family of mechanical systems and observers different from the ones proposed here, but also presenting jumps at the impact times.

In order to choose vector \(G_c\), the approach in [18] is followed. Let \(A_1 = \{\lambda_1, \ldots, \lambda_{2n-2}\}\), \(\text{Re}(\lambda_h) < 0, \ h = 1, \ldots, 2n - 2\), be a fixed set of desired eigenvalues, and let \(G_1\) be the vector guaranteeing that, setting \(A_{C_1} := A_a - G_1 C_a\), one has \(\sigma(A_{C_1}) = A_1\), with \(\sigma(\cdot)\) denoting the spectrum of the matrix at argument; notice that the existence and the uniqueness of \(G_1\) are guaranteed by the observability of pair \((C_a, A_a)\) and by the fact that the measured output \(y(t)\) is scalar. Now, let \(\varepsilon \in (0, 1]\) be a suitably small number, which will be chosen according to the subsequent Theorems 3 and 4, let \(A_{c} = \{\lambda_1/\varepsilon, \ldots, \lambda_{2n-2}/\varepsilon\}\), and let \(G_c\) be the vector...
guaranteeing that, setting $A_{O,\varepsilon} := A_o - G_C C_a$, one has $\sigma(A_{O,\varepsilon}) = A_o$. Under the assumption that a lower and an upper bound on the length of the intervals between adjacent impact times are known, independently of the choice of set $A_1$, it will be now proven that:

- when the masses $m_{n-1}$ and $m_n$ are equal, for any value of $k_{n,n}$, by choosing a suitably small $\varepsilon$, it is possible to guarantee the global exponential stability of the error dynamics, and an arbitrary rate of convergence to zero of the error dynamics itself (see Theorem 3);

- when the masses $m_{n-1}$ and $m_n$ are different, if $k_{n,n} \geq 0$ (i.e., if the subsystem having state vector $x_n$ is not exponentially unstable), it is possible to guarantee the global exponential stability of the error dynamics (see Theorem 4).

To render these statements more formal, let $\tilde{q}_n := q_n - \hat{q}_n$, $\tilde{v}_n := v_n - \hat{v}_n$, $\tilde{q}_a := q_a - \hat{q}_a$ and $\tilde{v}_a$ be the errors in the estimates of $q_n$, $v_n$, $q_a$, and $v_a$, respectively, denote by $\tilde{q}_n$ and $\tilde{v}_n$ the $n$-th components of $\tilde{q}_a$ and $\tilde{v}_a$, respectively, and let

$$
\tilde{x}_a := \begin{bmatrix} \tilde{q}_a \\ \tilde{v}_a \end{bmatrix}, \quad \tilde{x}_b := \begin{bmatrix} \tilde{q}_n \\ \tilde{v}_n \end{bmatrix}, \quad \tilde{x} := \begin{bmatrix} \tilde{x}_a \\ \tilde{x}_b \end{bmatrix}.
$$

Furthermore, let $\delta_i := t_i - t_{i-1}$, for all $i \in \mathbb{N}$, and consider the following assumption.

**Assumption 2** Assume that

(i) the impact times have an accumulation point at $+\infty$,

(ii) there exist two constants $\delta_{\min}$, $\delta_{\max} > 0$ such that for each $i \in \mathbb{N}$, $\delta_{\min} < \delta_i < \delta_{\max}$.

The following theorem can be used if $m_n = m_{n-1}$.

**Theorem 3.** Under the assumption that pair $(C_a, A_a)$ is observable and under Assumption 2, if $m_n = m_{n-1}$, for each $\alpha \in \mathbb{R}^+$, there exist two constants $\varepsilon_{\alpha} \in (0, 1)$ and $\mu_{\alpha} > 0$, such that, for any $\varepsilon \in (0, \varepsilon_{\alpha})$, for any initial condition $x(t_0) \in A$ of system (2), (3) and (6), and for any initial condition $\tilde{x}(t_0) \in \mathbb{R}^{2n}$ of the observer (7), the estimation error $\tilde{x}(t)$ satisfies:

$$
\|\tilde{x}(t)\| < \mu_{\alpha} e^{-\alpha t} \|\tilde{x}_a(t_0)\|, \quad \forall t \geq t_0.
$$

**Proof.** We first recall some properties of high gain observers, taken from [18] and the references therein. Let $P \in \mathbb{R}^{2(n-1) \times 2(n-1)}$ be the nonsingular matrix that transforms the pair $(C_a, A_a)$ into the canonical observer form:

$$
P^{-1} A_a P = \begin{bmatrix} -a_{2n-3} & 1 & 0 & \cdots & 0 \\
- a_{2n-4} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_1 & 0 & 0 & \cdots & 1 \\
a_0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad CP = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},
$$

where $a_0, a_1, ..., a_{2n-3}$ are the coefficients of the characteristic polynomial of $A_a$.

It is well known that, letting $D := -\max_{k=1, \ldots, 2n-2} \text{Re}(\lambda_{B}(A_{O,1}))$, with $\lambda_{B}(\cdot)$ denoting the $h$-th eigenvalue of the matrix at argument, there exists a positive constant $d$, depending on $A_a$ and $G_1$, such that

$$
\left\| e^{A_a \cdot \frac{\tilde{x}}{2}} \right\| \leq d e^{-D \frac{\tilde{x}}{2}}.
$$

Furthermore

$$
e^{A_a \cdot \delta} = P \text{diag} \left( 1, \frac{\varepsilon}{\varepsilon_{2n-3}}, \ldots, \frac{1}{\varepsilon_{2n-3}} \right) P^{-1} e^{A_a \cdot \frac{\tilde{x}}{2}} P \text{diag} \left( 1, \varepsilon, \ldots, \varepsilon_{2n-3} \right) P^{-1}.
$$

---

1 Here, and everywhere in the report, if $w$ is a vector, $\|w\|$ denotes the Cartesian norm of $w$.

2 Here, and everywhere in the report, if $W$ is a matrix, $\|W\|$ denotes the norm of $W$, using as matrix norm the one induced by the Cartesian norm of vectors.
From (8) and (9), it is easily seen that:
\[
\left\| e^{A_\infty \delta} \right\| < \frac{\| P \|^2 \| P^{-1} \|^2 d}{\varepsilon^{2n-3}} e^{-d \frac{z}{2}}.
\] (10)

Now, a simple consequence of the Jury test (see, e.g., [19]) is recalled. Let \( \theta \in (0, 1) \) and consider the polynomial \( p(z) = z^2 - c_1 z - c_0 \), where \( c_0 \) and \( c_1 \) are real coefficients, with \( c_1 > 0 \). It can be easily proven that the two roots of \( p(z) \) have both modulus smaller than \( \theta \) if and only if the following two conditions hold:
\[
c_1 < -\frac{1}{\theta} c_0 + \theta,
\]
\[
|c_0| < \theta^2.
\] (11a)

The proof is Liapunov-like. The positive definite function \( V := \| \hat{x} \|^2 \) of the error \( \hat{x} \) is used: although it is not a Liapunov function, because it can have positive jumps at the impact times, it will be helpful to prove the exponential convergence to zero of the error.

In view of equations (7c)-(7g), of the continuity of all the position variables and of equation (6), which in this case become simply \( v_{n-1}(t_{i-1}^+) = v_n(t_i^-) \) and \( v_n(t_i^+) = v_{n-1}(t_i^-) \), the following equations can be written:
\[
\tilde{q}_1(t_i^+) = 0,
\]
\[
\tilde{q}_h(t_i^+) = \tilde{q}_h(t_i^-), \quad h = 2, \ldots, n - 1,
\]
\[
\tilde{v}_h(t_i^+) = \tilde{v}_h(t_i^-), \quad h = 1, \ldots, n - 2,
\]
\[
\tilde{v}_{n-1}(t_i^+) = \tilde{v}_n(t_i^-),
\]
\[
\tilde{q}_n(t_i^+) = \tilde{q}_{n-1}(t_i^-),
\]
\[
\tilde{v}_n(t_i^+) = \tilde{v}_{n-1}(t_i^-).
\] (12a)

Let \( V_a := \| \tilde{x}_a \|^2 \) and \( V_b := \| \tilde{x}_b \|^2 \), so that \( V = V_a + V_b \). Equations (12) imply the following inequalities for each \( i \in \mathbb{N} \):
\[
V_a(t_i^+) \leq V_a(t_i^-) + V_b(t_i^-),
\]
\[
V_b(t_i^+) \leq V_a(t_i^-).
\] (13a)

Moreover, since the equations describing the error dynamics for \( t \in (t_i, t_{i+1}) \) are clearly:
\[
\dot{\tilde{x}}_a(t) = A_{\infty} \tilde{x}_a(t),
\]
\[
\dot{\tilde{x}}_b(t) = A_b \tilde{x}_b(t),
\]
for each \( i \in \mathbb{Z}^+ \),
\[
\tilde{x}_a(t_{i+1}^-) = e^{A_{\infty} \delta_{i+1}} \tilde{x}_a(t_i^+),
\]
\[
\tilde{x}_b(t_{i+1}^-) = e^{A_b \delta_{i+1}} \tilde{x}_b(t_i^+).
\] (14a)

By using (13b) and (14b), it can be seen that
\[
V_b(t_{i+1}^-) = \| \tilde{x}_b(t_{i+1}^-) \|^2
\]
\[
= \| e^{A_b \delta_{i+1}} \tilde{x}_b(t_i^+) \|^2
\]
\[
\leq \| e^{A_b \delta_{i+1}} \|^2 V_b(t_i^+)
\]
\[
\leq \| e^{A_b \delta_{i+1}} \|^2 V_a(t_i^-),
\] (15)
from which, by Assumption 2, it is clear that, in order to prove the exponential convergence to zero of $\tilde{x}_a$ it suffices to prove the exponential convergence to zero of $\tilde{x}_a$. In order to achieve this goal, by using (13a), (14a) and (15), write:

$$V_a(t_i^+)| = \|\tilde{x}_a(t_i^+)|^2$$

$$= \|e^{A b_r \epsilon_{i+1}} \tilde{x}_a(t_i^-)|^2$$

$$\leq \|e^{A b_r \epsilon_{i+1}} V_a(t_i^-)|^2$$

$$\leq \|e^{A b_r \epsilon_{i+1}} (2 V_a(t_i^-) + V_b(t_i^-))$$

$$\leq \|e^{A b_r \epsilon_{i+1}} V_a(t_i^-)|^2 + \|e^{A b_r \epsilon_{i+1}}|^2 \|e^{A b_r \epsilon_i}|^2 |V_a(t_i^-)).$$

(16)

Letting $V_{a,D}(i) := V_a(t_i^-)$ and $k = i - 1$, the inequality (16) can be rewritten as

$$V_{a,D}(k + 2) \leq c_1(k) V_{a,D}(k + 1) + c_0(k) V_{a,D}(k),$$

(17)

where the coefficients $c_1(k) := \|e^{A b_r \epsilon_{i+1}}|^2$ and $c_2(k) := \|e^{A b_r \epsilon_{i+1}}|^2 \|e^{A b_r \epsilon_i}|^2$ are non-negative. In view of Assumption 2, by letting

$$b_1 := \max_{\delta \in [\delta_{\text{min}}, \delta_{\text{max}}]} \{\|e^{A b_r \epsilon_{i+1}}|\|^2\},$$

$$b_0 := \max_{\delta \in [\delta_{\text{min}}, \delta_{\text{max}}]} \{\|e^{A b_r \epsilon_{i+1}}|\|^2\} \cdot \max_{\delta \in [\delta_{\text{min}}, \delta_{\text{max}}]} \{\|e^{A b_r \epsilon_i}|\|^2\},$$

it follows that $c_1(k) \leq b_1$ and $c_0(k) \leq b_0$, for all $k \in \mathbb{Z}^+$. Consider the following difference equation:

$$w(k + 2) = b_1 w(k + 1) + b_0 w(k).$$

(18)

By taking into account that $V_{a,D}(k)$ is a non-negative variable, it is easy to prove that, if $w(k)$ is the solution of (18) from the initial conditions $w(1) = V_a(t_1^-)$, $w(2) = V_a(t_2^-)$ then $V_{a,D}(k) \leq w(k)$, for all $k \geq 1$. Then, the exponential convergence to zero of the discrete-time variable $w(k)$, will prove the theorem. In particular, let $\varphi := e^{-2 \delta_{\text{d,max}}}.$

To see that it is possible to render (11) satisfied with $c_1$ and $c_0$ replaced by $b_1$ and $b_0$, respectively, by choosing a suitably small number $\varepsilon$, let $\mu_0 := \max_{\delta \in [\delta_{\text{min}}, \delta_{\text{max}}]} \{\|e^{A b_r \epsilon_i}|\|^2\}$.

Notice that, for any fixed $\delta$, the right hand side of inequality (10) tends to zero as $\varepsilon$ tends to zero. Then, there exist values $\varepsilon_a$, $\varepsilon_b \in (0, 1)$ such that $\max_{\delta \in [\delta_{\text{min}}, \delta_{\text{max}}]} \{\|e^{A b_r \epsilon_i}|\|^2\} < \frac{\varepsilon_a}{2} \|e^{A b_r \epsilon_{i+1}}|^2 \|e^{A b_r \epsilon_i}|^2 < \frac{\varepsilon_b}{2} \|e^{A b_r \epsilon_i}|^2 \|e^{A b_r \epsilon_{i+1}}|\|^2$, and $\max_{\delta \in [\delta_{\text{min}}, \delta_{\text{max}}]} \{\|e^{A b_r \epsilon_{i+1}}|\|^2\} < \frac{\varepsilon_a}{2} \|e^{A b_r \epsilon_i}|^2 \|e^{A b_r \epsilon_{i+1}}|\|^2$, for all $\varepsilon \leq \varepsilon_a$.

Let $\varepsilon_a := \min \{\varepsilon_a, \varepsilon_b\}$. It is easily seen that, for any $\varepsilon < \varepsilon_a$, conditions (11) are satisfied with $c_1$ and $c_0$ replaced by $b_1$ and $b_0$, respectively. In the remainder of the proof, assume that a value of $\varepsilon < \varepsilon_a$ has been chosen. In view of such a choice of $\varepsilon$, since conditions (11) are satisfied with $c_1$ and $c_0$ replaced by $b_1$ and $b_0$, respectively, the two roots $z_1$ and $z_2$ of the polynomial $p_w(z) = z^2 - b_1 z - b_0$ satisfy $|z_a| \leq R$, $h = 1, 2$, with $R < \varphi$. Then, in view of well known results for linear discrete-time systems, since the characteristic polynomial of the difference equation (18) is just $p_w(z)$, there exists a positive constant $r$ such that

$$|w(k)| \leq r R^k \left\|\frac{w(0)}{w(1)}\right\|, \forall k \geq 1.$$  

(19)

Let $\mu_a := \max_{\delta \in [\delta_{\text{min}}, \delta_{\text{max}}]} \{\|e^{A b_r \epsilon_i}|\|^2\}$. Since $w(0) = V_a(t_0) = \|\tilde{x}_a(t_0)|\|^2$, and $w(1) = V_a(t_1^-) = \|e^{A b_r \epsilon_{i+1}} \tilde{x}_a(t_0)|\|^2$, it follows that $\left\|\frac{w(0)}{w(1)}\right\| \leq (1 + \mu_a^2) \|\tilde{x}_a(t_0)|\|^2$, whence, by (19)

$$V_a(t_i^+) \leq r R t \left(1 + \mu_a^2\right) \|\tilde{x}_a(t_0)|\|^2.$$  

(20)

Let $\mu_0 := \max \{\sqrt{\mu_a}, \sqrt{\mu_b}\}$. In view of equations (14),

$$\|\tilde{x}(i)\| \leq \mu_0 \|\tilde{x}(t_i^+)|\|, \forall t \in (t_i, t_i+1),$$  

(21)
from which, taking into account also (13), (15) and (20), one can write

$$
\|\tilde{x}(t)\| \leq \mu_0 \|\tilde{x}(t_i^+)\|
$$

\[
\leq \mu_0 \sqrt{V_a(t_i^+) + V_b(t_i^-)}
\]

\[
\leq \mu_0 \sqrt{2 V_a(t_i^-) + V_b(t_i^-)}
\]

\[
\leq \mu_0 \sqrt{2 V_a(t_i^-) + \mu_k^2 V_a(t_{i-1}^-)}
\]

\[
\leq \mu_0 \sqrt{(2 r R^i + \mu_k^2 r R^{i-1}) (1 + \mu_k^2) \|\tilde{x}_a(t_0)\|^2}
\]

\[
\leq \mu_0 \sqrt{r \left(2 + \frac{\mu_k^2}{R}\right) (1 + \mu_k^2) \sqrt{R^i \|\tilde{x}_a(t_0)\|}}, \quad \forall t \in (t_i, t_{i+1}).
\]

(22)

Now, consider that, for all \( t \in (t_i, t_{i+1}), t \leq (i + 1)\delta_{\max}, \) which implies \( e^{-2 \alpha \delta_{\max}} \left(e^{-2 \alpha \delta_{\max}}\right)^t \leq e^{-2 \alpha t}. \) From the definition of \( \rho \), it follows that \( \rho^i \leq e^{2 \alpha \delta_{\max} e^{-2 \alpha t}} \), whence, since \( R < \rho \), from inequality (22) (which can be written for any \( i \in \mathbb{Z}^+ \)), it follows that

$$
\|\tilde{x}(t)\| \leq \mu_0 \sqrt{r \left(2 + \frac{\mu_k^2}{R}\right) (1 + \mu_k^2) \rho^i \|\tilde{x}_a(t_0)\|}.
$$

The proof of the theorem is complete, with \( \mu_{\alpha} := \mu_0 \sqrt{r \left(2 + \frac{\mu_k^2}{R}\right) (1 + \mu_k^2) \rho^i} \). \( \blacksquare \)

The following theorem deals with the case in which the masses of the two impacting bodies are possibly different.

**Theorem 4.** Let \( \alpha_L := -\frac{1}{\delta_{\max}} \ln(|M_{2,2}|) \). Under the assumption that pair \((C_a, A_a)\) is observable and under Assumption 2, if \( k_{\alpha,n} \geq 0, \) for each \( \alpha \in \mathbb{R}^+, \alpha < \alpha_L, \) there exist two constants \( \varepsilon_{\alpha} \in (0, 1) \) and \( \mu_{\alpha} > 0 \) such that, for any \( \varepsilon \in (0, \varepsilon_{\alpha}), \) for any initial condition \( x(t_0) \in \mathcal{A} \) of system (2), (3) and (6), and for any initial condition \( \tilde{x}(t_0) \in \mathbb{R}^{2n} \) of the observer (7), the estimation error \( \tilde{x}(t) \) satisfies:

$$
\|\tilde{x}(t)\| \leq \mu_{\alpha} e^{-\alpha t} \|\tilde{x}(t_0)\|, \quad \forall t \geq t_0.
$$

**Remark 4.** Notice that \(|M_{2,2}| < 1 \). Hence \( \alpha_L \) is a positive number, which represents the limit on the admissible decay rates \( \alpha \). Notice that such a limit depends on the difference between the masses of the impacting bodies: if the difference tends to zero, \( \alpha_L \) tends to \( +\infty \), thus partially recovering the result stated in Theorem 3. \( \square \)

**Proof of Theorem 4.** As in the proof of Theorem 3, equations (12) can be written, except for equations (12d) and (12f) which are replaced by:

\[
\tilde{v}_{n-1}(t_i^+) = M_{1,1} \tilde{v}_{n-1}(t_i^-) + M_{1,2} \tilde{v}_n(t_i^-), \tag{23a}
\]

\[
\tilde{v}_n(t_i^+) = M_{2,1} \tilde{v}_{n-1}(t_i^-) + M_{2,2} \tilde{v}_n(t_i^-). \tag{23b}
\]

The more complicated form of such jumps renders less convenient the quasi-Liapunov approach, so that in this case the theorem will be proven with direct reference to the estimation errors. Equations (14) still hold, but in this case it is needed to write equation (14b) more in detail. In particular, consider that, if \( k_{\alpha,n} = 0 \):

\[
e^{A_{\alpha} \delta} = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}.
\]

(24)
whereas if $k_{n,n} > 0$, letting $\omega := \sqrt{\frac{k_{n,n}}{m_n}}$:

$$e^{A_n \delta} = \begin{bmatrix} \cos(\omega \delta) & \frac{1}{\omega} \sin(\omega \delta) \\ -\omega \sin(\omega \delta) & \cos(\omega \delta) \end{bmatrix}. \quad (25)$$

Notice that, by taking the limit for $\omega \to 0^+$ of the right hand side of (25), the right hand side of (24) is obtained; hence, for brevity, the proof will be carried out in the (slightly more complicated) case of $k_{n,n} > 0$, but the results will hold true for $k_{n,n} = 0$ by simply substituting $\omega$ with 0. Using equations (12e), (23b), (14b) and (25), it can be seen that

$$\begin{align*}
\tilde{q}_n(t_{i+1}^-) &= \cos(\omega \delta_{i+1}) \tilde{q}_{n-1}(t_i^-) + \frac{1}{\omega} \sin(\omega \delta_{i+1}) \left(M_{2,1} \tilde{v}_{n-1}(t_i^-) + M_{2,2} \tilde{v}_n(t_i^-)\right), \quad (26a) \\
\tilde{v}_n(t_{i+1}^-) &= -\omega \sin(\omega \delta_{i+1}) \tilde{q}_{n-1}(t_i^-) + \cos(\omega \delta_{i+1}) \left(M_{2,1} \tilde{v}_{n-1}(t_i^-) + M_{2,2} \tilde{v}_n(t_i^-)\right). \quad (26b)
\end{align*}$$

From equations (7c)-(7g), it is clear that the discrete-time variable $\tilde{q}_n(t_{i+1}^-)$ plays no special role in the discrete-time convergence analysis (as in the proof of Theorem 3 happened for the whole vector $\tilde{x}_n(t_{i+1}^-)$), since, on one hand the value of the estimate $\hat{q}_n(t_{i+1}^-)$ is not used after time $t_{i+1}^-$ by our algorithm, and, moreover, from equation (26a) it is clear that the convergence to zero of $\tilde{q}_n(t)$ can be deduced from the convergence to zero of $\tilde{q}_{n-1}(t_i^-)$, $\tilde{v}_{n-1}(t_i^-)$ and $\tilde{v}_n(t_i^-)$. As for the estimation error $\tilde{x}_n$, it can be seen that

$$\tilde{x}_n(t_{i+1}^-) = e^{A_{\delta_{i+1}}} \begin{bmatrix} 0 \\
\tilde{q}_2(t_i^-) \\
\vdots \\
\tilde{q}_{n-1}(t_i^-) \\
\tilde{v}_1(t_i^-) \\
\vdots \\
\tilde{v}_{n-2}(t_i^-) \\
M_{1,1} \tilde{v}_{n-1}(t_i^-) + M_{1,2} \tilde{v}_n(t_i^-) \end{bmatrix}, \quad (27)$$

from which, by taking into account that $|M_{1,1}| < 1$,

$$||\tilde{x}_n(t_{i+1}^-)|| \leq ||e^{A_{\delta_{i+1}}}|| \left(||\tilde{x}_n(t_i^-)|| + |M_{1,2}| ||\tilde{v}_n(t_i^-)||\right). \quad (28)$$

On the other hand, from (26b),

$$||\tilde{v}_n(t_{i+1}^-)|| \leq (|M_{2,1}| + \omega)||\tilde{x}_n(t_i^-)|| + |M_{2,2}| ||\tilde{v}_n(t_i^-)||. \quad (29)$$

Letting $V_{1,D}(i) := ||\tilde{x}_n(t_i^-)||$ and $V_{2,D}(i) := ||\tilde{v}_n(t_i^-)||$ the inequalities (28) and (29) can be rewritten as

$$\begin{align*}
V_{1,D}(k + 1) &\leq b_{1,1}(k) V_{1,D}(k) + b_{1,2}(k) V_{2,D}(k), \quad (30a) \\
V_{2,D}(k + 1) &\leq b_{2,1}(k) V_{1,D}(k) + b_{2,2}(k) V_{2,D}(k), \quad (30b)
\end{align*}$$

where $b_{2,1} := |M_{2,1}| + \omega$, $b_{2,2} := |M_{2,2}|$, and also $b_{1,1}(k)$ and $b_{1,2}(k)$ are non-negative. In view of Assumption 2, by letting $b_{1,1} := \max_{i \in [a_n, a_{\infty}]} \{||e^{A \delta_i}||\}$, $b_{1,2} := b_{1,1} |M_{1,2}|$, $b_{1,1}(k) \leq b_{1,1}$ and $b_{1,2}(k) \leq b_{1,2}$, for all $k \in \mathbb{Z}^+$. Consider the discrete-time system:

$$\begin{align*}
w_1(k + 1) &= b_{1,1} w_1(k) + b_{1,2} w_2(k), \quad (31a) \\
w_2(k + 1) &= b_{2,1} w_1(k) + b_{2,2} w_2(k), \quad (31b)
\end{align*}$$
By taking into account that both \( V_{1,D}(k) \) and \( V_{2,D}(k) \) are non negative variables, it is easy to prove that, if \( w(k) := [w_1(k) \ w_2(k)]^T \) is the solution of (31) from the initial conditions \( w_1(1) = \|\tilde{x}_n(t^*_n)\| \), \( w_2(1) = \|\tilde{v}_n(t^*_n)\| \), then \( V_{h,D}(k) \leq w_h(k) \), for \( h = 1, 2 \) for all \( k \geq 1 \). Then, the exponential convergence to zero of the discrete-time vector variable \( w(k) \) will prove the theorem. Notice that the characteristic polynomial of the discrete-time system (31) is:

\[
p_w(z) = z^2 - (b_{1,1} + b_{2,2}) z - (b_{1,2}b_{2,1} - b_{1,1}b_{2,2}).
\]

Let \( \varrho := e^{-2\alpha \delta_{\max}} \), and notice that \( \alpha < \alpha_L \) implies \( \varrho > |M_{2,2}| \). By repeating a reasoning wholly similar to the one in the proof of Theorem 3, it is easily seen that, in order to achieve the desired exponential convergence to zero of the error dynamics, it is sufficient to obtain that conditions (11) are satisfied with \( c_1 \) and \( c_0 \) replaced by \( (b_{1,1} + b_{2,2}) \) and \( (b_{1,2}b_{2,1} - b_{1,1}b_{2,2}) \), respectively, by choosing a suitably small number \( \varepsilon \). In particular, simple computations show that it is sufficient to satisfy the following two inequalities:

\[
\max_{\delta \in [\delta_{\min}, \delta_{\max}]} \left\{ \|e^{A\delta_{\sigma,n}}\| \right\} \leq \frac{\varrho (\varrho - |M_{2,2}|)}{|M_{1,1}| (|M_{2,1} + \omega|) + \varrho - |M_{2,2}|},
\]

\[
\max_{\delta \in [\delta_{\min}, \delta_{\max}]} \left\{ \|e^{A\delta_{\sigma,n}}\| \right\} \leq \frac{\varrho^2}{|M_{1,1}| (|M_{2,1} + \omega|) - |M_{2,2}|}.
\]

### 3.4 An application of the asymptotic observer to a 3 masses \((2,1)\) system.

Consider a 3 masses system, composed by two decoupled subsystems (see Figure 1). The first one is constituted by two masses connected by a spring with elastic constant \( k_{12} \); the mass \( m_1 \) is actuated by means of the external force \( u(t) \). The second body is a single mass \( m_3 \), connected to the environment by a spring with elastic constant \( k_{33} \). Only the position of the mass \( m_1 \) is measured. The possibility of estimating all the state variables of the system is guaranteed by the constraint \( q_2(t) \leq q_3(t) \), which causes non-smooth impacts between the second and the third mass (which are the only impacts considered in this example).

![Fig. 1: The system considered in Chapter 3.4.](image)

The results of a simulation are reported in Figures 2-3. The following values of the parameters have been assumed: \( m_1 = 1 \text{kg} \), \( m_2 = 0.2 \text{kg} \), \( m_3 = 1 \text{kg} \), \( k_{1,2} = 1 \text{Nm}^{-1} \), \( k_{3,3} = 0.5 \text{Nm}^{-1} \), and a constant force \( u(t) = 1N \) has been applied. The matrix \( G_\varepsilon \) has been chosen so that the eigenvalues of the error dynamics are \( \lambda_1 = -2 + i \), \( \lambda_2 = -2 - i \), \( \lambda_3 = -3 \), \( \lambda_4 = -4 \). Starting from the initial conditions \( q_1(0) = 1m \), \( q_2(0) = -2m \), \( q_3(0) = 0m \), \( v_1(0) = 0ms^{-1} \), \( v_2(0) = 0ms^{-1} \), \( v_3(0) = 1ms^{-1} \), for the system and from null initial conditions for the observer, with the mentioned constant input, 17 impacts occur in the interval \((0, 30)\). Thanks to such impacts, as it is possible to see in the plots, the estimation errors are reduced to negligible values in about 20 seconds.

### 3.5 Dead-beat observer design

In this section, a dead-beat observer is proposed for the case \( n_a = n - 1 \), \( n_b = 1 \), again assuming that the pair \((C_a, A_a)\) is observable. Some notations are introduced first. In particular, with reference
Fig. 2: Time behavior of the position and velocity variables (continuous lines) and of their estimates (dashed lines) for the simulation described in Chapter 3.4. In the lower left-side plot the difference $f_{2,3}(t) = q_2(t) - q_3(t)$ is reported, in order to emphasize the impacts.

Fig. 3: Time behavior of the estimation errors for the simulation described in Chapter 3.4.

to the row vectors $M_1$ and $M_2$ given at the beginning of this section, let $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$, define the
square matrix $M \in \mathbb{R}^{2n \times 2n}$ as follows:

$$
M := 
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
I_{n-2} & 0_{n-2,n-2} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0_{n-2,n-2} & I_{n-2} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
$$

and define the matrix $\overline{M} \in \mathbb{R}^{2n \times 2(n-1)}$ as the matrix obtained by removing the first and the $n$-th columns of $M$ (i.e., the only two null columns of $M$). Define the matrices $\mathcal{L} \in \mathbb{R}^{2(n-1) \times 2n}$ and $\mathcal{L}^t \in \mathbb{R}^{2(n-1) \times 2n}$ as follows:

$$
\mathcal{L} := 
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
I_{n-2} & 0_{n-2,n} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix},
\mathcal{L}^t := 
\begin{bmatrix}
0 & 0 \\
I_{n-2} & 0_{n-2,n} \\
0 & 0 \\
\end{bmatrix},
$$

and notice that $\mathcal{L}^t \mathcal{L} = I_{2(n-1)}$.

The same notations as in Section 3.3 are used for the vectors $\hat{x}_a$, $\hat{x}_b$, $\hat{x}_a$, $\hat{x}_b$ and $\hat{x}$, as well as for their components. In the following, however, also the “reordered” estimation error defined as

$$
e(t) := [\hat{q}_1(t) \quad \hat{q}_2(t) \quad \hat{q}_n(t) \quad \hat{v}_1(t) \quad \hat{v}_2(t) \quad \hat{v}_n(t)]^T,
$$

will be used, which, obviously, is related to $\hat{x}(t)$ by a permutation matrix $P$: $e = P \hat{x}$.

The first step for observer design is to compute an output injection gain vector $G \in \mathbb{R}^{2(n-1) \times 1}$ such that the matrix $A_a - GC_a$ has all its eigenvalues in the open left half plane. Then, for later use, define the square matrix $\tilde{A} \in \mathbb{R}^{n \times n}$ as follows:

$$
\tilde{A} := P \begin{bmatrix} A_a - GC_a & 0_{2(n-1) \times 2} \\ 0_{2,2(n-1)} & A_b \end{bmatrix} P^{-1}.
$$

Moreover, for each $i \in \mathbb{N}$, recalling that $\delta_i = t_i - t_{i-1}$, denote with $\Gamma_{i,1}$, $\Gamma_{i,n-1}$ and $\Gamma_{i,n} \in \mathbb{R}^{1,n}$, respectively, the first, the $(n-1)$-th and the $n$-th rows of the matrix $e^{\Delta \delta_i}$.

The proposed observer is described by:

\begin{align}
\dot{x}_a(t) &= A_a \hat{x}_a(t) + B_a u(t) + G (y(t) - C_a \hat{x}_a(t)), \quad t \in (t_i, t_{i+1}), \\
\dot{x}_b(t) &= A_b \hat{x}_b(t), \quad t \in (t_i, t_{i+1}), \\
\dot{\hat{q}}_a(t_i^+) &= \begin{bmatrix} \hat{q}_1(t_i^-) \\ \vdots \\ \hat{q}_{n-1}(t_i^-) \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta q_i \end{bmatrix}, \quad i \in \mathbb{N}, \\
\dot{\hat{q}}_b(t_i^+) &= \hat{q}_{n-1}(t_i^-) + \Delta q_{i,n-2}, \quad i \in \mathbb{N}, \\
\begin{bmatrix} \hat{v}_a(t_i^+) \\ \hat{v}_b(t_i^+) \end{bmatrix} &= \begin{bmatrix} \hat{v}_1(t_i^-) \\ \vdots \\ \hat{v}_{n-2}(t_i^-) \end{bmatrix} + \Delta v_i, \quad i \in \mathbb{N},
\end{align}

for $i \in \mathbb{N}$ and $t_i \in (t_{i-1}, t_i)$.
where $\Delta q_{i,n-2}$ denotes the last component of the vector $\Delta q_i$ and the jumps $\Delta q_i \in \mathbb{R}^n$ are defined as the solution of the following difference equation from an arbitrary initial condition ($\Delta q_{-(n-2)}, \Delta v_{-(n-2)}, \ldots, \Delta q_1, \Delta v_1, \Delta q_0, \Delta v_0$):

$$
\begin{bmatrix}
\Delta q_{k+n-1} \\
\Delta v_{k+n-1}
\end{bmatrix} = \mathcal{L}_z \left( \mathcal{M} e^{\hat{A} t_{k+n-1}} \mathcal{M} e^{\hat{A} t_{k+n-2}} \ldots \mathcal{M} e^{\hat{A} t_{k+1}} \mathcal{M} z \right) \nonumber \\
- \sum_{j=0}^{n-2} \mathcal{M} e^{\hat{A} t_{k+n-1}} \mathcal{M} e^{\hat{A} t_{k+n-2}} \ldots \mathcal{M} e^{\hat{A} t_{k+1}} \mathcal{L} \left[ \Delta q_{k+j} \right. \\
\left. \Delta v_{k+j} \right] 
$$

where the vector $z \in \mathbb{R}^{2(n-1)}$ is given below as a function of the following quantities (all known at time $t_i^+ = t_{k+n-1}^+$, if $i \geq n$, or otherwise to be set to arbitrary initial values):

$$
\delta_{k+1}, \ldots, \delta_{k+n-1}, \begin{bmatrix}
\Delta q_k \\
\Delta v_k \\
\hat{q}_1(t_{k+1}) \\
\hat{q}_1(t_{k+1}) \\
\hat{q}_{n-1}(t_{k+1}) \\
\hat{q}_{n-1}(t_{k+1}) \\
\hat{q}_n(t_{k+1}) \\
\hat{q}_n(t_{k+1})
\end{bmatrix}, \begin{bmatrix}
\Delta q_{k+n-2} \\
\Delta v_{k+n-2}
\end{bmatrix}, \begin{bmatrix}
y(t_{k+1}), \ldots, y(t_{k+n-1})
\end{bmatrix}
$$

Define the matrix $\Xi_{k+n-1} \in \mathbb{R}^{2(n-1) \times 2n}$ as follows

$$
\Xi_{k+n-1} := \begin{bmatrix}
\Gamma_{k+1,1} & 0 \\
\Gamma_{k+1,1} & \mathcal{M} e^{\hat{A} t_{k+1}} \\
\Gamma_{k+2,1} & 0 \\
\Gamma_{k+2,1} & \mathcal{M} e^{\hat{A} t_{k+1}} \\
\vdots & \vdots \\
\Gamma_{k+n-1,1} & 0 \\
\Gamma_{k+n-1,1} & \mathcal{M} e^{\hat{A} t_{k+1}}
\end{bmatrix}
$$

and notice that the knowledge of $\delta_i$ in computing matrix $\Xi_i$ is needed only for the computation of the last two rows, so that the matrix $(\Xi_{i+1})_{1:2(n-2)} \in \mathbb{R}^{2(n-2) \times 2(n-1)}$ constituted by the first $2(n-2)$ rows of $\Xi_{i+1}$ can be already computed at time $t_{i+1}^+$. For a similar reason, for any $j = 1, \ldots, n-2$, the matrix $(\Xi_{i+j})_{1:2(n-1-j)} \in \mathbb{R}^{2(n-1-j) \times 2(n-1)}$ constituted by the first $2(n-1-j)$ rows of $\Xi_{i+j}$ can be already computed at time $t_{i+j}^+$. Finally, the description of the observer is completed by the following formula, which allows the vector $z$ to be computed at time $t_{k+n-1}^+$:

$$
z = \begin{bmatrix}
\begin{bmatrix}
y(t_{k+1}) - \hat{q}_1(t_{k+1}) \\
\hat{q}_1(t_{k+1}) - \hat{q}_{n-1}(t_{k+1}) \\
y(t_{k+2}) - \hat{q}_1(t_{k+2}) \\
\hat{q}_1(t_{k+2}) - \hat{q}_{n-1}(t_{k+2}) \\
y(t_{k+n-1}) - \hat{q}_1(t_{k+n-1}) \\
\hat{q}_1(t_{k+n-1}) - \hat{q}_{n-1}(t_{k+n-1})
\end{bmatrix}
\end{bmatrix}
+ \sum_{j=0}^{n-2} \left[ (\Xi_{k+n-1+j})_{1:2(n-1-j)} \right] \mathcal{L} \left[ \Delta q_{k+j} \right. \\
\left. \Delta v_{k+j} \right] 
$$

Assumption 3 The impact times have one and only one accumulation point at $+\infty$, and there exists an impact time $t_i^+$, with $i^+ \geq n$, such that the matrix $\Xi_i \mathcal{M}$, computed on the basis of $\delta_i$, $\delta_{i-1}$, $\ldots$, $\delta_{i-(n-1)}$, is nonsingular.
Theorem 5. Assume that the pair \((C_n, A_n)\) is observable. Under Assumption 3, for any initial
state \(x(t_0) \in \mathcal{A}\) of system (2), (3) and (6), and for any initial state \(\bar{x}(t_0)^T \bar{x}(t_0)^T \in \mathbb{R}^{2n}\) of
the observer (32), (33) and (34), the following dead-beat property is guaranteed for the estimation
errors:
\[
\begin{bmatrix}
\bar{x}_a(t) \\
\bar{x}_b(t)
\end{bmatrix} = 0, \quad \forall t > t^*.
\]

Proof. Let \(z : = \begin{bmatrix} q_2(t_K^+) & \ldots & q_{n-1}(t_K^+) & \tilde{v}_1(t_K^-) & \ldots & \tilde{v}_n(t_K^-) \end{bmatrix}^T\) be a \((2n - 1)\) dimensional vector
such that, letting \(z_q \in \mathbb{R}^{n-2}\) denote the vector of its first \(n - 2\) elements, and \(z_v \in \mathbb{R}^n\) denote
the vector of its last \(n\) elements, immediately before the \(k\)-th impact, \(i.e.,\) at time \(t_K^-\), it is possible to write
\(e(t_K^-) = [z_q^T \ast z_v^T]^T\) (* stands for an element whose value has no importance). Assume the
observer is applied starting from a generic time \(t_k\). It is now shown that, at each impact time, it is
possible to write two scalar equations in the unknown \(z\) in such a way that at time \(t_{k+n-1}^-\), \(i.e.,\)
after \(n - 1\) impacts, if Assumption 3 holds with \(i^* = k + n - 1\), it is possible to derive \(z\) (equation
(34)) and use it to set the jumps so that the subsequent estimation errors become 0 (equation
(33)). Observe that, in view of the definitions of matrices \(L\) and \(M\), and of equations (32c)-(32e),
the following equations hold:
\[
e(t_K^-) = Me(t_K^-) - L \begin{bmatrix} \Delta q_k \\
\Delta v_k
\end{bmatrix}.
\]  
(35)
Furthermore, at time \(t_{k+1}^-\), \(i.e.,\) immediately before the impact \(k + 1\), the estimation error can be
expressed as:
\[
e(t_{k+1}^-) = e^{A_{k+1}}e(t_K^-).
\]  
(36)
Now, the first element of \(e(t_{k+1}^-)\) is known and given by \(y(t_{k+1}) - \hat{y}_1(t_{k+1})\). Moreover, it is known
that, in order for \(t_{k+1}\) to be an impact, it is needed that \(\hat{q}_{n-1}(t_{k+1}^-) = q_n(t_{k+1}^-),\ i.e.,\) in terms of
estimates and estimation errors:
\[
\hat{q}_{n-1}(t_{k+1}^-) + \hat{q}_{n-1}(t_{k+1}^-) = \hat{q}_n(t_{k+1}^-) + \hat{q}_n(t_{k+1}^-).
\]
Hence, taking into account the definition of the row vectors \(\Gamma_{i,1}, \Gamma_{i,n-1}\) and \(\Gamma_{i,n}\) given above, it is
possible to write at the first impact after time \(t_k\) the two following scalar equations:
\[
\Gamma_{k+1,1} Me^{A_{k+1}}z = \Gamma_{k+1,1} L \begin{bmatrix} \Delta q_k \\
\Delta v_k
\end{bmatrix} + y(t_{k+1}) - \hat{q}_1(t_{k+1}),
\]  
(37a)
\[
(\Gamma_{k+1,n-1} - \Gamma_{k+1,n}) Me^{A_{k+1}}z = (\Gamma_{k+1,n-1} - \Gamma_{k+1,n}) L \begin{bmatrix} \Delta q_k \\
\Delta v_k
\end{bmatrix} + \hat{q}_n(t_{k+1}^-) - \hat{q}_{n-1}(t_{k+1}^-).
\]  
(37b)
Iterating similar computations, for every impact \(k+j\), with \(j = 2, \ldots, n-1\), it is possible to obtain
two more equations in the unknown \(z\):
\[
\Gamma_{k+j,1} Me^{A_{k+j-1}} \ldots Me^{A_{k+1}}z =
\Gamma_{k+j,1} \left(\sum_{h=0}^{j-1} Me^{A_{k+j-1}} \ldots Me^{A_{k+h+1}} L \begin{bmatrix} \Delta q_{k+h} \\
\Delta v_{k+h}
\end{bmatrix} + y(t_{k+j}) - \hat{q}_1(t_{k+j})\right),
\]  
(38a)
\[
(\Gamma_{k+j,n-1} - \Gamma_{k+j,n}) Me^{A_{k+j-1}} \ldots Me^{A_{k+1}}z =
(\Gamma_{k+j,n-1} - \Gamma_{k+j,n}) \left(\sum_{h=0}^{j-1} Me^{A_{k+j-1}} \ldots Me^{A_{k+h+1}} L \begin{bmatrix} \Delta q_{k+h} \\
\Delta v_{k+h}
\end{bmatrix} + \hat{q}_n(t_{k+j}^-) - \hat{q}_{n-1}(t_{k+j}^-)\right).
\]  
(38b)
Collecting all such equations for all times \(t_{k+j}, j = 1, 2, \ldots, n-1, 2(n-1)\) equations in the \(2(n-1)\)
unknown elements of \(z\) are obtained. If Assumption 3 holds with \(i^* = k + n - 1\), the linear system
of all these equations is solved by equation (34), which gives the unknown \( z \) as a function of known quantities. Now, by using again equations (35) and (36), at the impact \( k + n - 1 \), it is possible to write the error \( \tilde{x}(t_{k+n-1}) \) as a function of \( z \), of \( (\Delta q_k, \Delta v_i) \) for \( i = 0, \ldots, n-1 \), and of other known quantities. By requiring that \( \tilde{x}(t_{k+n-1}) = 0 \), it is possible to derive the values to be assigned to the jumps \( \Delta q_k \) and \( \Delta v_i \), which are given by equation (33).

If no disturbances act on the system, it can be verified that the estimation error remains at 0 for all \( t > t_1 \) if, on the contrary, some disturbance affects the system at some given time, the null error estimate will be recovered at the (n - 1)-th impact after the disturbance has vanished (if at such an impact Assumption 3 holds, or at the first impact after the (n - 1)-th where Assumption 3 holds).

**Remark 5.** Notice that the assumption that the impact times have a accumulation point at +∞ is needed only if, apart from the dead-beat convergence property of the estimation error stated in Theorem 5, also the capability of recovering from the effects of disturbances is desired. This will be clarified by the simulation example reported in Section 4.4, which is related to an observer wholly similar to the one described by (32), (33) and (34). \( \square \)

4 A mass impacting against a multi-degrees-of-freedom body in free motion (the case \( n_a = 1 \) and \( n_b = n - 1 \))

Consider the mechanical system of Chapter 2, with \( n_a = 1 \) and \( n_b \geq 1 \), that is the case of an actuated mass impacting against a multi-degree-of-freedom body in free motion (i.e., no external forces act on the second body). The mechanical system considered in this section is then constituted by \( n = 1 + n_b \) masses, that is the first mass is constrained to be on the left of the second one, whereas the other masses are not constrained at all.

Equations (4b)-(4c) take the following form:

\[
\begin{bmatrix}
v_1(t_i^-) \\
v_2(t_i^-)
\end{bmatrix} =
\begin{bmatrix}
M_1 \\
M_2
\end{bmatrix}
\begin{bmatrix}
v_1(t_i^+) \\
v_2(t_i^+)
\end{bmatrix}.
\]

(39)

4.1 Observability

According to Definitions 1 and 2, the following theorem gives a sufficient condition for the observability of the considered mechanical system in the \((1, n - 1)\) case.

**Theorem 6.** The mechanical system (2), (3) and (39) is observable if \( E \neq 0 \) and the pair \((C_b, A_b)\) is observable, where \( C_b \in \mathbb{R}^{1 \times n_b}, C_b := [1 \ 0 \ \ldots \ 0] \).

**Proof.** First of all, observe that the sub-system constituted by the first mass is fully observable and reachable (since \( E \neq 0 \)) for any initial state \( x(t_0) \in A \), it is straightforward to construct an input function \( \bar{u}(\cdot) \) such that there exists at least a non-degenerate impact time \( t_1 > t_0 \).

Now, let \( x_1, x_2 \in A \), with \( x_1 \neq x_2 \), be two arbitrary states. Let the input function \( \bar{u}(\cdot) \) be computed taking \( x_1 \) as initial state. Let \( t_{1,1} \) be the first impact time corresponding to \( \bar{u}(\cdot) \) when the initial state is \( x_1 \) (such an impact time certainly exists by construction), and let \( t_{1,2} \) be the first impact time (when it exists) corresponding to \( \bar{u}(\cdot) \) when the initial state is \( x_2 \), or otherwise let \( t_{1,2} = +\infty \). Let \( t_1 := \min(t_{1,1}, t_{1,2}) \). Denoting by \( T > t_1 \) a time such that there are no impacts in \((t_1, T)\) corresponding to \( \bar{u}(\cdot) \) starting from both \( x_1 \) and \( x_2 \), it is now shown that the two states are distinguishable in the interval \([t_0, T]\), that is

\[
y(\cdot, x_1, \bar{u}(\cdot))|_{[t_0, T]} \neq y(\cdot, x_2, \bar{u}(\cdot))|_{[t_0, T]},
\]

where the symbol \( \varphi(\cdot)|_{[t_i, t_f]} \) denotes the function \( \varphi(\cdot) \) restricted to the interval \([t_i, t_f]\) at subscript.

Let \( x_1 = \begin{bmatrix} x_{a,1} \\
x_{b,1} \end{bmatrix} \) and \( x_2 = \begin{bmatrix} x_{a,2} \\
x_{b,2} \end{bmatrix} \) and denote by \( q_{h,j}(t) \), \( v_{h,j}(t) \), \( h = 1, 2, \ldots, n \), the time behavior
of the mechanical system when the initial condition is \( x_j, j = 1, 2 \); if \( x_{a,1} \neq x_{a,2} \), then, since pair \((C_a, A_a)\) is observable, taking into account that there are no impacts in the interval \([t_0, t_1]\), the two states \( x_1 \) and \( x_2 \) are distinguishable in the interval \([t_0, t_1]\). Hence, only the case \( x_{a,1} = x_{a,2}, x_{b,1} \neq x_{b,2} \) will be considered next.

First, suppose that \( t_1 = t_{1,1} \) and \( t_{1,2} > t_{1,1} \). From the restitution rule at time \( t_1 \), if the initial state is \( x_1 \) (whence, \( t_1 \) is a non degenerate impact time), one has:

\[
\begin{bmatrix}
q_{1,1}(t_1^+) \\
v_{1,1}(t_1^+)
\end{bmatrix} = M_1 \begin{bmatrix}
q_{1,1}(t_1^-) \\
v_{1,1}(t_1^-)
\end{bmatrix},
\]

(with \( M_1 = \begin{bmatrix}
m_1-m_2 & 2m_2 \\
m_1+m_2 & m_1+m_2
\end{bmatrix} \)) whereas, if the initial state is \( x_2 \) (whence, \( t_1 \) is not an impact time), then

\[
\begin{bmatrix}
q_{1,2}(t_1^+) \\
v_{1,2}(t_1^+)
\end{bmatrix} = \begin{bmatrix}
q_{1,1}(t_1^-) \\
v_{1,1}(t_1^-)
\end{bmatrix}.
\]

Since \( t_1 \) is a non degenerate impact time when the initial state is \( x_1, v_{1,1}(t_1^+) \neq v_{1,1}(t_1^-) \), which implies \( v_{1,1}(t_1^+) \neq v_{1,1}(t_1^-) \). This, in view of the observability of the pair \((C_a, A_a)\), implies \( y(\cdot, x_1, \hat{u}(\cdot))|_{[t_1, T]} \neq y(\cdot, x_2, \hat{u}(\cdot))|_{[t_1, T]} \).

If \( t_1 = t_{1,2} \) and \( t_{1,1} > t_1 \), it is possible to repeat a reasoning similar to the one carried out in the case \( t_1 = t_{1,1} \) and \( t_{1,2} > t_1 \).

Assume now \( t_{1,1} = t_{1,2} \). It is easy to see, for the continuity of the motion of the first mass, that by modifying the input function \( \bar{u}(\cdot) \), with initial condition \( x_1 \), it is possible to get as time impact a time very close to \( t_{1,1} \). If \( t_{1,1} \neq t_{1,2} \), it is possible to proceed as above. Assume, by contradiction, that by modifying the input function it will always be \( t_{1,1} = t_{1,2} \), with \( t_{1,1} \) ranging with continuity in an interval \([t_a, t_b]\). We are using the first mass as a probe for measuring the position of the second mass, and use the input function to evaluate this position on a finite interval. If \( t_{1,1} = t_{1,2} \) with \( t_{1,1} \in [t_a, t_b] \), this means that, both under \( x_1 \) and \( x_2 \), the position of the second mass is the same in the interval \([t_a, t_b]\), whence, for the observability of the pair \((C_b, A_b)\), this implies \( x_{b,1} = x_{b,2} \), against the assumption.

### 4.2 Reconstructibility

According to Definition 3, it will be proven, as a corollary of the theorem provided in the next section, that, if the pair \((C_b, A_b)\) is observable, the mechanical system considered in this section is reconstructable under some conditions on the input function, regarding the presence of a sufficient number of impacts in the reconstruction interval.

### 4.3 Dead-beat observer design

Since the position of the first body can be measured at any time \( t \geq t_0 \), and the pair \((C_a, A_a)\) is observable, the estimation of \( x_a \) can be accomplished through a classical Luenberger observer. Let \( \hat{x}_a(t) \) be the estimate of \( x_a(t) \) at time \( t \) and \( \hat{x}_a(t) = x_a(t) - \hat{x}_a(t) \) be the estimation error. Then, it remains to design an observer for the state variables \( x_b(t) \) of the second subsystem. Let \( \hat{x}_b(t) \) be the estimate of \( x_b(t) \), \( \hat{x}_b(t) = \hat{q}_2(t) \ldots \hat{q}_n(t) \hat{v}_2(t) \ldots \hat{v}_n(t)^T \), and let \( \hat{x}_b(t) = x_b(t) - \hat{x}_b(t) \) denote the estimation error. In the following, the “reordered” estimation error defined as

\[
e(t) := [\hat{q}_1(t) \hat{q}_2(t) \ldots \hat{q}_n(t) \hat{v}_1(t) \hat{v}_2(t) \ldots \hat{v}_n(t)]^T,
\]

will be used, which is related to \( [\hat{x}_a(t)^T \hat{x}_b(t)^T]^T \) by a permutation matrix \( P \), \( e(t) = P \begin{bmatrix} \hat{x}_a(t) \\ \hat{x}_b(t) \end{bmatrix} \).
Let $\delta_i := t_i - t_{i-1}$, $i \geq 1$. For each $k \geq 1$, define $A_{D,k} := Pe^{A_G \delta_k} P^{-1}$, where 

\[
A_G := \begin{bmatrix}
A_a & GC_a & 0 \\
0 & A_b & 0
\end{bmatrix},
\]

with $A_a - GC_a$ Hurwitz. Partition $A_{D,k}$ as follows:

\[
A_{D,k} = \begin{bmatrix}
\ast \\ * & A_k
\end{bmatrix}, \quad \Gamma_k \in \mathbb{R}^{2 \times 2(n-1)}, \quad A_k \in \mathbb{R}^{(n-1) \times 2(n-1)}.
\]

The proposed observer is described by:

\[
\begin{align*}
\dot{x}_a(t) &= A_a \dot{x}_a(t) + B_a u(t) + G (y(t) - C_a \dot{x}_a(t)), \quad t \in (t_i, t_{i+1}), \\
\dot{x}_b(t) &= A_b \dot{x}_b(t), \quad t \in (t_i, t_{i+1}), \\
\dot{q}(t_i^+) &= \begin{bmatrix} y(t_i) \\ y(t_i) \\ \vdots \\ \dot{q}_n(t_i^-) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \Delta v_i, \quad i \in \mathbb{N}, 
\end{bmatrix},
\end{align*}
\]

\[
\dot{\hat{v}}(t_i^+) = \begin{bmatrix} M_{1,1} \dot{v}_1(t_i^-) + M_{1,2} \dot{v}_2(t_i^-) \\ M_{2,1} \dot{v}_1(t_i^-) + M_{2,2} \dot{v}_2(t_i^-) \\ \vdots \\ \dot{\hat{v}}_n(t_i^-) \end{bmatrix} + \Delta v_i, \quad i \in \mathbb{N},
\]

where $M_{1,1} = \frac{m_1 - m_2}{m_1 + m_2}$, $M_{1,2} = \frac{2m_2}{m_1 + m_2}$, $M_{2,1} = \frac{2m_1}{m_1 + m_2}$, $M_{2,2} = \frac{m_2 - m_1}{m_1 + m_2}$ and the jumps $\Delta q_i \in \mathbb{R}^{n-2}$ and $\Delta v_i \in \mathbb{R}^n$ are defined as the solution of the following difference equation from an arbitrary initial condition $(\Delta q_{-(n-2)}, \Delta v_{-(n-2)}, \ldots, \Delta q_1, \Delta v_1, \Delta q_0, \Delta v_0)$:

\[
\begin{bmatrix}
\Delta q_{k+1-n} \\
\Delta v_{k+1-n}
\end{bmatrix} = P_0 A_k A_{k+1-n} \cdots P_0 A_{k+1} \begin{bmatrix}
\Delta q_k \\
\Delta v_k
\end{bmatrix} + \begin{bmatrix}
P_0 z - \begin{bmatrix}
\Delta q_k \\
\Delta v_k
\end{bmatrix} \\
- P_0 A_{k+1-n} P_0 A_{k+2-n} \cdots P_0 A_{k+2} \begin{bmatrix}
\Delta q_{k+1} \\
\Delta v_{k+1}
\end{bmatrix} \cdots - P_0 A_{k+1} \begin{bmatrix}
\Delta q_{k+n-2} \\
\Delta v_{k+n-2}
\end{bmatrix},
\]

where $P_0 \in \mathbb{R}^{2(n-1) \times 2(n-1)}$, $P_0 := \begin{bmatrix}
I_{n-2} & 0 & 0 \\
0 & M & 0 \\
0 & 0 & I_{n-2}
\end{bmatrix}$, and the vector $z \in \mathbb{R}^{2(n-1)}$ is given below as a function of the following quantities (all known at time $t_i^+ = t_{k+n-1}^+$, if $i \geq n$, or otherwise to be set to arbitrary initial values):

\[
\begin{bmatrix}
\delta_{k+1} \\
\vdots \\
\delta_{k+n-1} \\
\Delta q_k \\
\Delta v_k \\
\Delta q_{k+2} \\
\Delta v_{k+2} \\
\ldots \\
\Delta q_{k+n-2} \\
\Delta v_{k+n-2} \\
y(t_{k+1}) \\
y(t_{k+2}) \\
\ldots \\
y(t_{k+n-1}) \\
\Gamma_{k+1} \\
\Gamma_{k+2} \\
\Gamma_{k+3} \\
\ldots \\
\Gamma_{k+n-1} \\
A_{k+1} \\
A_{k+2} \\
\ldots \\
A_{k+n-1}
\end{bmatrix},
\]

Define the matrix $\Xi_{k+n-1} \in \mathbb{R}^{2(n-1) \times 2(n-1)}$ as follows

\[
\Xi_{k+n-1} := \begin{bmatrix}
\Gamma_{k+1} \\
\Gamma_{k+2} P_0 A_{k+1} \\
\Gamma_{k+3} P_0 A_{k+2} P_0 A_{k+1} \\
\vdots \\
\Gamma_{k+n-1} P_0 A_{k+n-2} P_0 \ldots P_0 A_{k+1}
\end{bmatrix},
\]

where $\Gamma_k := \begin{bmatrix} \Gamma_k^- \\ \Gamma_k^+ \end{bmatrix}$.
and notice that the knowledge of $\delta_i$ in computing matrix $\Xi_i$ is needed only for the computation of the last two rows, so that the matrix $(\Xi_{i+1})_{2(n-2)} \in \mathbb{R}^{2(n-2) \times 2(n-1)}$ constituted by the first $2(n-2)$ rows of $\Xi_{i+1}$ can be already computed at time $t_i^-$. For a similar reason, for any $j = 1, \ldots, n-2$, the matrix $(\Xi_{i+j})_{1:2(n-1-j)} \in \mathbb{R}^{2(n-1-j) \times 2(n-1)}$ constituted by the first $2(n-1-j)$ rows of $\Xi_{i+j}$ can be already computed at time $t_i^+$. Finally, the description of the observer is completed by the following formula, which describes the vector $W := P_hz$ to be computed at time $t_{k+n-1}^+$:

$$W = \begin{cases} \Xi_{i+1}^{-1} & \begin{bmatrix} y(\tau_{k+1}) - \dot{q}_1(\tau_{k+1}) \\ y(\tau_{k+1}) - \dot{q}_2(\tau_{k+1}) \\ \vdots \\ y(\tau_{k+n-1}) - \dot{q}_1(\tau_{k+n-1}) \\ y(\tau_{k+n-1}) - \dot{q}_2(\tau_{k+n-1}) \end{bmatrix} + \\
\sum_{j=1}^{n-2} \begin{bmatrix} 0_{2j \times (n-1-j)} \\ (\Xi_{i+n-1-j})_{1:2(n-1-j)} \end{bmatrix} \begin{bmatrix} \Delta q_{k+j} \\ \Delta v_{k+j} \end{bmatrix} \end{cases} \quad (42)$$

Assumption 4 The impact times have one and only one accumulation point at $+\infty$, and there exists an impact time $t_{i^*}$, with $i^* \geq n$, such that the matrix $\Xi_{i^*}$, computed on the basis of $\delta_{i^*}, \delta_{i^*-1}, \ldots, \delta_{i^*-(n-1)}$, is nonsingular.

Theorem 7. Assume that the pair $(C_k, A_k)$ is observable. Under Assumption 4, for any initial condition $z(t_0) \in A$ of system (2), (3) and (39), and for any initial condition $[\tilde{x}_a(t_0) \tilde{x}_b(t_0)] \in \mathbb{R}^{2n}$ of the observer (40), (41) and (42), the following dead-beat property is guaranteed for the estimation errors:

$$\begin{bmatrix} \tilde{x}_a(t) \\ \tilde{x}_b(t) \end{bmatrix} = 0, \quad \forall t > t_{i^*}.$$ 

Proof. Let $z := [\dot{q}_1(t_k^-) \ldots \dot{q}_n(t_k^-) \dot{v}_1(t_k^-) \dot{v}_2(t_k^-) \ldots \dot{v}_{n-2}(t_k^-)]^T$ be a $2(n-2)$ dimensional vector such that, immediately before the $k$-th impact, i.e. at time $t_k^-$ it is possible to write $e(t_k^-) = [z]_{[**z^T]}^T$. Assume the observer is applied starting from a generic time $t_k$. At each impact time it is possible to write two scalar equations in the unknown $z$ in such a way that at time $t_{k+n-1}^-$, i.e., after $n-1$ impacts, if Assumption 4 holds with $i^* = k + n - 1$, it is possible to compute $z$ (equation (42)) and use it to set the jumps such that the subsequent estimation errors become 0 (equation (41)).

First of all observe that

$$e(t_k^+) = \begin{bmatrix} 0 \\ 0 \\ P_hz \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \Delta q_k \\ \Delta v_k \end{bmatrix}.$$ 

Hence, at time $t_{k+1}^+$, i.e., immediately before the impact $k+1$, the estimation error can be expressed as:

$$e(t_{k+1}^-) = \left[ \begin{bmatrix} \Gamma_{k+1} \\ A_{k+1} \end{bmatrix} \right] \begin{bmatrix} 0 \\ P_hz \end{bmatrix} - \begin{bmatrix} \Delta q_k \\ \Delta v_k \end{bmatrix}.$$ 

Now, the first two elements of $e(t_{k+1}^-)$ are known and given by $[y(t_{k+1}) - \dot{q}_1(t_{k+1}^-) \ y(t_{k+1}) - \dot{q}_2(t_{k+1}^-)]^T$, since at each impact time both $q_1(t_{k+1})$ and $q_2(t_{k+1})$ are known and coincident with the measured output $y(t_{k+1})$. Hence, it is possible to write at the first impact after time $t_k$ the two following scalar equations:

$$\Gamma_{k+1} \begin{bmatrix} 0 \\ P_hz \end{bmatrix} - \begin{bmatrix} \Delta q_k \\ \Delta v_k \end{bmatrix} = \begin{bmatrix} y(t_{k+1}) - \dot{q}_1(t_{k+1}^-) \\ y(t_{k+1}) - \dot{q}_2(t_{k+1}^-) \end{bmatrix}.$$ 

(43)
Doing the same for every impact \( k + j \), with \( j = 1, 2, \ldots, n - 1 \), it is possible to express \( e(t_{k+j}^-) \) as a function of \( z \) and of the jumps \([\Delta q_{k+i}, \Delta v_{k+i}]^T, i = 1, 2, \ldots, j - 1\):

\[
e(t_{k+j}^-) = \left[ \begin{array}{c} \Gamma_{k+j}^+ \\
A_{k+j}^+ 
\end{array} \right] e(t_{k+j-1}^+),
\]

with

\[
e(t_{k+j-1}^+) = \begin{bmatrix}
0_{2 \times 2(n-1)} & 0_{2 \times 2(n-1)} \\
P_0 & 0
\end{bmatrix} e(t_{k+j-1}^-) - \begin{bmatrix}
0 \\
0
\end{bmatrix} \begin{bmatrix}
\Delta q_{k+j-1}^- \\
\Delta v_{k+j-1}^-
\end{bmatrix}.
\]

By substituting \( e(t_{k+j-1}^-) \) as a function of \( z \) computed at the previous iteration, at the impact \( k + j \) two more equations in the unknown \( z \) can be considered:

\[
\begin{aligned}
\Gamma_{k+j}^+ P_0 A_{k+j-1} \cdots P_0 A_{k+1} \left( P_0 z - \begin{bmatrix}
\Delta q_k \\
\Delta v_k
\end{bmatrix} \right) &- \Gamma_{k+j}^+ P_0 A_{k+j-1} \cdots P_0 A_{k+2} \begin{bmatrix}
\Delta q_{k+1} \\
\Delta v_{k+1}
\end{bmatrix} \\
- \Gamma_{k+j}^+ P_0 A_{k+j-1} \cdots P_0 A_{k+3} \begin{bmatrix}
\Delta q_{k+2} \\
\Delta v_{k+2}
\end{bmatrix} &- \cdots - \Gamma_{k+j}^+ \begin{bmatrix}
\Delta q_{k+j-1} \\
\Delta v_{k+j-1}
\end{bmatrix} = \begin{bmatrix}
g (t_{k+j}) - \hat{q}_1^\infty (t_{k+j}) \\
g (t_{k+j}) - \hat{q}_2^\infty (t_{k+j})
\end{bmatrix}.
\end{aligned}
\]

Doing this for all the impact times \( t_{k+j}, j = 1, 2, \ldots, n - 1, 2(n-1) \) equations in the \( 2(n-1) \) unknown elements of \( z \) are obtained. If Assumption 4 holds with \( \hat{\sigma} = k + n - 1 \), the system of all these equations is solved by equation (42), where the solution is given as a vector \( W = P_0 z \). Now, at the impact \( k + n - 1 \), it is possible to write

\[
e(t_{k+j+n-1}^+) = \begin{bmatrix}
0_{1 \times 2(n-1)} \\
P_0 A_{k+n-1} \cdots P_0 A_{k+1} \\
0_{1 \times 2(n-1)} \\
P_0 A_{k+n-1} \cdots P_0 A_{k+2}
\end{bmatrix} \begin{bmatrix}
\Delta q_k \\
\Delta v_k \\
\Delta q_{k+1} \\
\Delta v_{k+1}
\end{bmatrix} - \begin{bmatrix}
0 \\
0 \\
\Delta q_{k+n-1} \\
\Delta v_{k+n-1}
\end{bmatrix}.
\]

Since \( W = P_0 z \) has been just derived and all the previous jumps \([\Delta q_{k+i}, \Delta v_{k+i}] \) are known, it is possible to set the jumps \([\Delta q_{k+n-1}, \Delta v_{k+n-1}] \) in the previous equation to have \( e(t_{k+n-1}^-) = 0 \). This is performed in equation (41). If no disturbances act on the system, it is straightforward to verify that the estimation error remains at 0 for all \( t > t_{k+n-1} \), with \( \Delta q_i = \Delta v_i = 0 \) for all \( i > k + n - 1 \). On the contrary, if some disturbance affects the system at some given time, the null error estimate will be recovered at the \((n - 1)\)-th impact after the disturbance has vanished (if at such an impact Assumption 4 holds, otherwise at the first impact after the \((n - 1)\)-th where the Assumption holds).

**Remark 6.** Notice that the assumption that the impact times have an accumulation point at \(+\infty\) is needed only if, apart from the dead-beat convergence property of the estimation error stated in Theorem 7, also the capability of recovering from the effects of disturbances is desired, as shown in the application to follow. □

As a consequence of the dead-beat property of the observer realized above, it is possible to reconstruct the state of system (2), (3) and (39) in any time interval \([t_0, T_f]\), provided that some assumptions are satisfied. The following corollary formally states such a property, which was anticipated in Chapter 4.2.

**Corollary 1.** Let \( T_f > t_0 \) and assume that the pair \((C_0, A_0)\) is observable. System (2), (3) and (39) is reconstructible in the interval \([t_0, T_f]\) with respect to all the input functions \( u(t) \) such that there are at least \( n \) impact times in \([t_0, T_f]\) and Assumption 4 holds for at least one of the impacts after the \((n - 1)\)-th.
4.4 An application to a 4 masses (1,3) system.

Consider a 4 masses system, composed by two decoupled subsystems (see Figure 4). The first one comprises a single body whose position can be measured at any time. The second one comprises three masses coupled each other by 3 springs: the elastic constant is $k_{ij}$ for the spring coupling body $i$ with body $j$. The positions and the velocities of the masses of the second subsystem cannot be directly measured and can be estimated thanks to the impacts between the masses $m_1$ and $m_2$, which are the only ones that may impact.

![Fig. 4: The system considered in Chapter 4.4.](image)

The results of a simulation including the observer proposed in this section are reported in Figures 5 and 6. The considered system is characterized by the following parameters: $m_1 = 1kg$, $m_2 = 2kg$, $m_3 = 0.5kg$, $m_4 = 2kg$, $k_{23} = 2Nm^{-1}$, $k_{24} = 3Nm^{-1}$ and $k_{34} = 1Nm^{-1}$ and by the following initial conditions: $q_1(0) = 0m$, $q_2(0) = 0.5m$, $q_3(0) = 0m$, $q_4(0) = 1m v_1(0) = 0ms^{-1}$, $v_2(0) = 2ms^{-1}$, $v_3(0) = 0ms^{-1}$, $v_4(0) = 0ms^{-1}$. A non measured (and therefore not taken into account in the observer) constant disturbance force $d(t) = -2N$ acts on the third body in the time interval (40, 50). Notice that the observer is able to bring again the estimation errors to zero when such a disturbance ceases, actually after time $t \approx 62s$. The simulation has been carried out by considering an input $u(t) = 1N$ for all $t$.

5 Conclusions

A class of mechanical systems, constituted by two composite bodies colliding with each other, has been considered. Since only one position variable of the first body is measurable, such systems would be unobservable in the absence of impacts between them. Each body has been schematically represented as constituted by several masses connected by linear springs, so that the internal deformations due to the impacts are taken into account. Two extreme cases have been dealt with, namely the $(n-1,1)$ case and the $(1,n-1)$ case. For each of the two cases, sufficient conditions for observability and reconstrucibility of the whole system have been provided. An asymptotic observer has been proposed for the $(n-1,1)$ case, which (if there is an infinite sequence of impacts) is able to asymptotically estimate all the non-measurable state variables, including those of the second mass that would be unobservable in absence of impacts. Moreover, for both the $(n-1,1)$ case and the $(1,n-1)$ case, a dead-beat observer has been proposed, which guarantees dead-beat convergence to zero of the estimation error.

Simulation tests show the effectiveness of the proposed algorithms.

It is interesting to compare the two kinds of observers (asymptotic or dead-beat) from the point of view of robustness against parametric uncertainties. Notice, first, that, even for a simple
All the observers proposed in the report (asymptotic or dead-beat) produce estimation errors $\varepsilon$ small parameter variations, in the case of bounded $(\varepsilon)$ that are continuous with respect to parameter uncertainties (and, therefore, remain small for bounded observers is that the estimation errors remain “small” for small parameters variations, in presence of the loop to stabilize the system). Then, the most that seems reasonable to require for the proposed observer, in some cases (in which the impacting masses are either equal, or exactly known), it is possible to obtain smaller estimation errors (at least for some variables) for large times by decreasing the value of $\varepsilon$, at the expense of a worse transient behavior.

Further work will be devoted to study the applicability of the proposed observers for the control of the considered systems, i.e., to the proof of a sort of “separation property”.

Fig. 5: Time behavior of the position and velocity variables (continuous lines) and of their estimates (dashed lines) for the simulation described in Chapter 4.4. In the lower left-side plot the difference $f_{1,2}(t) = q_1(t) - q_2(t)$ is reported, in order to emphasize the impacts.

Fig. 6: Time behavior of the estimation errors for the simulation described in Chapter 4.4.
References

3

Tracking control of complementarity Lagrangian systems*

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Abstract. In this paper we study the tracking control of Lagrangian systems subject to frictionless unilateral constraints. The stability analysis incorporates the hybrid and nonsmooth dynamical feature of the overall system. The difference between tracking control for unconstrained systems and unilaterally constrained ones, is explained in terms of closed-loop desired trajectories and control signals. This work provides details on the conditions of existence of controllers which guarantee stability. It is shown that the design of a suitable transition phase desired trajectory, is a crucial step. Some simulation results provide information on the robustness aspects. Finally the extension towards the case of multiple impacts, is considered.

1 Introduction

The focus of this paper is the tracking control of a class of nonsmooth fully actuated Lagrangian systems subject to frictionless unilateral constraints on the position. Let \( X \in \mathbb{R}^n \) denote the vector of generalized coordinates. Roughly speaking, trajectory tracking means that when properly initialized, all trajectories \( X(\cdot) \) have to converge, or remain close to, some desired trajectory \( X_d(\cdot) \) which is designed off-line. The Lyapunov stability of the fixed point of the transformed error system with state vector the tracking error \( (X - X_d, \dot{X} - \dot{X}_d) \) is often required to get a robust and implementable scheme. The stabilisation problem consists of choosing \( X_d \) constant. For nonlinear mechanical systems, tracking is known to be significantly more difficult than stabilisation, even for unconstrained systems [Lozano et al.(2000)Lozano, Brogliato, Egeland & Maschke]. The stabilisation problem for a class of nonsmooth systems, including Lagrangian systems with unilateral constraints, has been analysed in [Brogliato(2003a)] [Goeleven et al.(2003)Goeleven, Motreanu & Motreanu]. Applications may be found in manipulators performing tasks such as grinding, deburring [Komanduri(1993)] [Ramachandran et al.(1994)Ramachandran, Pande & Ramakrishnan], filamentary brushing tools for surface finishing [Shia et al.(1998)Shia, Stango & Heinrich], which have considerable importance in machining, disassembly robotic systems [Studny et al.(1999)Studny, Rittel & Zussman], etc, and more generally all mechanical systems performing tasks involving contact/impact phenomena.

The nonsmooth complementarity systems we deal with in this paper, may a priori evolve in three different phases of motion :

- **i)** A free motion phase, where the mechanical system is not subject to any constraints (i.e. \( F(X) > 0 \), where \( F(\cdot) \) is some \((m\text{-vector}) \) function representing the “distance” between the system and the constraint).
- **ii)** A permanently constraint phase where the dynamical system is subject to holonomic constraints \( (F_i(X) = 0 \) during a non-zero time interval and for some indexes \( i \in \{1, \cdots, m\} \)).
- **iii)** A transition phase whose goal is to stabilize the system on some surface \( \Sigma_T = \cap_{i \in I} \Sigma_i \), where \( I \) is some subset of \( \{1, \cdots, m\} \) and \( \Sigma_i = \{X|F_i(x) = 0\} \). In other words a transition

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control has to assure that $F_i(X(t)) = 0$ and $\nabla F_i(t))X(t^+) = 0$ for all $i \in \mathcal{I}$, where $t$ is a finite time for obvious practical reasons.

In the first phase the system is described by a set of ordinary differential equations (ODE). The tracking control problem has been solved by several feedback controllers assuring the global asymptotic stability (feedback linearization, adaptive control, robust control, passivity-based control, etc [Lozano et al.(2000)Lozano, Brogliato, Egeland & Maschke]). The second phase concerns the control of a differential-algebraic equation (DAE) by so-called force/position controllers, and has been solved in [McClamroch & Wang(1988)] and [Yoshikawa(1987)]. It reduces to a motion control problem plus an algebraic equality for contact force equilibrium when suitable coordinates are chosen. During the transition phase the system is subject to unilateral constraints, and collisions occur. These collisions will generate rebounds, which are generally seen as disturbances. On the contrary, during the transition phase the system is subject to unilateral constraints, and collisions occur.

The aim of this paper is to study a control scheme which guarantees some stability properties of the closed-loop system during general motions involving the three above phases. It provides an interpretation of the specific feature of tracking control for unilateral ly constrained systems in terms of some invariant closed-loop trajectories and some signals entering the control input (usually known as the desired trajectory). With respect to the results in [Brogliato et al.(1997)Brogliato, Niculescu & Orhant] and [Brogliato et al.(2000)Brogliato, Niculescu & Monteiro-Marques] we give accurate conditions under which various types of stability are assured, which were missing in these references. For instance the $n$-degree-of-freedom case with $n \geq 2$ is solved in [Brogliato et al.(1997)Brogliato, Niculescu & Orhant] only if a certain matrix is a Jacobian, which is quite restrictive as simple examples show [Brogliato(1999), §8.6]. In [Brogliato et al.(2000)Brogliato, Niculescu & Monteiro-Marques] the existence of a specific transition phase closed-loop trajectory is assumed, without proof. These two points are addressed in this paper, as well as the transition between permanent constraint phases and free-motion phases. We also study the robustness of this control scheme with respect to the knowledge of constraints position.

Finally we extend this work to the case of nonscalar frictionless unilateral constraints, which may generate so-called multiple impacts.

Glossary:

ODE: Ordinary Differential Equation, DAE: Differential Algebraic Equation, LCP: Linear Complementarity Problem, DES: Discrete Event System.

For a $m$-vector $X$, $X \geq 0$ means that $X_i \geq 0$ for all components of $X$, $1 \leq i \leq m$. The maximum and minimum eigenvalues of a matrix $M$ are denoted as $\lambda_{\text{max}}(M)$ and $\lambda_{\text{min}}(M)$ respectively. If a function $F(.)$ has a simple discontinuity at $t$, the right and left-limits are denoted as $F(t^+)$ and $F(t^-)$ respectively. The jump is denoted as $\sigma_F(t) = F(t^+) - F(t^-)$. The Lebesgue measure of an interval $[a, b]$ is denoted by $\lambda[a, b]$.

1.1 Dynamics

The systems we study in this paper belong to the complementarity hybrid dynamical systems [van der Schaft & Schumacher(2000)], a class of systems which generalizes that of nonsmooth mechanical systems [Moreau(1983)]. They are complementarity Lagrangian systems, with Lagrangian function $L = \frac{1}{2}X^TM(X)X - U(X)$, where $T(X, X) = \frac{1}{2}X^TM(X)X$ is the kinetic energy, $U(X)$ is the differentiable potential energy. The dynamics may be written as:

\[
\begin{align*}
M(X)\ddot{X} + C(X, \dot{X})\dot{X} + G(X) &= u + \nabla F(X)\lambda_X \\
F(X) \geq 0, \quad F(X)^T\lambda_X &= 0, \quad \lambda_X \geq 0 \\
\text{Collision rule}
\end{align*}
\] (1)

3 The reason why the right limit of the velocity is indicated will be made clear later when solutions are given a precise meaning.
where $X \in \mathbb{R}^n$ is a vector of generalized coordinates, $M(X) = M^T(X) \in \mathbb{R}^{n \times n}$ is the positive definite inertia matrix, $F(X) \in \mathbb{R}^m$ represent the distance to the constraints, $\lambda_X \in \mathbb{R}^m$ are the Lagrangian multipliers associated to the constraints, $u \in \mathbb{R}^n$ is the vector of generalized torque inputs, $C(X, \dot{X})$ is the matrix of Coriolis and centripetal forces, $G(X)$ contains conservative forces. $\nabla$ denotes the Euclidean gradient, i.e. $\nabla F_i(X) = (\frac{\partial F_i}{\partial x_1}, \ldots, \frac{\partial F_i}{\partial x_n})^T \in \mathbb{R}^n$ and $\nabla F(X) = (\nabla F_1(X), \ldots, \nabla F_m(X)) \in \mathbb{R}^{m \times n}$. The impact times will be denoted generically as $t_k$ in the following. We assume that the functions $F_i(\cdot)$ are continuously differentiable and that $\nabla F_i(X(t_k)) \neq 0$ for all $t_k$.

A major discrepancy of complementarity systems compared to systems with switching vector fields, is that their state may be discontinuous, and that they may live on lower-dimensional spaces. This creates serious difficulties in their study [Brogliato(2003b)] [Heemels & Brogliato(2003)].

The Lagrangian system in (1) is fully actuated, i.e. $\dim(u) = \dim(X)$. This excludes for instance lumped joint flexibilities. In case $\dim(u) < \dim(X)$ the system is said to be underactuated and the control problem is much harder to solve. The first instance in the Control and Robotics literature where such a complementarity model has been used, is in [Huang & McClamroch(1988)]. One very specific feature of systems as in (1) is their intrinsic nonsmoothness, which hampers one to tangentially linearize them in the neighborhood of trajectories. Consequently linear controllers generally fail to stabilise such complementarity systems, and nonlinear feedback controllers have to be designed.

### 1.2 Admissible domain

The admissible domain $\Phi$ is a closed domain in the configuration space where the system can evolve, i.e.

$$\Phi = \{ X | F(X) \geq 0 \} = \bigcap_{1 \leq i \leq m} \Phi_i, \quad \Phi_i = \{ X | F_i(X) \geq 0 \}$$

For obvious reasons it is assumed that $\Phi \neq \emptyset$, and even more: it contains a closed ball of positive radius. This allows us to get rid of meaningless models. A motion like the one in items i), ii), iii) above can then be defined. The boundary of $\Phi$ is denoted as $\partial \Phi$.

**Definition 1.** A singularity of $\partial \Phi$ is the intersection of two (or more) surfaces $\Sigma_i = \{ X | F_i(X) = 0 \}$.

As alluded to above, the goal of the control problem during transition phases is to stabilise the system on the boundary $\partial \Phi$. When $m \geq 2$ this may be a singularity (i.e. a codimension $\alpha \geq 2$ surface) of the boundary. In this study we restrict ourselves to domains which have non-differentiable boundaries but which are convex around such non-differentiable points (like on Fig.
1.a). The unilateral constraints are expressed by the relation $F(X) \geq 0$, which can be translated locally into: $CX + D \geq 0$ for some matrices $C$ and $D$. Clearly the non-convex example of Fig. 1.(b) cannot be expressed as the intersection of convex domains $\Phi_i$. This case is named a reintrant corner in the literature, and modelling issues are not yet fixed for reintrant corners [Glocker(2001)] [Frémond(2002)]. This restriction on singular non-convex points does not mean that the whole space must be convex. For example the domain of the Fig. 2 is non-convex but can be described as $\Phi$ above. Such sets are called regular [Clarke(1990)]. For regular sets convexity holds locally and can be recovered by a suitable generalized coordinates change (diffeomorphic hence preserving the Lagrangian structure).

Fig. 2: Example of a regular non-convex domain

1.3 Impact model

A collision rule is needed to integrate the system in (1) and to render the set $\Phi$ invariant. A collision rule is a relation between the post-impact velocities and the pre-impact velocities. In this work, it is chosen as in [Moreau(1988)]:

$$
\begin{align*}
\dot{X}(t^+_k) &= -e_n \dot{X}(t^-_k) \\
+ (1 + e_n) \arg \min_{z \in T\Phi(X(t_k))} \frac{1}{2} [z - \dot{X}(t^-_k)]^T M(X(t_k)) [z - \dot{X}(t^-_k)] 
\end{align*}
$$

(2)

where $\dot{X}(t^+_k)$ is the post impact velocity, $\dot{X}(t^-_k)$ is the pre-impact velocity, $T\Phi(X(t))$ the tangent cone to the set $\Phi$ at $X(t)$ (see Figs. 1-2 where the sets $X, T\Phi(X)$ are depicted) and $e_n$ is the restitution coefficient, $e_n \in [0, 1]$. Notice that if the angle $(\Sigma_1, \Sigma_2) \leq \pi$ then in the neighborhood of $X$ one has $\Phi \approx T\Phi(X)$ when $X \in \Sigma_1 \cap \Sigma_2$. The tangent cone is defined as the cone which is polar to the normal cone $N\Phi(X(t))$, see [Clarke(1990)] [Hiriart-Urruty & Lemaréchal(1996)] [Moreau(1988)]. Both are always convex sets. They generalize the tangent and normal subspaces to the configuration space to which velocities and contact forces belong, in bilaterally constrained systems. When $m = 1$, the rule in (2) is the Newton’s law $\dot{X}_n(t^+_k) = -e_n \dot{X}_n(t^-_k)$, where $\dot{X}_n$ is the normal component of the velocity. The restitution mapping in (2) can be equivalently rewritten as [Mabrouk(1998)]:

$$
\dot{X}(t^+_k) = \dot{X}(t^-_k) - (1 + e_n) \text{prox}_{M(X(t_k))}[M^{-1}(X(t_k))]N\Phi(X(t_k)); \dot{X}(t^-_k)] 
$$

(3)

where the $\text{prox}_{M(X(t_k))}$ means the proximation in the metric defined by the kinetic energy at time $t_k$, and $N\Phi(X(t_k))$ is the normal cone to $\Phi$ at $X(t_k)$. The form in (3) will be useful for some calculations in stability proofs. It can also be written using a suitable generalized momentum
transformation [Brogliato(1999), Chapter 6]. See also [Glocker(2002)] for a nice geometrical interpretation of this rule. The restitution mapping in (2) yields a kinetic energy loss at the impact times given by [Mabrouk(1998)]:

\[ T_L(t_k) = -\frac{1}{2} \left[ 1 - e_n \right] \left( \ddot{X}(t_k^+) - \ddot{X}(t_k^-) \right) M(q(t_k)) \left[ \dddot{X}(t_k^+) - \dddot{X}(t_k^-) \right] \leq 0 \] (4)

Clearly this particular choice is arbitrary, and other models exist in the literature. However Moreau’s collision rule is chosen here because it is mathematically sound, numerically tractable because it relies on Gauss’ principle of Mechanics [Brogliato et al.(2002)] and is a direct extension of Newton’s law (which is quite valid as long as friction is not considered). Moreover it lends itself very well to possible extensions towards more complex collision rules as the ones developed in [Frémond(2002)], which are based on the use of super-potentials of dissipation [Moreau(1968)].

### 1.4 Model well-posedness

The most general result on existence and uniqueness of solutions for mechanical systems as in (1) can be found in [Ballard(2000)] [Ballard(2001)]. Under the condition that all data entering (1) are piecewise real analytic, then existence and uniqueness of a solution to (1) with \( X(\cdot) \) absolutely continuous and \( \ddot{X}(\cdot) \) right-continuous of local bounded variation, is assured. Then the acceleration is a measure and so is the multiplier \( \lambda_X \). We shall always assume that the required conditions are fulfilled in this paper. Multiple impacts (see definitions 1 and 5) generally render solutions discontinuous with respect to the initial conditions \((X(0),\dot{X}(0+))\), except in particular cases (plastic impacts and kinetic angle between the constraint surfaces less or equal to \( \pi \) \[Paoli(2002)\]), or kinetic angle equal to \( \frac{\pi}{2} \) [Ballard(2000)]. When \( m = 1 \) then continuity holds whatever \( e_n \) [Schatzman(1998)].

Due to the fact that velocities may be time discontinuous, but that their right-limit (and left-limit as well) exist everywhere, models as in (1) may be named prospective, because during the integration one looks for \( X(t^+) \) [Moreau(2003)].

### 1.5 Cyclic task

In this paper we restrict ourselves to a specific task, or trajectory: a succession of free and constrained phases \( \Omega_k \). During the transition between a free and a constrained phase, the dynamic system passes into a transition phase \( I_k \). As we shall see, transitions between constrained and free motion are monitored by a Linear Complementarity Problem (see Appendix C for a definition).

\[
\Omega_{2k} \xrightarrow{I_k} \Omega_{2k+1} \xrightarrow{\text{LCP}(\lambda)} \Omega_{2k+2}
\]

In the time domain one gets a representation as :

\[
\mathbb{R}^+ = \bigcup_{\text{cycle 0}} \bigcup_{\text{cycle } k} \Omega_0 \cup I_0 \cup \Omega_2 \cup I_2 \cup \cdots \cup \Omega_{2k-1} \cup I_k \cup \Omega_{2k+1} \cup \cdots
\] (5)

where \( \Omega_{2k} \) denotes the time intervals associated to free-motion phases and \( \Omega_{2k+1} \) those for constrained-motion phases. The transition \( \Omega_{2k+1} \longrightarrow \Omega_{2k+2} \) does not define a specific phase (or DES mode) because it does not give rise to a new type of dynamical system, as we shall see in Sec. 3.3. The order of the phases is important but the initial phase may be \( \Omega_0 \) or \( I_0 \) or \( \Omega_1 \), see remark 2. Before passing to the description of the stability framework which will enable us to design a feedback controller for tracking, let us investigate more deeply how (5) may be seen as a consequence of the basic control objectives i), ii) and iii) listed in the introduction.

First of all, let us notice that despite the problem involves contact and consequently generalized forces in the control objectives (during phases \( \Omega_{2k+1} \) the contact force should have some desired
value), the control problem remains primarily a motion control problem. Indeed the contact force, i.e. the Lagrange multiplier $\lambda_X$ in (1), is not part of the system’s state $(X, \dot{X})$. Its value is only a consequence of the motion (in fact its value has to be calculated with a LCP, which is assured to always possess at least one solution for frictionless constraints, see Brogliato(1999), theorem 5.4). For instance in a one degree-of-freedom system the contact force control simply reduces to an algebraic equation $\lambda = \lambda_d$ for some signal $\lambda_d$ (possibly time-varying). However this is not a stabilisation problem, this is a static equilibrium. Therefore the force/position control problem should rather be called a motion-control/force-equilibrium problem in such a case. During such a static equilibrium phase, motion tracking drastically simplifies to triviality. This is going to be the same in higher dimensions, in the normal direction to $\partial \Phi$.

More precisely, the items i) and ii) in the introduction imply that the trajectory of the unconstrained system that has to be tracked, denoted as $X^{t, nc}(\cdot)$ possesses the generic form shown in Fig. 3. More exactly the orbit of this trajectory in the configuration space is depicted on Fig. 3. It is clear that in particular item ii) implies that $F(X^{t, nc}(t)) < 0$ for some $t(\in \Omega_{2k+1})$, otherwise there would be a zero contact force when the system perfectly tracks the desired motion. Roughly speaking, the system has to have the tendency to violate the constraints in order to assure a non-zero contact force. In the same spirit item i) implies that $F(X^{t, nc}(t)) > 0$ for some $t(\in \Omega_{2k})$. Consequently there exists a point $A$ in the configuration space, at which contact is made with $\partial \Phi$. This gives rise to a transition phase whose role is as in item iii). In the same way there is a point $B$ at which $F(X^{t, nc}(t)) = 0$ and detachment is monitored by a LCP. The central issue in the present control problem, is the design of such transition phases. A first idea is to impose a tangential contact, i.e. with $\nabla F(X^\ast_d)^T X^\ast_d = 0$, where $X^\ast_d(\cdot)$ is a signal entering the control input and playing the role of the desired trajectory during some parts of the motion (the difference between $X^\ast_d(\cdot)$, and $X^{t, nc}(\cdot)$ will be made clear below). However

- $\alpha$) Due to non-zero initial tracking errors $X(0) - X^\ast_d(0) \neq 0$, $\dot{X}(0) - \dot{X}^\ast_d(0) \neq 0$, impacts may occur.
- $\beta$) This is not a robust strategy since a bad estimation of the constraint position, may result to no stabilisation at all on $\partial \Phi$. Consequently it is a much better strategy to impose collisions for stabilisation on $\partial \Phi$.
- $\gamma$) In any case, collisions have to be incorporated into the stability analysis.
- $\delta$) The best strategy for stabilisation on $\partial \Phi$ is to impose closed-loop dynamics which mimics the bouncing-ball dynamics $\ddot{X} = -g, X > 0$:

  - $\delta_1$) This is very robust with respect to the constraint position uncertainties.
  - $\delta_2$) As we will see it lends itself very well to Lyapunov stability of some closed-loop Poincaré map.
Secondly, we will see in the next section that the type of stability we desire is based on a single Lyapunov-like function \( V(X, \dot{X}, t) \). Then difficulties arise due to the following:

- **a)** There are non-zero couplings between “tangential” and “normal” coordinates in the inertia matrix \( M(X) \) (this will be formulated more rigorously later).
- **b)** This unique function \( V(X, \dot{X}, t) \) has to work for all phases, i.e. for \( \Omega_{2k} \) (ODE), \( \Omega_{2k+1} \) (DAE), and \( I_k \) (the dynamics may then be seen as a Measure Differential Equation [Brogliato(1999)])
- **c)** If \( V \equiv 0 \) then any velocity jump \( \dot{q}(t_k^+) \neq \dot{q}(t_k^-) \) implies a positive jump \( V(t_k^+) - V(t_k^-) > 0 \) in the Lyapunov function. This means that impacts will generally preclude asymptotic stability
- **d)** The function \( V \) has to satisfy \( V = 0 \) when the desired trajectory of the closed-loop system is perfectly tracked, according to the definition of a Lyapunov function. This implies that the desired set of the complete (constrained) system must be used in the definition of \( V \).

One therefore realises that the control problem is itself subject to many constraints. The proposed strategy has to cope with these various and sometime antagonist facts (like \( \beta \) and \( \delta \)).

Item **c)** hampers the use as time goes to infinity of any controller that would switch at time \( t_k \) between a free-motion feedback input with \( F(X^i) > 0 \) to a transition phase controller with a “bouncing-ball” dynamics (i.e. such that \( F(X^c) \)). However such a discontinuous input can be used during the transient period. The idea of using a desired motion that would mimic the impacts so that \( V(t_k^+) - V(t_k^-) = 0 \) even when \( V(0) = 0 \) is not a good one. First of all items \( \beta \) and \( \delta \) are in force, and such a strategy requires also a perfect knowledge of \( e_n \) in (2). Secondly, proving the stability of such a trajectory is a hard task. We therefore disregard this sort of signals \( X_{d}^{*} \) for transition phases \( I_k \). In order to clarify these various notions let us consider a one degree-of-freedom system:

\[
\begin{cases}
(X - \dot{X}_d^* + \gamma_2(X - \dot{X}_d^*) + \gamma_1(X - X_d^*) = \lambda \\
0 \leq X \perp \lambda \geq 0 \\
\dot{X}(t^*_k) = -e_n\dot{X}(t^*_k)
\end{cases}
\]

(6)

where \( X_{d}^{*} \) is some twice differentiate function, \( \gamma_2 > 0, \gamma_1 > 0 \). The ”\( \perp \)” means that \( X \) and \( \lambda \) are orthogonal, i.e. \( X\lambda = 0 \). It is clear that \( X_{d}^{*} \) is the desired trajectory of the constrained system. Rather, this is going to be simply 0 on \( I \). Item **d)** means that the function \( V \) used for stability purpose (e.g. a quadratic function of the tracking error) is zero on \( \Omega_{2k+1} \) (constrained-motion phases). Therefore the Lyapunov function will be defined such that on \( I_k \) and on \( \Omega_{2k+1} \) one has \( V(X, \dot{X}, t) = 0 \). Since this is a tracking control problem and since the desired trajectory is equal to 0 on such phases (even the rebound phase), we conclude that the tracking error \( \dot{X} \) entering \( V \) has to satisfy \( \dot{X} = 0 \), so that \( V(X, \dot{X}, t) = \dot{X} = 0 \). Thus \( \dot{X} \) cannot be defined from \( X_{d}^{*} \) neither from \( X^{i,nc} \) but from a third signal which we shall denote as \( X_{d}^{\prime} \). Let us again clarify the difference between \( X_{d}^{*} \) and \( X_{d}^{\prime} \). Let us take a constant \( X_{d}^{\prime} = X_{d}^{*} \) but since the fixed point of the complementarity system is \( X \) then this must have \( V(X, X) = 0 \). \( X \) is then stable during these periods of time. In the following we shall denote \( X = X - X_d \) and \( \dot{X} = X - X_d^* \).

Finally in general \( X_{d}^{*} \neq X_{d}^{\prime} \) because \( X_{d}^{\prime} \) may be chosen to evolve from one transition phase \( I_k \) to the next one \( I_{k+1} \) whereas \( X_{d}^{i,nc} \) does not depend on the cycle index \( k \).

Such conditions appear quite stringent. Actually we are looking for the most direct extension of Lyapunov’s second method for complementarity systems as in (1) evolving as in (5). If the task is less complex than (5) and/or the dynamics possess some strong properties (see [Brogliato(1999), chapter 9]) then the stability analysis may simplify.

\[\footnote{This is mainly due to the fact that the controllers used on phases \( \Omega_k \) assure asymptotic convergence of the tracking errors towards zero, but do not possess any finite-time convergence properties.}\]
The control strategy which is developed in the sequel, takes all these features into account and especially imposes an desired trajectory $X^{i,nc}$ as depicted in Fig. 4. The orbits of the trajectories are depicted. Tangential contact is made at $A''$ when force control starts so that $X^{i,nc}$ jumps at $B$. In addition item $\beta$ is taken into account by imposing a “bouncing-ball” dynamics only during the transient period, i.e. on $I_k$ with $k < +\infty$. In other words the trajectory $X^{i,nc}(t)$ makes a tangential contact with $\partial \Phi$ because if initial data satisfy $X(0) - X^*_j(0) = 0$ and $\dot{X}(0) - \dot{X}^*_j(0) = 0$ on $\Omega_{2k}$, then $X(t) \equiv X^{i,nc}(t)$ for $t \in \Omega_{2k}$, but during the transient period the controller assures the existence of collisions on phases $I_k$. Therefore between points $A$ and $B$ on figure 4, one may have $X^*_j(\cdot)$ which violates the constraint during the transient period, and converges towards a tangential approach trajectory after a finite or infinite number of transition phases (or cycles $\Omega_{2k} \cup I_k \cup \Omega_{2k+1}$). Between $B$ and $C$ the phase $\Omega_{2k+1}$ occurs during which objective ii) is fulfilled. The dashed orbit $AA'B'$ on Fig. 4 represents $X^*_d(\cdot)$ during a transition phase with impacts. The system stabilizes on $\partial \Phi$ between $A$ and $B'$ when the controller is switched to a force control so that $X^{i,nc}(\cdot)$ and $X^*_d(\cdot)$ may jump to $B$. In the control scheme described later, the point $B'$ will converge (in a finite or infinite number of cycles) towards $A''$. We finally define the closed-loop desired trajectory of the complementarity system as $X^{i,c}(\cdot)$. On Fig. 4, $X^{i,c}(\cdot)$ is the curve $(CA'A'C)$ and $X^{i,c}(\cdot) \in \partial \Phi$ on $(A''C)$. It is an impactless trajectory. Let us assume that a periodic motion is desired. Then on Fig. 4 only the orbits of $X^{i,nc}(\cdot)$ (i.e. $AA'B'CA$) and $X^{i,c}(\cdot)$ (i.e. $AA''CA$) are fixed. The other two orbits may vary with the cycle index $k$. But on a single phase $I_k$ the fixed point of the closed-loop error system may indeed be a signal $X_d \in \partial \Phi$ ($A'A''$) which differs from $X^*_d \notin \Phi$ ($A'B'$). The orbits ($AA'B'$) and the point $A'$ generally vary from one cycle $\Omega_{2k} \cup I_k \cup \Omega_{2k+1}$ to the next cycle $\Omega_{2k+2} \cup I_{k+1} \cup \Omega_{2k+3}$. One can also interpret this as defining a desired trajectory $X^*_j(\cdot)$ on each cycle $\Omega_{2k} \cup I_k \cup \Omega_{2k+1}$, which is iterated from cycle $k$ to cycle $k + 1$ so that it converges towards $X^{i,c}(\cdot)$. The mixture between the DES and continuous dynamics clearly appears.

In summary the control strategy and stability analysis are led with four different trajectories: $X^*_d(\cdot)$ in the control input, $X_d(\cdot)$ in the Lyapunov function, $X^{i,c}(\cdot)$ and $X^{i,nc}(\cdot)$. Still referring to Fig. 4: when the system is initialised on $X^{i,c}(\cdot)$ between $C$ and $A$ (i.e. on $\Omega_2$), then $X_d(t) = X^{i,c}(t)$ on $(CA')$ and $X_d(t) \in \partial \Phi$ on $(A'C)$. If initially $X(0) \neq X^{i,c}(0)$ and/or $X(0) \neq X^{i,c}(0)$, then $X_d(\cdot)$ differs and is set to zero in the Lyapunov function at a time corresponding to the first impact. This is the major discrepancy compared to unconstrained motion control in which all four trajectories are the same, usually denoted as $X_d(\cdot)$ (see remark 3).

![Fig. 4: The closed-loop desired trajectories and control signals.](image-url)
2 Stability Framework

The stability criterion used in this paper is an extension of the Lyapunov second method adapted to closed loop mechanical system with unilateral constraints and has been proposed in [Brogliato et al. (1997)Brogliato, Niculescu & Orhant] and [Brogliato et al. (2000)Brogliato, Niculescu & Monteiro-Marques]. Let \( x(t) \) denote the state of the closed-loop system in (1) with some feedback controller \( u(X, \dot{X}, t) \).

**Definition 2 \((\Omega \text{-weakly stable system})\).** The closed-loop system is \( \Omega \)-weakly stable if for each \( \epsilon > 0 \), there exists \( \delta(\epsilon) > 0 \) such that \( \|x(0)\| \leq \delta(\epsilon) \Rightarrow \|x(t)\| \leq \epsilon \) for all \( t \geq 0 \), \( t \in \Omega = \cup_{k \geq 0} \Omega_k \).

Asymptotic weak stability holds if in addition \( x(t) \to 0 \) as \( t \to +\infty \), \( t \in \Omega \). Practical \( \Omega \)-weak stability holds if there is a ball centered at \( x = 0 \), with radius \( R > 0 \), and such that \( x(t) \in B(0, R) \) for all \( t \geq T ; T < +\infty, t \in \Omega, R < +\infty \).

Let us define the closed-loop impact Poincaré map that corresponds to the section \( \Sigma^- \) \( = \{x | F_1(x) = 0, \dot{X}^T \nabla F_1(x) < 0, i \in I \} \), which is a hypersurface of codimension \( \alpha = \text{card}(I) \). The pre-impact velocities are chosen to define \( P_{\Sigma^-} \) for a reason given after claim 3. We define:

\[
P_{\Sigma^-} : \Sigma^- \to \Sigma^- \nonumber
\]

\[
x_{\Sigma^-}(k) \mapsto x_{\Sigma^-}(k + 1). \quad (7)
\]

where \( x_{\Sigma^-} \) is the state of \( P_{\Sigma^-} \). Let us introduce the positive function \( V(\cdot) \) that will serve in the subsequent analysis. Let \( V_{\Sigma^-} \) denote the restriction of \( V \) to \( \Sigma^- \).

**Definition 3 \((\text{Strongly stable system})\).** The system is said strongly stable if: (i) it is \( \Omega \)-weakly stable, (ii) on phases \( I_k \), \( P_{\Sigma^-} \) is Lyapunov stable with Lyapunov function \( V_{\Sigma^-} \), and (iii) the sequence \( \{t_k\}_{k \in \mathbb{N}} \) has a finite accumulation point \( t_\infty < +\infty \).

Clearly \( P_{\Sigma^-} \) has a fixed point \( x_{\Sigma^-}^* \in \partial \Phi \). Let \( V(\cdot) \) satisfy \( \beta(||x||) \geq V(x, t) \geq \alpha(||x||) \), \( \alpha(0) = 0 \), \( \beta(0) = 0 \), \( \alpha(\cdot) \) and \( \beta(\cdot) \) strictly increasing. Let \( I_k = [r^k_0, t^k_1] \).

**Claim 1 \((\Omega \text{-Weak Stability} [Brogliato et al. (1997)Brogliato, Niculescu & Orhant])\)** Assume that the task is as in (5), and that

- (a) \( \lambda(\Omega) = +\infty \),
- (b) \( \lambda(I_k) < +\infty \),
- (c) \( V(x(t^k_0), t^k_1) \leq V(x(r^k_0), r^k_0) \),
- (d) \( V(x(\cdot), \cdot) \) uniformly bounded on each \( I_k \).

If on \( \Omega \), \( V(x(t), t) \leq 0 \) and \( \sigma_V(t_k) \leq 0 \) for all \( k \geq 0 \), then the closed-loop system is \( \Omega \)-weakly stable. If \( V(x(t), t) \leq -\gamma(||x||), \gamma(0) = 0 \), \( \gamma(\cdot) \) strictly increasing, then the system is asymptotically \( \Omega \)-weakly stable.

This accommodates for other types of motions than the one as in (5), see [Brogliato et al. (1997)Brogliato, Niculescu & Orhant]. Let us assume that \( t_\infty < +\infty \). It is noteworthy that from [Ballard (2001), proposition 4.11] this implies \( e_n < 1 \) (because if \( e_n = 1 \) impact times satisfy \( t_{k+1} - t_k \geq \beta_k > 0 \) with \( \sum_{k \geq 0} \beta_k \) unbounded, so that \( t_\infty = +\infty \)).

**Claim 2 \((\Omega \text{-Weak Stability})\)** Let us assume that (a) and (b) in claim (1) hold, and that

- (a) outside phases \( I_k \) one has \( \dot{V}(t) \leq -\gamma V(t) \) for some \( \gamma > 0 \),
- (b) inside phases \( I_k \) one has \( V(t_{k+1}) - V(t_k) \leq 0 \), for all \( k \geq 0 \),
- (c) the system is initialized on \( I_0 \) with \( V(r^0_0) \leq 1 \),
- (d) \( \sum_{k \geq 0} \sigma_V(t_k) \leq K V(r^0_0) + \epsilon \) for some \( K \geq 0 \), \( K \geq 0 \) and \( \epsilon \geq 0 \).

Then there exists a constant \( N < +\infty \) such that \( \lambda_t^{\infty, t^k \jmath} = N \), for all \( k \geq 0 \) (the cycle index), and such that:
(i) - If \( \kappa \geq 1 \), \( \epsilon = 0 \) and \( N = \frac{1}{\gamma} \ln \left( \frac{1 + K}{\delta} \right) \) for some \( 0 < \delta < 1 \), then \( V(t_0^k) \leq \delta V(t_0^k) \). The system is asymptotically weakly stable.

(ii) - If \( \kappa < 1 \), then \( V(t_0^k) \leq \delta(\gamma) \), where \( \delta(\gamma) \) is a function which can be made arbitrarily small by increasing \( \gamma \). The system is practically \( \Omega \)-weakly stable with \( R = \alpha^{-1}(\delta(\gamma)) \).

Let us notice that the upperbound in (d) is the key point of the analysis. It characterizes the uncertainty that is allowed in the variation of function \( V(.) \).

**Proof**

From assumption (a) of claim 2, one has

\[
V(t_f^k) \leq V(t_\infty) e^{-\gamma(t_f^k - t_\infty)}
\]

From assumptions (b) and (d) of claim 2, one has

\[
V(t_{\infty}) \leq V(t_0^k) + \sum_{k=0}^{\infty} \sigma V(t_k) + \sum_{k=0}^{\infty} V(t_{k+1}^-) - V(t_k^+)
\]

\[
\leq V(t_0^k) + KV^\kappa(t_0^k) + \epsilon
\]

Inequalities (9) and (8) give

\[
V(t_f^k) \leq e^{-\gamma(t_f^k - t_\infty)}[V(t_0^k) + KV^\kappa(t_0^k) + \epsilon]
\]

Let us now analyse two cases:

(i) If \( \kappa \geq 1 \), then \( V(t_0^k) \geq V^\kappa(t_0^k) \). If \( \epsilon = 0 \), Eq. (10) becomes

\[
V(t_f^k) \leq e^{-\gamma(t_f^k - t_\infty)(1 + K)V(t_0^k)}
\]

If we want to have \( V(t_f^k) \leq \delta V(t_0^k) \), we must choose \( \lambda[t_f^k - t_\infty] \) such that:

\[
e^{-\gamma(t_f^k - t_\infty)}(1 + K) \leq \delta
\]

This is assured by choosing \( \lambda[t_f^k - t_\infty] = N \) with

\[
N = \frac{1}{\gamma} \ln \left( \frac{1 + K}{\delta} \right)
\]

Clearly if \( \delta > 0 \), then \( N < +\infty \), which proves the first item.

(ii) If \( \kappa \leq 1 \) then \( V(t_0^k) \leq V^\kappa(t_0^k) \leq 1 \). Inequality (10) becomes

\[
V(t_f^k) \leq e^{-\gamma(t_f^k - t_\infty)}(1 + K + \epsilon) = \delta(\gamma)
\]

The term \( \delta(\gamma) \) can be made as small as desired by increasing \( \gamma \) (or increasing \( \lambda[t_f^k - t_\infty] \)). The proof is complete since \( \alpha(\|x\|) \leq V(x,t) \) for all \( x \) and \( t \).

**Claim 3 (Strong Stability)** The system is strongly stable if in addition to the conditions in claim 1 one has:

- \( V(t_{k+1}^-) \leq V(t_k^-) \);
- \( V \) is uniformly bounded and time continuous on \( I_k - \cup_k \{t_k\} \).
Then the system is strongly stable in the sense of definition 3. □

Sufficient conditions for strong stability are that \( \sigma V(t_k) \leq 0 \) and \( V(t_{k+1}) < V(t_k) \), but this framework permits \( \sigma V(t_k) \geq 0 \) provided \( V(t_{k+1}) < V(t_k) - \delta \) for some large enough \( \delta > 0 \). Notice also that \( \dot{V}(t) \) needs not to be \( \leq 0 \) along the system’s trajectories on the whole of \( (t_k, t_{k+1}) \). The reason why we have chosen \( \Sigma^- \) and not \( \Sigma^+ \) in (7) is that it allows us to take into account the value \( V(t_0^-) \) in the stability analysis. Notice that \( \dot{q}(t_n) = \dot{q}(t_n^-) \).

In order to summarize the consequences of what is stated in Secs. 1-2, let us propose the following:

**Proposition 1.** Let the Lagrangian complementarity system as in (1) perform a motion as in (5), with the closed-loop requirements as in i), ii), iii). Let us assume that asymptotic tracking controllers are used on phases \( \Omega_k \). Then the asymptotic stability in the sense of definitions 2 and 3 implies that:

- The asymptotically stable closed-loop desired trajectory \( X^{i,c}(-) \) is impactless.
- During the transient period the feedback controller has to guarantee the existence of collisions with \( \partial \Phi \) and a finite-time stabilisation on \( \partial \Phi \).
- Contrary to the unconstrained motion case (\( \Phi = \mathbb{R}^n \)), the signals \( X_d(-) \) entering the Lyapunov function, \( X_d^*(-) \) in the controller, and \( X^{i,c}(-) \), are not equal to a single so-called desired trajectory. □

This proposition is a consequence of items i), ii), iii), a) through \( \delta \), a) through d), as well as of definitions 2 and 3.

### 3 Tracking Controller Framework

#### 3.1 Controller structure

To make the controller design easier the dynamical equations (1) are considered in the generalized coordinates introduced in [McClamroch & Wang(1988)]. After transformation in the new coordinates \( q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} \), \( q = Q(X) \in \mathbb{R}^n \), the dynamic system is as follows:

\[
\begin{align*}
M_{11}(q) \ddot{q}_1 + M_{12}(q) \ddot{q}_2 + C_1(q, \dot{q}) \dot{q} + g_1(q) &= T_1(q)u + \lambda \\
M_{21}(q) \ddot{q}_1 + M_{22}(q) \ddot{q}_2 + C_2(q, \dot{q}) \dot{q} + g_2(q) &= T_2(q)u \\
q_i &\geq 0, \quad q_i \lambda_i = 0, \quad \lambda_i \geq 0, \quad 1 \leq i \leq m \\
\text{Collision rule}
\end{align*}
\]

(15)

where the set of complementarity relations can be written more compactly as \( 0 \leq \lambda \perp Dq \geq 0 \) with \( D = [I_m; 0] \in \mathbb{R}^{m \times n}, I_m \) is the identity matrix. Clearly \( M_{21}(q) = M_{12}^T(q) \in \mathbb{R}^{(n-m) \times m} \), \( M_{11}(q) \in \mathbb{R}^{m \times m}, M_{22}(q) \in \mathbb{R}^{(n-m) \times (n-m)} \). In the new coordinates \( q \) one therefore has \( \Phi = \{ q | Dq \geq 0 \} \). The tangent cone \( T_\Phi(q_1 = 0) = \{ v | Dv \geq 0 \} \) is the space of admissible velocities on the boundary of \( \Phi \). The polar cone to \( T_\Phi(\cdot) \) is the normal cone \( N_\Phi(q) = \{ v | v \perp \exists v \in T_\Phi, z^Tv \leq 0 \} \). In case \( q \in \partial \Phi \), one gets \( N_\Phi(q) = \{ v | v = D^T \lambda, \lambda \leq 0 \} \). [Hirai & Lemaréchal(1996)]. Obviously from (15) the generalized contact force \( P_0 = D^T \lambda \in -N_\Phi(q) \). The controller developed in this paper uses three different low-level control laws for each phase \( \Omega_{2k}, \Omega_{2k+1} \) and \( I_k (3) \):

---

3 With some abuse of notations we assimilate the time domains to the modes that correspond to the three phases in (5).
\[
T(q)u = \begin{cases} 
U_{nc} & \text{for } t \in \Omega_{2k} \\
U_t & \text{for } t \in I_k \\
U_c & \text{for } t \in \Omega_{2k+1} 
\end{cases}
\]

where \( T(q) = \begin{pmatrix} T_1(q) \\ T_2(q) \end{pmatrix} \in \mathbb{R}^{m \times n} \). A supervisor switches between these three control laws, and is described below (see Fig. 8). The stability of this controller is analyzed by using the criteria proposed in Sec. 2. The asymptotic stability of this scheme makes the system land on the constraint surfaces tangentially after enough cycles of constraints/free motions (one cycle = \( \Omega_{2k} \cup I_k \cup \Omega_{2k+1} \)). Asymptotically the transitions between free motion phases and permanently constraint phases are done without any collision.

Remark 1 (Dynamic coupling effects). From (15) it follows that \( \sigma \dot{q}_2(t_k) = M_{22}^{-1}M_{21}\sigma \dot{q}_1(t_k) \). Apply for instance a feedback linearizing input \( u \) in (15) so as to get the dynamics

\[
\begin{align*}
\dot{q}_1 &= v_1 + \lambda \\
\dot{q}_2 &= v_2
\end{align*}
\]  

(16)

where \( v_1 \) and \( v_2 \) are the new inputs. One is then tempted to mimic the one degree-of-freedom case, see [Brogliato et al. (1997)]. However except if \( V(t) = T(t) \) (the kinetic energy) at time \( t = t_k \), then there is few chance to get \( \sigma V(t_k) \leq 0 \) (because the controller does not decouple the dynamics at impact times!). This precludes the use of any controller with Lyapunov function not resembling the system’s mechanical energy. In the sequel we will use a Lyapunov function which is very close to the nonsmooth global energy of the system. This will help us a lot in the stability analysis.

Let us choose:

\[
V(t, \dot{q}, \ddot{q}) = \frac{1}{2} \dot{q}^T M(q) \ddot{q} + \frac{1}{2} \gamma_1 \ddot{q}^T \dddot{q}
\]  

(17)

with \( \ddot{q}(\cdot) = q(\cdot) - \dot{q}(\cdot) \). The control law used in this scheme is based on the controller presented in [Paden & Panja (1988)], originally designed for free-motion position and velocity global asymptotic tracking. Let us propose:

\[
T(q)u = \begin{cases} 
U_{nc} = M(q)\ddot{q}_q + C(q, \dot{q})\dot{q}_q + g(q) - \gamma_1(q - \dot{q}_q) - \gamma_2(\ddot{q} - \ddot{q}_q) \\
U_t = U_{nc} & \text{before the first impact} \\
U_t = g(q) - \gamma_1(q - \dot{q}_q) - \gamma_2 \ddot{q} & \text{after the first impact} \\
U_c = U_{nc} - P_d + K_f(P_q - P_d) 
\end{cases}
\]  

(18)

where \( \gamma_1 > 0, \gamma_2 > 0, K_f > 0 \), \( P_d = D^T \lambda_d \) is the desired force we want for the permanently constraint motion. The signals \( \dot{q}_q \) and \( q_q \) will be defined later, as well as the switching conditions between the controllers in (18). The overall structure of the controller is depicted in Fig. 5. One sees that the controller structure is constant. Discontinuities are a consequence of the feedforward part only. The switchings may be event-based, or open-loop, see Fig. 8 which depicts how the supervisor is designed. The interest for choosing this controller is that the function \( V(t, \ddot{q}, \dot{q}) \) in (17) is very close to the total energy of the system. Notice that \( u \) in (18) is independent of the restitution coefficient \( e_r \). From (18) the third condition in claim 1 can be replaced by \( V(t_f^k) \leq V(t_0^k) \) since \( V(t_0^k) \leq V(\tau_0^k) \).
Remark 2. It is noteworthy that in order for the system to track a sequence of modes as in (5), some conditions on the initial state and the selected input are required. This is called synchronicity of the high-level controller and the system’s modes defined in (5) in [Brogliato et al. (2000) Brogliato, Niculescu & Monteiro-Marques].

As observed in the introduction, a control strategy which consists of attaining the surface $\partial \Phi$ tangentially and without incorporating impacts in the stability analysis, cannot work in practice due to its lack of robustness. In view of this, the control law for the transition phase is defined in order:

- To make the system hit the constraint surface (and then dissipate energy during impacts) if the tracking error is not zero.
- To make the system approach the constraint surface tangentially (without rebound) if the tracking is perfect.

This two situations are conflicting. On the other hand the coupling between $q_1$ and $q_2$ in (15), and the stability framework in claims 1 and 3, make the asymptotic stability quite difficult to obtain if velocities are subject to discontinuities. Indeed as indicated in item c) in Sec. 1.5, any velocity jump at $t_k$ implies $\sigma_V(t_k) > 0$ when $V \equiv 0$. Hence if the transition phase is constructed with impacts, one has to find a manner to get $V(t_{k+1}) = 0$ in order to force the system to remain on the desired trajectory $X_d(t)$ (here $q_d(t)$). This is not obvious in general (see remark 1) and defining $q_d^*(t)$ as done below is a way to get the result.

Remark 3. If the system is unconstrained (i.e. $\Phi = \mathbb{R}^n$) then motion control is assured by setting $T(q)u \equiv U_{nc}$ and the trajectory $q^*_d(t)$ is the unique closed-loop invariant. It is globally uniformly asymptotically stable in this case, see [Paden & Panja (1988)]. As we indicated in the introduction, many other controllers can be used in this case which all guarantee the same tracking properties.

3.2 Design of the desired trajectory on phases $I_k$.

During the transition phase $q^*_d(t)$ is defined as follows (see figure 6 for $q^*_d(t)$, where $A, A', B', B$ and $C$ correspond to Fig. 4):

Let us note that the indices $k$ for the phases $\Omega_k$ and $I_k$ and for the impact times $t_k$, are not related. They are dummy variables. To avoid possible confusion, all superscripts ($.)^k$ will refer to cycle $k$ in (5). Let us define:

- $\tau_0^k$ is the chosen by the designer as the start of the transition phase $I_k$,
- $t_0^k$ is the time corresponding to $q^*_d(t_0^k)=0$, 

![Fig. 5: Structure of the controller](image-url)
- $t_0$ is the first impact,
- $t_\infty$ is the finite accumulation point of the sequence $\{t_k\}_{k\geq0}$,
- $t^k_k$ is the end of the transition phase $I_k$,
- $\tau^k_k$ is such that $q^*_d(\tau^k_k) = -\alpha V(\tau^k_0)$ and $\dot{q}^*_d(\tau^k_k) = 0$ (4).
- $\Omega_{2k+1} = [t^k_k, t^k_f]$ will be defined in Sec. 3.3 (see Fig. 7).

One has $I_k = [\tau^k_0, t^k_f]$, $\Omega_{2k+1} = [t^k_f, t^k_d]$. On $[\tau^k_0, t_0]$, we impose that $q^*_d(t)$ is twice differentiable, and $q^*_d(t)$ decreases towards $-\alpha V(\tau^k_0)$ on $[\tau^k_0, \tau^k_1]$. In order to cope with the coupling between $q_1$ and $q_2$, the signal $q^*_d(t) \in C^2(\mathbb{R}^+)$ is frozen during the transition phase, i.e.:

- $q^*_d(t) = q^*_d(t^k_f), \dot{q}^*_d(t^k_f) = 0$ on $[\tau^k_0, t_\infty]$.
- $q^*_d(t)$ is defined on $[\tau^k_0, t^k_0]$ such that $\dot{q}^*_d(t^k_0) = 0$.

On $(t_0, t_f^k)$, we define $q_d$ and $q^*_d$ as follows:

$$q_d = \begin{pmatrix} 0 \\ q^*_d \end{pmatrix}, \quad q^*_d = \begin{pmatrix} -\alpha V(\tau^k_0) \\ q^*_d \end{pmatrix}$$

On $[t^k_f, t^k_d]$ we set $q_d = \begin{pmatrix} 0 \\ q^*_d(t) \end{pmatrix}$ and $q^*_d = 0$. Therefore on $(t^k_f, t^k_d)$ one has $q_d(t) = q^*_d(t)$. The purpose of $q^*_d$ is to create a “virtual” potential force which stabilizes the system on $\partial \Phi$ even if the

---

4 In [Brogliato(1999)] [Brogliato et al.(2000)Broglia, Niculescu & Monteiro-Marques] it is implicitly assumed in the stability proofs that $\tau^k_1 < t_0$, which is a shortcoming that we avoid in this paper.
position of the constraint is uncertain. Consequently the fixed point \((q_d, \dot{q}_d)\) of the complementarity system is used in the expression of the Lyapunov function \((\ddot{q} = q - q_d)\), whereas the unreachable fixed point \(q^*_d\) is used in the control law \((\ddot{q} = q - q^*_d\) with \(q^*_d\) as in (19)). In \(U_{nc}\) in (18) we have \(q^*_d(t) = q_d(t)\) since \(q^*_d(t) = q_d(t)\) for \(t \in \Omega_{2k} \cup [\tau_k^0, t_0]\). In summary, after the first impact at \(t_0\), \(q_{id}(\cdot)\) is set to zero while in case \(\tau_k^b > t_0\), \(q_{id}(\cdot)\) is set to \(-\alpha V(\tau_k^b)\) (in other words \(U_d\) switches as indicated in (18)). Since \(\dot{q}_{id}(t_0^+) \neq 0\) and \(q_{id}(t_0^+) \neq 0\) in general, the trajectory \(q_{id}(\cdot)\) behaves like in a sort of plastic collision \((\epsilon_n = 0)\). With respect to Fig. 4, one has \(t_k^0\) at A, \(t_\infty\) at \(B'\), \(t_b^k\) at \(A'\), \(t_d^k\) at C, and \(B\) at \(t_f^k\) (the term \(-P_d - K_f P_d\) defines the signal \(X^*_d(\cdot)\) between \(B\) and \(C\) on Fig. 4). If \(V(\tau_k^b) = 0\) then \(A''\) corresponds to the time \(\tau_k^b\).

The piece of curve \(A A'\) on Fig. 4 is normal to \(\partial \Phi\) (which in coordinates \(q\) is the codimension-\(m\) plane \(q_1 = 0\)). The closed-loop desired trajectory \(X^{c,c}(\cdot)\) is defined as \(q^{c,c}(t) = q^*_d(t)\) on \(\Omega_{2k}\), \(q^{c,c}(t) = q_d(t)\) with \(\alpha = 0\) on \(I_k\), and \(q^{c,c}(t) = 0\) on \(\Omega_{2k+1}\), \(q^{c,c}(t) = q_{2d}(t)\) on \(R^+\). It is impactless.

The choice for \(q_{2d}(\cdot)\) is done essentially to get \(\sigma_V(t_k) \leq 0\) on \(I_k\).

**Remark 4.** It is noteworthy that the proposed strategy implies that \(U_c\) is switched only after stabilisation on \(\partial \Phi\) is achieved. This implies that the period at which a cycle \(\Omega_{2k} \cup I_k \cup \Omega_{2k+1}\) is performed, is lower-bounded by \([t_\infty - t_0]\). If impacts are plastic \((\epsilon_n = 0)\) then the speed of a cycle can be increased while if \(\epsilon_n\) is close to 1 the programmed speed must be smaller. This is logical from an intuitive point of view since this is a consequence on how much kinetic energy impacts dissipate.

**Remark 5.** Due to the fact that we want \(V_{S_d}\) to act as a Lyapunov function for \(P_{S_d}\) in (7) and since the Poincaré mapping fixed point satisfies \(q^*_{S_d,1} = 0\), we have to set \(q_d\) to zero and \(q_{2d}\) constant on the transition phase. However the approach trajectory \((A A')\) on Fig. 4 is not so easy to design. This is what claim 5 below solves.

### 3.3 Conditions for take-off

In the previous subsection we designed the trajectory \(q_{2d}(\cdot)\) to stabilize the system on \(\partial \Phi\). We now deal with the conditions on the control signal \(U_c(q_d, \dot{q}_d, \ddot{q}_d, P_d)\) for take-off at the end of \(\Omega_{2k+1}\). On \([t_f^k, t_d^k]\), the dynamics of the system is defined by:

\[
M(q) \ddot{q} + F(q, \dot{q}) = U_c + D^T \lambda
\]

\[
0 \leq q_1 \perp \lambda \geq 0
\]

with \(F(q, \dot{q}) = C(q, \dot{q}) \ddot{q} + G(q)\). On \([t_f^k, t_d^k]\), the system is permanently constrained, i.e. \(q_1(\cdot) = 0\) and \(\dot{q}_1(\cdot) = 0\). Then (20b) implies \([\text{Glocker}(2001)]\):

\[
0 \leq \dot{q}_1 \perp \lambda \geq 0
\]

(22)

There is take-off at \(t_d^k\) if \(\dot{q}_1(t_d^{k+}) > 0\). From (22) a necessary condition to have \(\dot{q}_1(t_d^{k+}) > 0\) is that \(\lambda(t_d^{k-}) = 0\).

**Claim 4** Consider the closed-loop system (20) (18), during the permanently constraint phase \([t_f^k, t_d^k]\). Detachment is assured if

\[
b(q, \dot{q}, U_{nc}, \lambda_d) > 0
\]

with \(b(q, \dot{q}, U_{nc}, \lambda_d) = DM^{-1}(q) [-F(q, \dot{q}) + U_{nc} - D^T (1 + K_f) \lambda_d]\).

**Proof.**

Let us detail the expression of the Linear Complementarity Problem (LCP) in (22). With the notation of Sec. 3.1, (22) can be rewritten as
\[ 0 \leq D\ddot{q} \perp \lambda \geq 0 \] (23)

From (20a) and (18), one has:

\[ \ddot{q} = M^{-1}(q)[-F(q, \dot{q}) + U_c + D^T \lambda] \]

\[ = M^{-1}(q)[-F(q, \dot{q}) + U_{nc} + (1 + K_f)(D^T \lambda - P_d)] \] (24)

By inserting (24) in (23), one obtains the following LCP:

\[ 0 \leq DM^{-1}(q)[-F(q, \dot{q}) + U_{nc} - (1 + K_f)D^T \lambda_d] + (1 + K_f)DM^{-1}(q)D^T \lambda \perp \lambda \geq 0 \] (25)

which we rewrite more compactly as

\[ 0 \leq b(q, \dot{q}, U_{nc}, \lambda_d) + A(q)\lambda \perp \lambda \geq 0 \] (26)

Let us study the LCP in (26). Since \( A(q) > 0 \) there is a unique solution:

- If \( b(.) > 0 \), then \( b(.) + A(q)\lambda > 0 \) and the orthogonality condition \( b(.) + A(q)\lambda \perp \lambda \) implies \( \lambda = 0 \).
- If \( b(.) < 0 \) then the condition \( 0 \leq b(.) + A(q)\lambda_1 \) and the orthogonality imply \( \lambda = -A^{-1}(q)b(.) > 0 \).
- If \( b(.) = 0 \) then (26) becomes \( 0 \leq A(q)\lambda \perp \lambda \geq 0 \) and \( \lambda = 0 \).

In conclusion, \( \lambda = 0 \) if and only if \( b(q, \dot{q}, U_{nc}, \lambda_d) \geq 0 \). From (24) and (25)

\[ \ddot{q}_1(t) = b(q, \dot{q}, U_{nc}, \lambda_d) + A(q)\lambda \]

If \( \lambda = 0 \), then \( \ddot{q}_1(t) = b(q, \dot{q}, U_{nc}, \lambda_d) \), and a sufficient condition for detachment is:

\[ b(q, \dot{q}, U_{nc}, \lambda_d) > 0 \]

\[ \text{3.4 Control strategy to assure detachment} \]

The only parameter we can tune to force take-off without influencing the variation of the Lyapunov function \( V(.) \) is \( \lambda_d(t) \). By inserting (18) in the expression of \( b(q, \dot{q}, U_{nc}, \lambda_d) \), one gets:

\[ b(q, \dot{q}, U_{nc}, \lambda_d) = DM^{-1}(q)[M(q)\ddot{q}_d - C(q, \dot{q})\dot{q}_d - \gamma_1\dot{q}_1 - \gamma_2\dot{q}_2 - D^T(1 + K_f)\lambda_d] \] (27)

After some computation, (27) and the result of claim 4 provide a sufficient condition for take-off (time argument is dropped in (28)):

\[ \ddot{q}_1 = \left( [M^{-1}_{(q)}]_{11}C_{11}(q, \dot{q}) + [M^{-1}_{(q)}]_{12}C_{21}(q, \dot{q}) \right) \dot{q}_{1d} + \gamma_2[M^{-1}_{(q)}]_{11}\dot{q}_{1d} + \gamma_1[M^{-1}_{(q)}]_{11}\dot{q}_{1d} \]

\[ - \left( [M^{-1}_{(q)}]_{21}C_{11}(q, \dot{q}) + [M^{-1}_{(q)}]_{22}C_{21}(q, \dot{q}) \right) \dot{q}_2 - \gamma_2[M^{-1}_{(q)}]_{21}\dot{q}_2 - \gamma_1[M^{-1}_{(q)}]_{21}\dot{q}_2 \] (28)

with the decomposition of matrix \( M^{-1}(q) \) and \( C(q, \dot{q}) \) as:

\[ M^{-1}(q) = \begin{pmatrix} [M^{-1}_{(q)}]_{11} & [M^{-1}_{(q)}]_{12} \\ [M^{-1}_{(q)}]_{21} & [M^{-1}_{(q)}]_{22} \end{pmatrix} \]

and \( C(q, \dot{q}) = \begin{pmatrix} C_{11}(q, \dot{q}) & C_{12}(q, \dot{q}) \\ C_{21}(q, \dot{q}) & C_{22}(q, \dot{q}) \end{pmatrix} \)
Depending of the sign of $\ddot{q}_2$ and $\dot{q}_2$, $b(.)$ is not necessarily positive with $\lambda_d = 0$. Therefore we have to choose a profile for $\lambda_d(t)$ which is continuously decreasing until $b(q, \dot{q}, U_{nc}, \lambda_d) > 0$, even if a negative desired force is meaningless because it is not reachable (see Fig. 7). The time $t^k_d$ is defined as the first instant such that $\ddot{q}_1(t^k_d) > 0$. Since all signals are bounded, from (28) $t^k_d$ is guaranteed to be bounded as well.

Now we have to assure that the system does not make contact again with $\partial \Phi$ when the control law switches from $U_c(t^k_d)$ to $U_{nc}(t^k_d)$ at the take-off. Then $U_{nc}(t^k_d)$ has to be chosen to guarantee $\ddot{q}_1(t^k_d) > 0$.

At $t^k_d$, the control law is $U_c$ and $q_{1d}(t^k_d) = 0$, $\dot{q}_{1d}(t^k_d) = 0$ and $\ddot{q}_{1d}(t^k_d) = 0$. Therefore (28) is simplified to:

$$\ddot{q}_1(t^k_d) = b(q, \dot{q}, U_{nc}, \lambda_d)$$

$$= - \left( [M^{-1}(q)]_{21} C_{11}(q, \dot{q}) + [M^{-1}(q)]_{22} C_{21}(q, \dot{q}) \right) \ddot{q}_2 - \gamma_2 [M^{-1}(q)]_{21} \dot{q}_2$$

$$- \gamma_1 [M^{-1}(q)]_{21} \ddot{q}_2 - [M^{-1}(q)]_{11}(1 + K_f) \lambda_d(t^k_d) > 0$$

At $t^{k+}_d$, the control law is $U_{nc}$ so that $\lambda_d(k^+_d) = 0$ in $b(q, \dot{q}, U_{nc}, \lambda_d)$ evaluated at $t^k_d$. Since the desired trajectory has to be twice differentiable, let us choose $q_{1d}(t^{k+}_d) = 0$ and $\dot{q}_{1d}(t^{k+}_d) = 0$. We obtain:

$$\ddot{q}_1(t^{k+}_d) = b(q, \dot{q}, U_{nc}, 0)$$

$$= - \left( [M^{-1}(q)]_{21} C_{11}(q, \dot{q}) + [M^{-1}(q)]_{22} C_{21}(q, \dot{q}) \right) \ddot{q}_2 - \gamma_2 [M^{-1}(q)]_{21} \dot{q}_2$$

$$- \gamma_1 [M^{-1}(q)]_{21} \ddot{q}_2 + \ddot{q}_{1d}(t^{k+}_d)$$

Finally the condition to guarantee $q_1(t) > 0$ on $(t^k_d, t^{k+}_d + \epsilon)$, for some $\epsilon > 0$, is that the term $\ddot{q}_{1d}(t^{k+}_d)$ in (30) compensates the loss of the term $- [M^{-1}(q)]_{11}(1 + K_f) \lambda_d$ in (29) due to the switching from $U_c$ to $U_{nc}$. The condition on the desired trajectories at the beginning of the free-motion phase $\Omega_{2k+2}$ is:

$$\ddot{q}_{1d}(t^{k+}_d) \geq \max \left( 0, -[M^{-1}(q(t^{k+}_d))]_{11}(1 + K_f) \lambda_d(t^{k+}_d) \right)$$

(31)

Fig. 7: Trajectory $\lambda_d(t)$
Remark 6. It is interesting to notice that the two transitions $\Omega_{2k} \longrightarrow I_k \longrightarrow \Omega_{2k+1} \longrightarrow \Omega_{2k+2}$, are monitored by desired signals $q_{1d}$ and $\lambda_d$ which violate the complementarity conditions, as shown in Fig. 7.

3.5 Closed-loop stability analysis

The closed-loop dynamical system is now completely defined. It consists of a somewhat complex dynamical system, with complementarity conditions, impact law, and switching torque input.

The aim is now to prove that this dynamical system, seen as an error system with state the vector $(\tilde{q}, \dot{\tilde{q}})$, is stable in the sense of definitions 2 and 3. As we saw this means that asymptotically the trajectory $q^{โส}(\cdot)$ is attained. The closed-loop state can be chosen as $x = (\tilde{q}, \dot{\tilde{q}})$, according to definition 2 which concerns only phases $\Omega_k$.

**Definition 4.** $(CI)$ is the subspace of initial conditions $x(0)$ which assure $t_0 \geq \tau^{k}_{1}$ uniformly along a motion as in (5).

The foregoing developments hold independently of $m$. Let us assume that $m = 1$ now. We will come back to the case $m \geq 2$ later on. $(CI)$ contains the initial data guaranteeing that no impact
occurs before the signal $q_{11}^2(\cdot)$ is frozen. This is very useful because it can then be proved that the conditions for asymptotic strong stability are fulfilled. However in general $x(0) \notin \{CI\}$, so that an impact occurs before $q_{11}^2(\cdot)$ is frozen (i.e. $q_{11}^2(t_0^+) \neq 0$, see figure 6). A specific analysis (completing the one in [Brogliato et al. (2000)Brogliato, Niculescu & Monteiro-Marques]) has to be done.

Assumption 5 The controller $U_t$ in (18) assures that a sequence $\{t_k\}_{k \geq 0}$ of impact times exists, with $\lim_{k \to +\infty} t_k = t_\infty < +\infty$.

One difficulty in the stability analysis along a cycle like in (5), is to assure that initial tracking errors do not increase from one cycle $\Omega_2 \cup I_k \cup \Omega_{2k+1}$ to the next, due to the impacts. As we shall see next, one key point in the stability is the value of the first jump in $V(\cdot)$, i.e. $\sigma_V(t_0)$. Let us calculate the value of the jumps in $V(\cdot)$ at $t_k$:

$$
\sigma_V(t_k) = T_L(t_k) - \frac{1}{2} \gamma_1 q_{d_1}^2(t_k^-) - \frac{1}{2} \dot{q}_d(t_k^-)^T M(q(t_k^-)) \dot{q}_d(t_k^-) + \dot{q}(t_k^-)^T M(q(t_k^-)) \dot{q}(t_k^-)
$$

where $T_L(t_k)$ is the loss of kinetic energy at impact $t_k$, and we used the fact that $\dot{q}_d(t_k^+) = 0$, $q_{d_2}(t_k^+) = 0$, $q_{d_2}(t_k^-) = q_{d_2}(t_k^+)$ and $q_{d_1}(t_k^-) = 0$.

For $k \geq 1$, one has $q_{d_2}(t_k^+) = 0$ and $\dot{q}_d(t_k^-) = 0$. From the above definition of $q_{d_1}(\cdot)$ it is assumed that $t_k^0 < t_0$, so that $\dot{q}_{d_2}(t_0^+) = 0$. If this is not the case then $q_{d_2}(\cdot)$ can be frozen earlier in the process to assure that at the first impact $\dot{q}_{d_2}(t_0) = 0$. Then one has:

$$
\begin{align*}
\sigma_V(t_k) &= T_L(t_k) - \frac{1}{2} \gamma_1 q_{d_1}^2(t_k^-) - \frac{1}{2} \dot{q}_d(t_0^-)^T M(q(t_0^-)) \dot{q}_d(t_0^-) \\
&\quad + M_{11}(q(t_0^-)) \dot{q}_1(t_0^-) \dot{q}_{d_1}(t_0^-) + M_{21}(q(t_0^-)) \dot{q}_{d_2}(t_0^-)
\end{align*}
$$

It is noteworthy that the equalities in (33) hold independently of the chosen impact rule. The only assumption is that impacts dissipate kinetic energy. The above choice for $q_{d_1}(\cdot)$ and switching strategy, is done in order to possibly obtain $\sigma_V(t_0) \leq 0$ and $\sigma_V(t_k) \leq 0$ for $k \geq 1$. Let us now state the following:

Claim 5 Let assumption (5) hold. The system defined by (1) in closed-loop with the controller in (18) and $q_{d_1}(\cdot)$, $q_{d_2}(\cdot)$ as defined above, is:

(i) - Asymptotically strongly stable if $x(0) \notin \{CI\}$.

(ii) - Asymptotically strongly stable if $q_{d_1}^2(\cdot)$ is designed such that at the first impact time of each phase $I_k$ we have $[M_{11}(q(t_0^-)) \dot{q}_1(t_0^-) + \dot{q}_2(t_0^-)^T M_{21}(q(t_0^-))] \dot{q}_{d_1}(t_0^-) \leq 0$.

(iii) - Asymptotically strongly stable if $M_{12} = 0$ and $\epsilon_n = 0$.

(iv) - Asymptotically weakly stable if $M_{12} = 0$ and $0 \leq \epsilon_n < 1$.

Proof. (i) The proof of the first item can be found in [Brogliato et al. (2000)Brogliato, Niculescu & Monteiro-Marques]. Instances for which $\{CI\} \neq \emptyset$ can be calculated in simple cases like one degree-of-freedom systems. They occur under somewhat stringent conditions.

(ii) It follows immediately from (33) that if $[M_{11}(q(t_0^-)) \dot{q}_1(t_0^-) + \dot{q}_2(t_0^-)^T M_{21}(q(t_0^-))] \dot{q}_{d_1}(t_0^-) \leq 0$ then $\sigma_V(t_0) \leq 0$. And then we can use the proof done in [Brogliato et al. (2000)Brogliato, Niculescu & Monteiro-Marques].

(iii) The proof of the third item follows the same line but in this case $\sigma_V(t_0)$ has to be shown to be non-negative because it is not equal to the kinetic energy loss. Let us consider Moreau’s collision rule as written in (3). Notice that since $m = 1$

$$
\text{prox}_{M(q(t_0))}[M^{-1}(q(t_0))N_{\Phi}(q(t_0)); \dot{q}(t_0^-)^T M(q(t_0))] n_q n_q
$$

(34)
where \( n_q = \frac{M^{-1}(q(t_0))D^T}{\sqrt{DM(q(t_0))D^T}} \in \mathbb{R}^{n \times 1} \) is the normal vector in the kinetic metric [Brogliato(1999), chapter 6] and \( D = [10 \ldots 0] \in \mathbb{R}^{n \times 1} \). One gets from (34) and using for instance the Schur complement to calculate \( M^{-1}(q(t_0)) \) [Horn & Johnson(1999), p.472]

\[
\text{prox}_{M(q(t_0))}[M^{-1}(q(t_0))N\phi(q(t_0)))\hat{q}(t_0)] = \hat{q}_1(t_0) \left( M_{22}^{-1}(q(t_0))M_{12}^T(q(t_0)) \right)
\]

(35)

Therefore from (3) one gets

\[
\begin{align*}
\sigma_q(t_k) &= -(1 + e_n)\hat{q}_1(t_k^-) \\
\sigma_q(t_k) &= (1 + e_n)M_{22}^{-1}(q(t_k))M_{12}^T(q(t_k))\hat{q}_1(t_k^-)
\end{align*}
\]

(36)

From (36) and (33), after some manipulations we arrive at the following:

\[
\begin{align*}
\sigma_V(t_0) &= \frac{c^2}{2}(M_{11}(q(t_0)) - M_{12}(q(t_0))M_{22}^{-1}(q(t_0))M_{12}^T(q(t_0))\hat{q}_1^2(t_0^-)) \\
&\quad - \frac{1}{2}M_{11}(q(t_0))\hat{q}_2^2(t_0^-) + M_{11}(q(t_0))\hat{q}_1(t_0^-)\hat{q}_1(t_0^-) \\
&\quad + \hat{q}_2(t_0^-)^T M_{21}(q(t_0))\hat{q}_1(t_0^-) - \frac{1}{2}\gamma_1 \hat{q}_1^2(t_0^-)
\end{align*}
\]

(37)

It follows immediately from (37) that if \( e_n = 0 \) and \( M_{21} = 0 \) then

\[
\sigma_V(t_0) = -\frac{1}{2}M_{11}(q(t_0))\hat{q}_1^2(t_0^-) - \frac{1}{\gamma_1} \hat{q}_1^2(t_0^-) \leq 0
\]

(38)

Hence strong stability is assured and the third item is proved.

(iv) If \( M_{12} = 0 \) and \( 0 \leq e_n < 1 \), one has

\[
V(t) = V_1(t) + V_2(t) = \frac{1}{2}M_{11}(q(t_0))\hat{q}_1^2(t) + \frac{1}{2}\hat{q}_2(t)^T M_{22}(q(t_0))\hat{q}_2(t)
\]

\[
+ \frac{1}{2}\gamma_1 \hat{q}_1^2(t) + \frac{1}{\gamma_1} \hat{q}_2^2(t)
\]

(39)

From (39), \( V_2(t) \) and \( V_1(t) \) are decoupled, then \( V_2(t) \) is a smooth function and \( \dot{V}_2(t) \leq 0 \) for all \( t \). Therefore \( V_2(t_\infty) \leq V_2(\tau_0^k) \). Since \( V_1(t_\infty) = 0 \leq V_1(\tau_0^k) \) one has:

\[
V(t_\infty) \leq V(\tau_0^k)
\]

(40)

Then item (iv) of claim 5 is proved.

\[ \square \]

4 A Weakly-Stable Scheme

It is of some interest to design a feedback control strategy whose closed-loop stability can be analyzed with claim 2. The control law used in this section has the same global structure than in Figs. 5-8. However the nonlinear controller block is based on the scheme presented in [Slotine & Li(1988)]. Let us propose the following:
If the controller $\mathbf{q}$ and with the particular choice of Assumption 6

Consider the closed-loop system (15) (41) on the time interval $\lim_{t \to \infty} t_k = t_\infty < +\infty$.

Let assumption 6 hold, where $K_{f} > 0$, $P_{d} = D^{T} \lambda_{d}$ is the desired contact force during permanently constraint motion.

\textbf{Assumption 6} The controller $U_{t}$ in (41) assures that a sequence $\{t_{k}\}_{k \geq 0}$ of impact times exists, with $\lim_{k \to +\infty} t_{k} = t_{\infty} < +\infty$.

Let us consider the following positive functions:

\begin{align*}
V_{1}(t, s) &= \frac{1}{2} s(t)^{T} M(q) s(t) \\
V_{2}(t, s) &= \frac{1}{2} s(t)^{T} M(q) s(t) + \gamma_{2} \gamma_{1} q(t)^{T} \tilde{q}(t)
\end{align*}

In case $\Phi = \mathbb{R}^{n}$, any of the two functions $V_{1}(\cdot)$ and $V_{2}(\cdot)$ can be used in order to prove the stability of the closed-loop system (15) (41) [Lozano et al.(2000)Lozano, Brogliato, Egeland & Maschke, §6.2.5] [Spong et al.(1990)Spong, Ortega & Kelly]. In the case of interest here $\Phi \subset \mathbb{R}^{n}$, things complicate and as we shall see, both functions are needed for the stability analysis. In particular one has $V_{1}(t) \leq 0$ and $V_{2}(t) \leq 0$ along the closed-loop system as long as $T(q)u = U_{nc}$ in (41), see [Lozano et al.(2000)Lozano, Brogliato, Egeland & Maschke] [Slotine & Li(1988)]. It is noteworthy that claim 6 is proved with $V_{2}(\cdot)$, while claim 7 is based on $V_{1}(\cdot)$ and the choice of the closed-loop state vector $x(t) = s(t)$.

\textbf{Claim 6 (upper-bounds)} Consider the closed-loop system (15) (41) on the time interval $[\tau_{k}^{+}, t_{0}]$, and with the particular choice of $q_{\tau_{k}^{+}}(\cdot)$ given in (55) (56) (57) in appendix A. One has:

1. \[ |q_{\tau_{k}^{+}}(t_{0})| \leq \sqrt{\frac{V_{2}(\tau_{k}^{+})}{\gamma_{2} \gamma_{1}}} \]
2. \[ |\tilde{q}_{\tau_{k}^{+}}(t_{0})| \leq K_{0} V_{2}^{1/4}(\tau_{k}^{+}) \]

where $K_{0} \geq 0$.

\textbf{Proof} The proof of claim 6 is provided in appendix A.

\textbf{Claim 7} Let assumption 6 hold, $e_{n} \in (0, 1)$ and $q_{\tau_{k}^{+}}$ be defined as in (55)-(57). Consider the system defined by (15) in closed-loop with the controller in (41).

1. \[ \text{If the controller } T(q)u \text{ in (41) assures that } ||\tilde{q}(\tau_{k}^{+})|| < \epsilon, \epsilon > 0 \text{ for all } k \text{ over the cycles, then the system initialized on } \Omega_{0} \text{ with } V_{2}(\tau_{0}^{+}) \leq 1 \text{ satisfies the requirements of claim 2 and is therefore practically } \Omega\text{-weakly stable with closed-loop state } x(\cdot) = s(\cdot). \]
2. \[ \text{If the controller } T(q)u \text{ in (41) assures that } ||\tilde{q}_{k}(t_{k+1})|| \leq ||\tilde{q}_{k}(t_{k})||, \text{ for all } t_{k} \text{ on } [t_{0}, t_{\infty}], \text{ then the system initialized on } \Omega_{0} \text{ with } V_{2}(\tau_{0}^{+}) \leq 1 \text{ satisfies the requirements of claim 2 and is therefore practically } \Omega\text{-weakly stable with closed-loop state } x(\cdot) = [s(\cdot), \tilde{q}(\cdot)]. \]
Notice that $\epsilon$ in (i) need not be small, it is however important that it does not depend on the cycle index in (5). Note also that $V_1(t) \leq V_2(t)$ for all $t \geq 0$ so that $V_1(\tau_0^1) \leq V_2(\tau_0^0) \leq 1$ in (i).

**Proof**
The proof of claim 7 is provided in appendix B.

**Claim 8** Consider the closed-loop system (15) (41). The tracking errors satisfy $\|\tilde{q}(t)\| \leq 2R$ and $\|\tilde{q}(t)\| \leq (1+2\gamma)R$ for all $t \in \Omega$, and $\|s(t)\| \leq R$ for all $t \in \Omega$, with $R = \left(\frac{2}{\lambda_{\min}(M(q))}e^{-\gamma(t_0^1-t_\infty)}(1+K+\epsilon')\right)^{\frac{1}{2}}$.

**Proof**
From the definition of $s(t)$ one has $\tilde{q} = \frac{1}{p+1}s$ where $p \in \mathfrak{C}$ is the Laplace variable. Then on $[t^*_k, t]$ with $t \in \Omega$, $\tilde{q}(t)$ is the response of a linear filter with input $s(.)$. One obtains:

$$\tilde{q}(t) = e^{-(t-t_k^*)}s(t^*_k) + \int_{(t^*_k,t]} e^{-(t-\tau)}s(\tau)d\tau \quad (43)$$

Equality (43) implies the following inequality:

$$\|\tilde{q}(t)\| \leq \|s(t^*_k)\| + e^{-t}(t-t_k^*)\|s\|_\infty \quad (44)$$

where $\|x\|_\infty = \sup_{t \geq 0} |x(t)|$ is the $L_\infty$ norm. From claim 7, one has $\|s\| \leq R$ so (44) becomes:

$$\|\tilde{q}(t)\| \leq \left[1 + e^{-t}(t-t_k^*)\right]R \leq 2R \quad (45)$$

From the definition of $s(t)$ one has $\tilde{q}(t) = s(t) - \gamma_2\tilde{q}(t)$ then

$$\|\tilde{q}(t)\| \leq \|s(t)\| + \gamma_2\|\tilde{q}(t)\| \quad (46)$$

By inserting (45) in (46), and using the fact that $\|s\| \leq R$, one obtains

$$\|\tilde{q}(t)\| \leq \left[1 + 2\gamma_2\right]R \quad (47)$$

**Claim 9 (plastic impact)** Let assumption 6 hold, $e_n = 0$ and $q_{1d}$ be defined as in (55)-(57). The system defined by (15) in closed-loop with the controller in (41) initialized on $\Omega_0$ with $V_2(\tau_0^0) \leq 1$ satisfies the requirements of claim 2 and is therefore practically $\Omega$-weakly stable with closed-loop state $x(.) = [s(.), \tilde{q}(.)]$.

**Proof**
As $e_n = 0$, there is only one impact per phase $I_k$, and then the item (b) of claim 2 is useless. Items (a) and (d) are proved in the proof of claim 7(ii).

Then the system (15) with the controller (41) satisfies all the requirements of claim 2 with $\epsilon \neq 0$. Consequently it is practically $\Omega$-weakly stable with $x(.) = [s(.), \tilde{q}(.)]$. 


5 Simulation Examples

The control scheme in (18) is tested in simulation on a 2-link planar manipulator for the simplest case of a scalar constraint. The constraint surface corresponds to the ground \((y = 0)\). The natural generalized coordinates so that the dynamics fits with (15), with \(m = 1\), are the work-space coordinates \((x, y)\). We take:

\[
q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}, \ y > 0
\]

![Fig. 9: 2-link planar manipulator](image)

5.1 Asymptotic convergence

Figure 10 shows the evolution of \(q_1(t)\) and \(q_2(t)\) during cyclic tasks as in (5). On the graph of \(q_1\), the asymptotic convergence of the controller is exhibited as the value of \(\alpha V(\tau_k^f)\) decreases exponentially. The graph of \(q_2\) shows the coupling between \(q_1\) and \(q_2\). At each impact time a jump in \(\dot{q}_2\) occurs. The periodic step on \(q_{2d}\) corresponds to the transition phase during which \(q_{2d}\) needs to be frozen.

5.2 Robustness

In this subsection, we study the robustness of the controller with respect to the uncertainty on the constraint position. The robustness of closed-loop systems is a crucial step towards their implementation. The work that is performed here is essentially numerical, but may provide useful informations on the controller robustness and its performance in practice. The location of the constraint surface is not known accurately. As seen on Fig. 11, two situations may be considered:

- If \(c < 0\), the estimated position of the constraint is lower than the real position. In this case the desired trajectories decrease toward \(q_{1d}(\tau_k^f) = -\alpha V(\tau_k^f) - |c|\) instead of \(q_{1d}(\tau_k^f) = -\alpha V(\tau_k^f)\).

The error \(c\) can be incorporated in the term \(-\alpha V(\tau_k^f)\) and the stability of the transition phase is not changed. During the constraint phase the controller is:

\[
U_c = U_{nc}^{ideal} - (P_d + \gamma_1 \begin{bmatrix} |c| \\ 0 \end{bmatrix}) + K_f(P_q - P_d)
\]

The error term \(\gamma_1|c|\) is added to the desired force \(P_d\) and contributes to keep the contact with the surface during the constrained phase.
The system remains stable but it loses its asymptotic stability: If the tracking is perfect $V(\tau_k^0) = 0$ and $q_{1d}^* = -|c|$, so that the system does not approach the surface tangentially and rebounds occur. Due to item c) in Sec. 1.5, asymptotic stability is not preserved. An example is depicted on Fig. 12.

- If $c > 0$, the estimated position of the constraint is above the real position. If the tracking is perfect $V(\tau_k^0) = 0$, the desired trajectory decreases toward $q_{1d} = c$ and the system never reaches the constraint. There is no convergence (see Fig. 13).

This problem can be solved by monitoring the time of stabilization. If there is no stabilization after an estimated time $t_{\infty}$, the estimated position of the constraint is refreshed as $\hat{q}_{1c}^{new} = \hat{q}_{1c}^{old} - \epsilon$. After a finite number of iterations, one gets $\hat{q}_{1c} < 0$. The system is in the previous
situation $c < 0$ and the stability is preserved. Figure 14 shows an example of self-adjustment of the estimated position of the constraint. When tracking is not perfect and $\alpha V(\tau^0_k) > c$, the transition phase is able to stabilize the system on the surface $\partial \Phi$. But during the constraint phase, the control law is:

$$U_c = U_{nc}^{ideal} - (P_d - \gamma_1 \begin{bmatrix} c \\ 0 \end{bmatrix}) + K_f (P_q - P_d)$$

$P_d$ must be chosen large enough compared to $\gamma_1 c$ to be sure that the system keeps the contact with the surface during the whole constraint phase.

6 Multiple Impacts

This section extends the previous controller framework to the case of multiple impact.
Definition 5 (Multiple impact). A multiple impact is an impact into a singularity as in definition 1. If the singularity has codimension $\alpha$, the multiple impact is named an $\alpha$-impact. We also denote the singularity as $\Sigma^\alpha$.

The difficulty created by stabilization at singularities of $\partial \Phi$, is that the way the system attains the singularity, may vary: the system may hit the singularity directly, or hit one or several surfaces $\Sigma_i$ (through a finite or infinite number of impacts) before attaining the singularity, as depicted on Fig. 15. Let us define $\theta^\text{kin}_{ij}$ as the kinetic angle between two surfaces $\Sigma_i$ and $\Sigma_j$, i.e. the angle in the kinetic metric defined as $x^T M(q)y$ for $n$-vectors $x$ and $y$. In the following we shall restrict ourselves to $m = 2$ (two constraints) and $\theta^{12}_{\text{kin}} \leq \frac{\pi}{2}$. The reasons for this choice are the following:

- Let us further assume that $e_n = 0$ in (2). As shown in [Paoli(2002)], the conditions $\theta^{12}_{\text{kin}} \leq \frac{\pi}{2}$ and $e_n = 0$ imply that trajectories (i.e. solutions of the closed-loop system) are continuous with respect to the initial conditions.
- Let us take $e_n \in [0, 1]$ and assume that the system performs a constrained motion phase on $\Sigma_1$ before hitting $\partial \Phi$ at $q$. Then $\dot{q}(t_k^-) \in N_\Phi(q)$ so that from (3) $\dot{q}(t_k^+) = -e_n \dot{q}(t_k^-)$. This means that after the shock the velocity is again tangent to $\Sigma_1$, and the state at $t_k^+$ is consistent with the constraint $q_1^1 = 0$.

The goal is to stabilize the system on the singularity $\Sigma^2 = \Sigma_1 \cap \Sigma_2$ during the transition phase. Several cases are examined next, and the controller in (18) is used.
6.1 Stabilisation with a 2-impact

In this case, the two surfaces are reached simultaneously. This means that at each impact time \( t_k \), one has \( q_1^i(t_k) = q_2^i(t_k) = 0 \), and the closed-loop analysis made in [Brogliato et al. (2000)] for a 1-impact can be adapted immediately to such a 2-impact. If \( \epsilon_n = 0 \) the continuity of solutions with respect to initial data allows us to further conclude that this strategy possesses some robustness properties. Indeed even if the system does not strike right at the singularity \( \Sigma^2 \), but in a neighborhood of it, then stabilisation still occurs with the same controller as depicted on Fig. 15 (b). If \( \epsilon_n > 0 \) then such a strategy does not seem amenable in practice due to its lack of robustness (because trajectories impacting in a neighborhood of \( \Sigma^2 \) may drastically differ from those impacting on \( \Sigma^2 \)).

6.2 Impact on one surface before a 2-impact

In this case the transition phase is decomposed into two main steps: a first subphase during which the system is stabilized on \( \Sigma_1 \) (without impact on \( \Sigma_2 \)). And a second subphase during which the system is stabilized on \( \Sigma^2 \). The property in the second item just above, assures that the system remains on \( \Sigma_1 \) during this second subphase. The proof of stability for the first phase is similar to the 1-impact case if we take \( q_1 = [q_1^1] \) and \( q_2 = \begin{bmatrix} q_2^1 \\ q_2^2 \end{bmatrix} \). During the second phase, the system is in a constraint motion, and the closed-loop dynamics is:

\[
M(q)\ddot{q} = -C(q, \dot{q})\dot{q} - \gamma_1 \dot{q} - \gamma_2 \dot{q} + (1 + K_{f1})(\lambda_{q1} - \lambda_{d1})\nabla_q q_1^i
\] (48)

The system is stabilized on \( \Sigma^2 \) using the signal \( q_{1d}^{i*} = \begin{bmatrix} 0 \\ q_{1d}^{i*} \end{bmatrix} \), where \( q_{1d}^{i*} \) has the same form as \( q_1^{i*} \) in the previous phase and decreases towards \(-\alpha_2 V(x_0^k)\).

With the same proof as before, we need to show that the inequality:

\[
V(x(t_{k+1}^-), t_{k+1}^-) - V(x(t_k^+, t_k^+)) \leq 0
\] (49)

holds. One obtains:

\[
V(x(t_{k+1}^-), t_{k+1}^-) - V(x(t_k^+, t_k^+)) \\
= \int_{(t_k, t_{k+1})} \dot{V}(t)dt \\
= \int_{(t_k, t_{k+1})} \dot{q}^T M \ddot{q} + \dot{q}^T \frac{M}{2} \ddot{q} + \gamma_1 \dot{q}^T \ddot{q}dt \\
= \int_{(t_k, t_{k+1})} \left( \dot{q}^T [-C\dot{q} - \gamma_1 \dot{q} - \gamma_2 \dot{q} + (1 + k_{f1})(\lambda_{q1} - \lambda_{d1})\nabla_q q_1^i] \\
+ \dot{q}^T \frac{M}{2} \ddot{q} + \gamma_1 \dot{q}^T \ddot{q} \right)dt \\
= \int_{(t_k, t_{k+1})} -\gamma_2 \dot{q}^T \ddot{q}dt + \gamma_1 \int_{(t_k, t_{k+1})} \dot{q}_1^T q_{1d}^{i*}dt \\
+ \int_{(t_k, t_{k+1})} \dot{q}^T (1 + k_{f1})(\lambda_{q1} - \lambda_{d1})\nabla_q q_1^i dt \\
= \int_{(t_k, t_{k+1})} -\gamma_2 \dot{q}^T \ddot{q}dt \leq 0
\]

The last but one equality is deduced from the preceding one using the property that the matrix \( 2C(q, \dot{q}) - M(q, \dot{q}) \) is skew-symmetric [Lozano et al. (2000)] Lozano, Brogliato, Egeland &
Maschke], and $\dot{q}^T \ddot{q} = \dot{q}^T q_1^*$. The last inequality is deduced from the preceding equality since $\dot{q}^T (1 + k_{f1}) (\lambda_{q_1} - \lambda_{d1}) \nabla_{\dot{q}} q_1^T = 0$ and $q_1^T q_1^{t_{k+1}} = 0$ since $q_1(t_k) = 0$ during the 2-impact. A proof similar to the 1-impact case allows one to conclude on asymptotic stability of this 2-impact tracking problem. However we have supposed that there is no impact on the second surface during the first transition subphase. This may not always be realizable in practice, and may also be seen as a lack of robustness for stabilisation in a neighborhood of singularities.

6.3 Case (c) : General case

In this case the system can collide indifferently the two surfaces. There are several 1-impacts on both surfaces before the 2-impact occurs. In this situation we do not have $q_1(t_k) = 0$ for all impacts (this true only during the 2-impact). The weak stability of the transition phase can be obtained by studying the variation of $V(q(t), \dot{q}(t), t)$ between two impacts on the same surface ($\Sigma_1$ or $\Sigma_2$).

![Fig. 16: General case](image)

Let us choose the following notations: $t_{2k}$ is for impacts on $\Sigma_2$, and $t_{2k+1}$ is for impacts on $\Sigma_1$. Let us also choose

$$q_1^* = \begin{bmatrix} q_1^* & q_1^* \\ q_1^* & q_1^* \end{bmatrix} = \begin{bmatrix} -\alpha_1 V(x(\tau_0^k), \tau_0^k) \\ -\alpha_2 V(x(\tau_0^k), \tau_0^k) \end{bmatrix}.$$  

Let us now calculate the following variation:

$$V(t_{2(k+1)}) - V(t_{2k})$$

$$= \int_{(t_{2k}, t_{2k+1})} \dot{V}(t) dt + \sigma V(t_{2k+1}) + \int_{(t_{2k+1}, t_{2(k+1)})} \dot{V}(t) dt$$

$$= \sigma V(t_{2k+1}) - \gamma_2 \int_{(t_{2k}, t_{2k+1})} \dot{q}^T \dot{q} dt - \gamma_2 \int_{(t_{2k+1}, t_{2(k+1)})} \dot{q}^T \dot{q} dt$$

$$+ \gamma_1 q_1^T T \begin{bmatrix} q_1^{t_{2k+1}} \\ q_1^{t_{2k+1}} \end{bmatrix} + \gamma_1 q_1^T T \begin{bmatrix} q_1^{t_{2k+1}} \\ q_1^{t_{2k+1}} \end{bmatrix}$$

$$= \Delta + \gamma_1 q_1^T T (q_1(t_{2(k+1)}) - q_1(t_{2k}))$$

$$= \Delta + \gamma_1 q_1^T T (q_1(t_{2(k+1)}) - q_1(t_{2k}))$$

$$= \Delta + \gamma_1 q_1^T T (q_1(t_{2(k+1)}) - q_1(t_{2k}))$$

where $\Delta$ is the sum of all negative terms in (50). Equality (51) is deduced from (50) since $q_1^+(t_{2k}) = 0$ for all $k$. With $\alpha_1 = 0$, we have $q_1^* = 0$ and then :

$$V(t_{2(k+1)}) - V(t_{2k}) < 0$$
The strategy is to take $\alpha_1 = 0$ (target A, see Fig. 16) at the beginning of the transition phase to stabilize the system on $\Sigma_2$, and to switch to $\alpha_2 = 0$, $\alpha_1 > 0$ (target B, see Fig. 16) when the system is on $\Sigma_2$ (or to switch to the previous case).

7 Conclusion

This paper deals with the tracking control of fully actuated Lagrangian systems subject to frictionless unilateral constraints. These dynamical systems are named complementarity systems because they involve complementarity conditions. They are nonsmooth because the velocity may possess discontinuities (at impact times), so that the acceleration and the contact force are measures. They may be seen as a complex mixture of ordinary differential equations, differential-algebraic equations, and measure differential equations. The extension of the tracking control of unconstrained (or persistently constrained) Lagrangian systems, towards complementarity Lagrangian systems, is not trivial. The aim of this paper is to study the design of a feedback controller for these specific nonsmooth systems, supposed to perform a general cyclic impacting task. First the stability framework dedicated to study these systems is recalled, and some definitions and claims are given. Then we focus on the condition of existence of closed-loop trajectories (usually called desired trajectories in unconstrained motion tracking control) which assure the asymptotic stability in closed-loop, i.e. the asymptotic convergence of the generalized coordinates towards some closed-loop invariant trajectory. The second part of this paper is devoted to numerically study an example: a 2-link planar manipulator subject to a single unilateral constraint. This example allows us to exhibit some results on the robustness of this control framework in term of uncertainty of the constraint surface position. The effect of measurement noise is also studied. It is shown that the proposed scheme possesses some interesting robustness properties. The last part of this paper is devoted to the case of so-called multiple impacts (an accurate definition is provided). Some specific difficulties related to the constraint boundary geometry, are highlighted, and some possible manners to extend the single constraint case are indicated.

References


A Proof of Claim 6

i) On \([\tau_0^k, t_0]\), one has \(V_2(t) \leq 0\), so that \(V_2(t_0^-) \leq V_2(\tau_0^k)\). Therefore from (17)

\[
V_2(\tau_0^k) \geq V_2(t_0^-) \geq \gamma_2 \gamma_1 \dot{q}(t_0^-)^T \ddot{q}(t_0^-) \geq \gamma_2 \gamma_1 \dot{q}^2(t_0^-)
\]

so that

\[
\sqrt[\gamma_2 \gamma_1]{V_2(\tau_0^k)} \geq |q_1(t_0) - q_1^*(t_0^-)| = |\dot{q}_1^*(t_0^-)|
\]

(54)

because \(q_1(t_0) = 0\). The desired trajectory \(q_1^*(\cdot)\) is chosen as a decreasing function, and from inequation (54) we have \(t_{\text{min}} \leq t_0 \leq t_{\text{max}}\), where \(q_1^*(t_{\text{min}}) = \sqrt[\gamma_2 \gamma_1]{V_2(\tau_0^k)}\) and \(q_1^*(t_{\text{max}}) = -\sqrt[\gamma_2 \gamma_1]{V_2(\tau_0^k)}\) (see Fig. 17).

Remark 7. From the value of \(t_{\text{max}}\), it follows that if \(\alpha V_1(\tau_0^k) > \sqrt[\gamma_2 \gamma_1]{V_2(\tau_0^k)}\), then \(t_0 \leq \tau_1^k\) on the cycle \(k\).

ii) The signal \(q_1^*(t)\) is a function decreasing toward \(-\alpha V_1(\tau_0^k)\). Let us use a degree 3 polynomial with limit conditions \((t_{\text{ini}} = \tau_0^k\) and \(t_{\text{end}} = \tau_1^k\)). After some manipulations we will exhibit an upper-bound of \(\dot{q}_1^*(t)\) on \([t_{\text{min}}, t_{\text{max}}]\). Since \(t_0 \in [t_{\text{min}}, t_{\text{max}}]\) then:

\[
\begin{align*}
q_1^*(t) &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 \\
\dot{q}_1^*(t) &= 3a_3 t^2 + 2a_2 t + a_1 \\
& \text{at } t_{\text{ini}} = \tau_0^k: \ q_1^*(t_{\text{ini}}) = q_1^*(\tau_0^k) \quad \text{and} \quad \dot{q}_1^*(t_{\text{ini}}) = 0 \\
& \text{at } t_{\text{end}} = \tau_1^k: \ q_1^*(t_{\text{end}}) = -\alpha V_1(\tau_0^k) \quad \text{and} \quad \dot{q}_1^*(t_{\text{end}}) = 0
\end{align*}
\]

(55)

To compute \(\max_{t \in [t_{\text{min}}, t_{\text{max}}]} |q_1^*(t)|\), let us make a time scaling transformation \(t' = t'(t)\), such that \(t'(\tau_0^k) = 0\) and \(t'(-\tau_0^k) = 1\), as \(t'(t) = \frac{t - \tau_0^k}{\tau_1^k - \tau_0^k}\). We obtain:

\[
\begin{align*}
a_3 &= 2[q_1^*(\tau_0^k) + \alpha V_1(\tau_0^k)] \\
a_2 &= -3[q_1^*(\tau_0^k) + \alpha V_1(\tau_0^k)] \\
a_1 &= 0 \\
a_0 &= q_1^*(\tau_0^k)
\end{align*}
\]

(56)
and the signal $q_{id}^*(t)$ is:

$$
q_{id}^*(t') = [q_{id}^*(\tau_0^k) + \alpha V_1(\tau_0^k)](2t'^3 - 3t'^2) + q_{id}^*(\tau_0^k)
$$

(57)

Then,

$$
\dot{q}_{id}^*(t') = -6[q_{id}^*(\tau_0^k) + \alpha V_1(\tau_0^k)](1 - t')t'
$$

From (57), we see that $q_{id}^*(t')$ is decreasing on $t' \in [0, 1]$. Consequently

$$
q_{id}(t'_0) \leq q_{id}(t'_{\text{min}}) \leq \sqrt{\frac{V_2(\tau_0^k)}{\gamma_2 \gamma_1}}
$$

(58)

By inserting (57) in (58), one obtains:

$$
[q_{id}^*(\tau_0^k) + \alpha V_1(\tau_0^k)](2t'^3 - 3t'^2) + q_{id}^*(\tau_0^k) \leq \sqrt{\frac{V_2(\tau_0^k)}{\gamma_2 \gamma_1}}
$$

(59)

Then,

$$
t'_0(3 - 2t'_0) \geq \frac{q_{id}^*(\tau_0^k) - \sqrt{\frac{V_2(\tau_0^k)}{\gamma_2 \gamma_1}}}{q_{id}^*(\tau_0^k) + \alpha V_1(\tau_0^k)}
$$

(60)

For $t \geq 0$, one has $t(2 - t) \geq t^2(3 - 2t)$, therefore:

$$
t'_0(2 - t'_0) \geq \frac{q_{id}^*(\tau_0^k) - \sqrt{\frac{V_2(\tau_0^k)}{\gamma_2 \gamma_1}}}{q_{id}^*(\tau_0^k) + \alpha V_1(\tau_0^k)}
$$

(61)

The root of $t(2 - t) = a$ is $t = 1 - \sqrt{1 - a}$, from which it follows that:

$$
t'_0 \geq 1 - \frac{1 - \frac{q_{id}^*(\tau_0^k) - \sqrt{\frac{V_2(\tau_0^k)}{\gamma_2 \gamma_1}}}{q_{id}^*(\tau_0^k) + \alpha V_1(\tau_0^k)}}{\frac{\alpha V_1(\tau_0^k) + \frac{V_2(\tau_0^k)}{\gamma_2 \gamma_1}}{\alpha V_1(\tau_0^k) + q_{id}^*(\tau_0^k)}} = t'_{\text{min}}
$$

(62)

On $[t_{\text{min}}, t_{\text{max}}]$, one has $|\dot{q}_{id}(t')| \leq |\dot{q}_{id}(t'_{\text{min}})|$. Thus:

$$
|\dot{q}_{id}(t'_0)| \leq -6(q_{id}^*(\tau_0^k) + \alpha V_1(\tau_0^k))(1 - t'_{\text{min}})t'_{\text{min}}
$$

$$
\leq 6(q_{id}^*(\tau_0^k) + \alpha V_1(\tau_0^k))\sqrt{\frac{\alpha V_1(\tau_0^k) + \frac{V_2(\tau_0^k)}{\gamma_2 \gamma_1}}{\alpha V_1(\tau_0^k) + q_{id}^*(\tau_0^k)}}
$$

(63)

$$
\leq 6\sqrt{(q_{id}^*(\tau_0^k) + \alpha V_1(\tau_0^k))(\alpha V_1(\tau_0^k) + \sqrt{\frac{V_2(\tau_0^k)}{\gamma_2 \gamma_1}})}
$$

Now we change back the time variable $t'$ to $t$.

$$
|\dot{q}_{id}(t_0)| \leq 6\frac{\tau_1 - \tau_0}{\tau_1 - \tau_0} \sqrt{(q_{id}^*(\tau_0^k) + \alpha V_1(\tau_0^k))(\alpha V_1(\tau_0^k) + \sqrt{\frac{V_2(\tau_0^k)}{\gamma_2 \gamma_1}})}
$$

(64)

From (42) one has $V_2(t) \geq V_1(t)$. Thus equation (64) becomes:

$$
|\dot{q}_{id}(t_0)| \leq 6\frac{\tau_1 - \tau_0}{\tau_1 - \tau_0} \sqrt{(q_{id}^*(\tau_0^k) + \alpha V_2(\tau_0^k))(\alpha V_2(\tau_0^k) + \sqrt{\frac{V_2(\tau_0^k)}{\gamma_2 \gamma_1}})}
$$

(65)
Let us define the parameter:

$$K_0 = \frac{6}{\tau^k - \tau^0} \left[ \alpha q_1^*(\tau^0) + q_1^*(\tau^k) \right] \sqrt{\frac{1}{\gamma_1 \gamma_2}} + \alpha^2 + \alpha \sqrt{\frac{1}{\gamma_2 \gamma_1}}$$  \hspace{1cm} (66)$$

If the system is initialized with $V_2(\tau^0) \leq 1$, then $V_2^{1/4} \geq V_2^{1/2} \geq V_2$ and inequality (65) becomes:

$$|\dot{q}^*_1(t^-)| \leq K_0 V_2^{1/4}(\tau^k)$$  \hspace{1cm} (67)$$

Then item (ii) of claim 6 is proved.
B Proof of Claim 7

(i) Proof of the first result of claim 7: Let us show that conditions (a) (b) and (d) in claim 2 are satisfied.

(a) Outside phase $I_k$ it can be computed that $\dot{V}_1(t) = -\gamma_1 s(t)^T s(t)$ [Slotine & Li(1988)], then from (42) one has:

$$\|s(t)\|^2 \geq \frac{2}{\lambda_{\text{max}}(M(q))} V_1(t)$$

(68)

where $\lambda_{\text{min}}(.)$ and $\lambda_{\text{max}}(.)$ denote the minimum and maximum eigenvalues, respectively. It follows that:

$$\dot{V}_1(t) \leq - \frac{2\gamma_1}{\lambda_{\text{max}}(M(q))} V_1(t)$$

(69)

Therefore condition (a) of claim 2 is satisfied with $\gamma = \frac{2\gamma_1}{\lambda_{\text{max}}(M(q))}$.

(b) After the first impact the closed-loop equation of the system defined by (41) and (15) is:

$$M(q)\dot{s}(t) + Cs(t) + \gamma_1 \ddot{s}(t) = 0$$

(70)

Let us calculate $\dot{V}_1(t)$ along trajectories of (70):

$$\dot{V}_1(t) = \frac{1}{2} s(t)^T \dot{M}(t) s(t) + s(t)^T M(q) \ddot{s}(t)$$

(71)

where $\dot{M}(t) = \frac{d}{dt}[M(q(t))]$. By introducing (70) in (71) and using the fact that $\dot{M}(t) - 2C(q, \ddot{q})$ is a skew-symmetric matrix [Lozano et al.(2000)Lozano, Brogliato, Egeland & Maschke, lemma 5.4] one obtains:

$$\dot{V}_1(t) = -\gamma_1 s(t)^T \ddot{s}(t)$$

(72)

After the first impact $q^*_d$ is constant, $\bar{q}$ and $\tilde{q}$ are defined from (19) as $\bar{q}(t) = \left( \begin{array}{c} q_1(t) - q^*_d \\ q_2(t) - q^*_d \\ \end{array} \right)$ and $\tilde{q}(t) = \left( \begin{array}{c} q_1(t) - q^*_d \\ q_2(t) - q^*_d \\ \end{array} \right)$. Then $\dot{\tilde{q}}(t) = \ddot{q}(t)$ and one has:

$$\ddot{s}(t) = \dot{\bar{q}}(t) + \gamma_2 \ddot{q}(t)$$

$$= \ddot{q}(t) + \gamma_2 \ddot{q}(t) - \gamma_2 \left( \begin{array}{c} q^*_d \\ 0 \\ \end{array} \right)$$

$$= \ddot{s}(t) - \gamma_2 \left( \begin{array}{c} q^*_d \\ 0 \\ \end{array} \right)$$

(73)

Introducing (73) into (72) one obtains:

$$\dot{V}_1(t) = -\gamma_1 s(t)^T s(t) + \gamma_1 \gamma_2 s(t)^T \left( \begin{array}{c} q^*_d \\ 0 \\ \end{array} \right)$$

$$= -\gamma_1 s(t)^T s(t) + \gamma_1 \gamma_2 s(t) q^*_d$$

$$= -\gamma_1 s(t)^T s(t) + \gamma_1 \gamma_2 \ddot{q}_1(t) q^*_d$$

$$= -\gamma_1 s(t)^T s(t) + \gamma_1 \gamma_2 \ddot{q}_1(t) q^*_d + \gamma_1 \gamma_2 q_1(t) q^*_d$$

$$= -\gamma_1 s(t)^T s(t) + \gamma_1 \gamma_2 \ddot{q}_1(t) q^*_d - \gamma_1 \gamma_2 q_1(t) q^*_d$$

(74)
Using the fact that \( q_1(t) \geq 0, q_1(t_k) = 0 \) and that \( q_{id}^* = -\alpha V(\tau_k^0) \leq 0 \), then between two impacts one has for all \( k \geq 0 \):

\[
V_i(t_{k+1}^-) - V_i(t_k^+) = \int_{t_k}^{t_{k+1}} \dot{V}_i(t) dt
\]

\[
= -\int_{t_k}^{t_{k+1}} \gamma_1 s(t)^T s(t) dt - \int_{t_k}^{t_{k+1}} \gamma_1 \dot{q}_i(t) |q_{id}^*| dt
\]

\[
+ \gamma_1 2|q_{id}^*| |q_1(t)| t_{k+1}^-
\]

\[
= -\int_{t_k}^{t_{k+1}} \gamma_1 s(t)^T s(t) dt - \int_{t_k}^{t_{k+1}} \gamma_1 \dot{q}_i(t) |q_{id}^*| dt
\]

\[
\leq 0
\]

Therefore condition (b) of claim 2 is satisfied.

**d** Let us start with the computation of \( \sigma_V(t_k) \). For \( k \geq 1 \), \( q_d(t_k^+) = q_d(t_k^-) \) and \( \dot{q}_d(t_k^+) = \dot{q}_d(t_k^-) = 0 \) see (19). Consequently one has:

\[
\sigma_V(t_k) = V_1(t_k^+) - V_1(t_k^-)
\]

\[
= \frac{1}{2} \left[ (\dot{q}(t_k^+) + \gamma_2 \ddot{q}(t_k^+))^T M_k \dot{q}(t_k^+) + \gamma_2 \ddot{q}(t_k^+) \right]
\]

\[
- (\dot{q}(t_k^-) + \gamma_2 \ddot{q}(t_k^-))^T M_k \dot{q}(t_k^-) + \gamma_2 \ddot{q}(t_k^-))
\]

\[
= \frac{1}{2} \dot{q}(t_k^+)^T M_k q_k(t_k^+) - \frac{1}{2} \dot{q}(t_k^-)^T M_k \dot{q}(t_k^-) + \gamma_2 [\ddot{q}(t_k^+)^T M_k \dot{q}(t_k^+)]
\]

\[
- (\dot{q}(t_k^-)^T M_k \dot{q}(t_k^-)]
\]

\[
= T_{L,k}(t_k) + \gamma_2 [\dot{q}(t_k^+)^T M_k \dot{q}(t_k^+)]
\]

where \( M_k \triangleq M(q(t_k)) \). Using the fact that \( q_1(t_k) = 0 \) and \( q_{id}(t) = 0 \) after the first impact see (19), one gets from (76):

\[
\sigma_V(t_k) = T_{L,k}(t_k) + \gamma_2 [\dot{q}(t_k^+)^T M_k \dot{q}(t_k^+)]
\]

Introducing (36) in (77) one obtains for all \( k \geq 1 \):

\[
\sigma_V(t_k) = T_{L,k}(t_k) \leq 0
\]

For \( k = 0 \), two cases have to be examined.

- If \( t_0 > \tau_k^0 \) then one has also \( q_d(t_0^-) = q_d(t_0^+) \) and \( \dot{q}_d(t_0^-) = \dot{q}_d(t_0^+) = 0 \), and one can use the result of Eq. (78) to obtain:

\[
\sigma_V(t_0) = T_{L}(t_0) \leq 0
\]

- If \( t_0 < \tau_k^0 \) then one has \( q_{id}(t_0^-) \neq q_{id}(t_0^+) \) and \( \dot{q}_{id}(t_0^-) \neq \dot{q}_{id}(t_0^+) \) = 0. One calculates the initial jump as follows:

\[
\sigma_V(t_0) = T_{L}(t_0) - \frac{1}{2} \dot{q}_{id}(t_0^-)^T M(q(t_0)) \dot{q}_{id}(t_0^-) - \frac{1}{2} \dot{q}_{id}(t_0^-)^T M(q(t_0)) q_{id}(t_0^-)
\]

\[
+ \gamma_2 [\ddot{q}_1(t_0^-) M_{11}(q(t_0)) + \ddot{q}_2(t_0^-) M_{21}(q(t_0))]
\]

\[
\leq \gamma_2 [\ddot{q}_1(t_0^-) M_{11}(q(t_0)) + \ddot{q}_2(t_0^-) M_{21}(q(t_0))]
\]

\[
\leq 0
\]

From (79), (80) and (78) one has:

\[
\sum_{k=0}^{\infty} \sigma_V(t_k) \leq \gamma_2 \|\ddot{q}(t_0^-)\| \|q_{id}(t_0^-)\| \|M_1(q(t_0))\| + \gamma_2 \|\ddot{q}_{id}(t_0^-)\|
\]

\[
\|M_{12}(q(t_0))\| \|\ddot{q}(t_0^-)\| + \gamma_2 \|q_{id}(t_0^-)\| \|M_{12}(q(t_0))\| \|\ddot{q}_2(t_0^-)\|
\]
where $M_1 = [M_{11}; M_{12}]^T$. Let us now prove that:

$$
\sum_{k=0}^{\infty} \sigma V_1(t_k) \leq KV_2^2(\rho_0^k) \tag{82}
$$

where $K > 0$. Let us calculate upper-bounds on $q_d(t_0^k)$, $\dot{q}_d(t_0^k)$, $\dot{\bar{q}}(t_0^-)$ and $\dot{q}_2(t_0^-)$. On $[\tau_0^k, t_0)$, one has $V_2(t) \leq 0$, so that $V_2(t_0) \leq V_2(\tau_0^k)$. Therefore from (42) we get:

$$
V_2(\tau_0^k) \geq V_2(t_0^-) \geq \gamma_2\gamma_1 \dot{\bar{q}}(t_0^-) \dot{\bar{q}}(t_0^-) \geq \gamma_2\gamma_1\|\dot{q}_2(t_0^-)\|^2 \tag{83}
$$

so that

$$
\|\dot{\bar{q}}(t_0^-)\| \leq \|\dot{q}(t_0^-)\| \leq \sqrt{\frac{V_2(\tau_0^k)}{\gamma_2\gamma_1}} \tag{84}
$$

From (42) one has $V_2(t) \geq \frac{1}{2}\|s(t)\|M(q)s(t)$. Consequently:

$$
\|s(t_0^-)\| \leq \sqrt{\frac{2V_2(\tau_0^k)}{\lambda_{min}(M)}} \tag{85}
$$

From (84), (85) and the definition of $s(t)$ one concludes that

$$
\|\dot{\bar{q}}(t_0^-)\| \leq \|\dot{s}(t_0^-)\| + \gamma_2\|\dot{q}(t_0^-)\| \leq \left[\sqrt{\frac{2}{\lambda_{min}(M(q))}} + \frac{72}{\gamma_1}\right] V_2^2(\tau_0^k) \tag{86}
$$

From (84), (86), the result of claim 6 and the fact that $V_2(\tau_0^k) \leq 1$ and the fact that $q_d(t_0^-) = q_d^*(t_0)$ and $\dot{q}_d(t_0^-) = \dot{q}_d^*(t_0)$, inequation (81) becomes:

$$
\sum_{k=0}^{\infty} \sigma V_1(t_k) \leq KV_2^2(\rho_0^k) \tag{87}
$$

with

$$
K = \left[\sqrt{\frac{2\gamma_2}{\gamma_1\lambda_{min}(M(q))}} + \frac{\gamma_2}{\gamma_1}\right] \|M_{11}(q(t_0))\| + \left[\frac{\gamma_2}{\gamma_1} \right] \|M_{12}(q(t_0))\| \tag{88}
$$

By inserting (42) in (87), one gets

$$
\sum_{k=0}^{\infty} \sigma V_1(t_k) \leq KV_2^2(\rho_0^k) + K(\gamma_2\gamma_1)^{\frac{3}{2}}\|\dot{\bar{q}}(\tau_0^k)\|^2 \tag{89}
$$

Therefore one has:

$$
\sum_{k=0}^{\infty} \sigma V_1(t_k) \leq KV_2^2(\rho_0^k) + \epsilon' \tag{90}
$$

for some $\epsilon' > 0$. Therefore condition (d) of claim 2 is satisfied. The system (15) with the controller (41) satisfies all the requirements of claim 2 with $\epsilon \neq 0$. Consequently it is pratically $\Omega$-weakly stable with $x(.) = s(.)$, and $R = \left(\frac{2}{\lambda_{min}(M(q))}e^{-\gamma(t_f^k - t_\infty)}(1 + K + \epsilon')\right)^{\frac{1}{2}}, \gamma = \frac{2\gamma_2}{\lambda_{max}(M(q))}$.

(ii) Proof of the second result of claim 7: Let us show that conditions (a) and (d) in claim 2 are satisfied.
(a) Outside phase $I_k$ it can be computed that [Spong et al.,(1990)]
\( \dot{V}_2(t) = -\gamma_1 \ddot{q}^T \dot{q} - \gamma_1 \dot{q}^T \ddot{q} \) 
(91)

Let us upper bound $V_2(t)$. From (42) one has
\[
V_2(t) \leq \frac{\lambda_{\max}(M(q))}{2} \|\dot{q}\|^2 + \frac{\lambda_{\max}(M(q))}{2} \|\ddot{q}\|^2 + \gamma_2 \lambda_{\max}(M(q)) \|\dddot{q}\| \|\ddot{q}\| + \gamma_1 \gamma_2 \|\ddot{q}\|^2
\]
(92)

Since $\|\dddot{q}\| \|\ddot{q}\| \leq \|\ddot{q}\|^2 + \|\dot{q}\|^2$ inequality (92) is rewritten:
\[
V_2(t) \leq \lambda_{\max}(M(q)) \frac{1 + 2\gamma_2}{2\gamma_1} \|\dot{q}\|^2 + \frac{\lambda_{\max}(M(q)) (\gamma_2 + 2) + 2\gamma_1}{2\gamma_1 \gamma_2} \gamma_1 \gamma_2 \|\ddot{q}\|^2
\]
(93)

With
\[
\gamma^{-1} = \max \left\{ \lambda_{\max}(M(q)) \frac{1 + 2\gamma_2}{2\gamma_1} : \frac{\lambda_{\max}(M(q)) (\gamma_2 + 2) + 2\gamma_1}{2\gamma_1 \gamma_2} \gamma_1 \gamma_2 \right\} > 0
\]
(94)

inequality (93) becomes
\[
V_2(t) \leq \gamma^{-1} \left[ \|\dot{q}\|^2 + \gamma_1 \gamma_2 \|\ddot{q}\|^2 \right]
\]
(95)

Inserting (91) in (95) yields
\[
\dot{V}_2(t) \leq -\gamma^{-1} \dot{V}_2(t)
\]
(96)

Then $\dot{V}_2(t) \leq -\gamma \dot{V}_2(t)$, and condition (a) of claim 2 is satisfied.

(d) As $V_2(t) = V_1(t) + \gamma_1 \gamma_2 \dot{q}^T \ddot{q}$ then
\[
\sigma_{V_2}(t_k) = \sigma_{V_1}(t_k) + \gamma_1 \gamma_2 \sigma_{\|\ddot{q}\|^2}(t_k)
\]
(97)

For $k \geq 1$, one has $q_d(t_k^+) = q_d(t_k^-)$, the position $q(t)$ is continuous, so that $\sigma_{\|\dot{q}\|^2}(t_k) = 0$ and
\[
\sigma_{V_2}(t_k) = \sigma_{V_1}(t_k) = T_L(t_k) \leq 0
\]
(98)

For $k = 0$, one has $q_d(t_0^+) \neq q_d(t_0^-)$. Let us upper bound $\sigma_{\|\ddot{q}\|^2}(t_0)$. One has
\[
\sigma_{\|\ddot{q}\|^2}(t_0) = \|\ddot{q}_1(t_0^-)\|^2 + \|\ddot{q}_2(t_0^-)\|^2 - \|\ddot{q}_1(t_0^-)\|^2 - \|\ddot{q}_2(t_0^-)\|^2
\]
(99)

As $q_2d(t_0^-) = q_2d(t_0^+)$, $q_1d(t_0^-) = 0$ and $q_1(t_0) = 0$ one obtains
\[
\sigma_{\|\ddot{q}\|^2}(t_0) = -\|q_1d(t_0^-)\|^2 \leq 0
\]
(100)

From (97), (98), (100) and (87) one has that
\[
\sum_{k=0}^{\infty} \sigma_{V_2}(t_k) \leq \sum_{k=0}^{\infty} \sigma_{V_1}(t_k) \leq KV_2^2(t_0^+)
\]
(101)

Therefore condition (d) of claim 2 is satisfied. The system (15) with the controller (41) satisfies all the requirements of claim 2(ii). Consequently it is pratically $\Omega$-weakly stable with $x(.) = [s(.), \ddot{q}(.)]$. ■
C Linear Complementarity Problem

A LCP is a system of the form [Murty(1997)]:

\[
\begin{align*}
\lambda & \geq 0 \\
A\lambda + b & \geq 0 \\
\lambda^T (A\lambda + b) & = 0
\end{align*}
\]

which can also be written as

\[
0 \leq \lambda \perp A\lambda + b \geq 0
\]

Such a LCP possesses a unique solution for all \( b \), if and only if \( A \) is a P-matrix (positive-definite matrices are P-matrices).
Complete stability analysis of a control law for walking robots with non-permanent contacts

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1 Introduction

One of the main specificities of walking robots is their non-permanent contact with the ground which impairs their stability. The only stability analyses of control laws that have been proposed so far for walking robots have been distinctly focusing on each contact phases, with the strong assumption that these phases are never perturbated [11, 13]. In this study we aim at analysing the stability of a regulation of the position of a walking robot without any assumption on the state of these contacts. In order to do so, we proposed in [2] to work in the framework of nonsmooth dynamical systems, what provides a general formulation of the dynamics that does not depend on the contact state. Classical stability theorems cannot be applied to this framework, so we needed to derive in [2] a Lyapunov stability theorem and a Lagrange Dirichlet theorem specifically for Lagrangian dynamical systems with non-permanent contacts. Based on these theorems, we will prove here the stability of a simple control action that realizes the regulation of the position and contact forces of a walking robot.

2 Position and force regulation law for walking robots

We first present in section 2.1 the model used to describe walking machines with non-permanent contacts. In sections 2.2 and 2.3 we will see that walking machines can be seen as underactuated systems, and thus a control law such as the one studied in [3] for a robotic manipulator cannot be used directly. Therefore, we propose in section 2.4 a control law designed in a similar way but adapted to the regulation of the position and contact forces of a walking robot.

2.1 Walking machines with non-permanent contacts

With $n$ the number of degrees of freedom of the walking robot, let us consider a time-variation of generalized coordinates $q : \mathbb{R} \rightarrow \mathbb{R}^n$ and the related velocity $\dot{q} : \mathbb{R} \rightarrow \mathbb{R}^n$:

$$\forall t, t_0 \in \mathbb{R}, \quad q(t) = q(t_0) + \int_{t_0}^t \dot{q}(\tau) d\tau.$$

We assume that the interaction between the feet of the walking machine and the ground is modelled by non-penetrating rigid bodies, which implies that the feet cannot penetrate or stick on the ground. That can be described by a set of unilateral constraints on the position of the system $\varphi(q) \geq 0, \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

The dynamics of walking robots can be described in this case by a classical Lagrange differential equation:

$$M(q) \ddot{q} + N(q, \dot{q}) \dot{q} + g(q) = u + r + f,$$

where $M(q)$ is the mass matrix, $N(q, \dot{q})$ is the Coriolis and Centripetal term, $g(q)$ is the gravity term, $u$, $r$, and $f$ are respectively the control input, the external forces, and the contact forces.
with \( M(q) \) the inertia matrix, \( N(q, \dot{q}) \dot{q} \) the corresponding nonlinear effects, \( g(q) \) the gravity forces, \( u \) the control action, \( r \) and \( f \) the normal and tangential contact forces (also referred as the reaction and friction forces). In the following, we are not going to precise here the model of these contact forces, we will only use the dissipativity property of friction: \( f^T \dot{q} \leq 0 \).

### 2.2 Walking machines as underactuated systems

First note that the vector of generalized coordinates \( q \) of a walking robot can be shown [14] to have the following structure,

\[
q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},
\]

with \( q_1 \) a vector gathering the position of the robot articulations and \( q_2 \) a vector describing the position and orientation of one solid of the robot with respect to the environment.

Since the actuators of the robot produce a torque \( \tau \) that acts only on the positions of its articulations, the actuation \( u \) can be shown therefore [15] to have the following structure

\[
u = \begin{bmatrix} \tau \\ 0 \end{bmatrix}.
\]

### 2.3 Contact forces at equilibrium positions

With the actuation (2), the robot dynamics (1) is then given by

\[
M(q) \frac{dq}{dt} + N(q, \dot{q}) \dot{q} + g(q) = \begin{bmatrix} \tau \\ 0 \end{bmatrix} + r + f,
\]

and at equilibrium points, when \( q = 0 \) and \( \frac{dq}{dt} = 0 \), it reduces to

\[
r + f = g(q) - \begin{bmatrix} \tau \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} r_1 + f_1 \\ r_2 + f_2 \end{bmatrix} = \begin{bmatrix} g_1(q) - \tau \\ g_2(q) \end{bmatrix}.
\]

Relation (3) shows an equilibrium condition between the three external forces: the contact forces (normal and tangential), the gravity forces and the actuation. Due to the structure of the actuation (2), \( \tau \) appears only in the upper part of this equilibrium condition. So the lower part of this equilibrium condition (3) relates only the reaction forces and the gravity forces independently of the actuation:

\[
r_2 + f_2 = g_2(q).
\]

This equilibrium condition can be related to more usual equilibrium conditions of mechanical systems [7, 15]. Indeed, the dynamics (1) can be shown to have a structure matching the one of the equilibrium condition (3), and as proved in [14], the second part of this dynamics corresponds to the Newton and Euler equations of the robot. In this way, the second part of equation (3) can be shown to provide an equilibrium condition between the position of the center of mass and the positions of the contacts with the ground, see [15] for more details.

The upper part of the equilibrium condition (3) relates the part \( r_1 + f_1 \) of the contact forces to the gravity forces and the the actuation

\[
r_1 + f_1 = g_1(q) - \tau.
\]

Though it seems that the contact forces \( r_1 + f_1 \) can be set to any value through the parameter \( \tau \), the inner structure of the friction and reaction forces (not precised here) entails that the parts \( r_1 + f_1 \) and \( r_2 + f_2 \) of the reaction forces are in fact completely intertwined, thus \( r_1 + f_1 \) also strongly depends on the equilibrium condition (4).
2.4 A control law through potential shaping

We want to realize a regulation of the position and contact forces of a walking robot to some desired position $q_d$ and desired contact forces $r_d + f_d$. The torque $\tau$ has to be designed to compensate the part $g_1(q_d)$ of the gravity forces so that we can obtain the desired reaction forces $f_{1d} + r_{1d}$ at the desired position $q_d$:

$$\tau(q_d) = g_1(q_d) - r_{1d} - f_{1d}. \tag{6}$$

In order to do so, we’ll consider a control law designed through potential shaping, and more precisely, following [12], we choose the following potential function

$$\tilde{P}(q_1) = \frac{1}{2}(q_1 - q_{1d})^T W (q_1 - q_{1d}) + (r_{1d} + f_{1d} - g_1(q_d))^T (q_1 - q_{1d}),$$

with $W$ a symmetric positive definite matrix and $q_{1d}$ the desired position of the robot articulations. The derivative of this potential function is

$$\frac{d\tilde{P}}{dq_1}(q_1) = W (q_1 - q_{1d}) + r_{1d} + f_{1d} - g_1(q_d),$$

to which we add a dissipative term $T\dot{q}_1$, with $T$ a positive definite matrix, in order to obtain the following Proportional Derivative control law with precompensation of the gravity and desired contact forces:

$$\tau = -W (q_1 - q_{1d}) + g_1(q_d) - r_{1d} - f_{1d} - T\dot{q}_1. \tag{6}$$

3 Nonsmooth dynamical systems

In order to prove the stability of the control law (6) without any assumption on the state of the contacts, we will use in section 3.3 some stability results for nonsmooth Lagrangian dynamical systems as proposed in [2]. The sections 3.1 and 3.2 aim therefore at presenting very briefly the framework of nonsmooth Lagrangian dynamical systems and its mathematical tools from convex analysis and measure theory that are unusual in control theory. For a more complete presentation, refer to [2].

3.1 Some geometry for systems with non-permanent contacts

We have seen in section 2.1 that the non-penetration of perfectly rigid bodies can be expressed as a constraint on the robot position, a constraint that will take the form of a closed set

$$\Phi = \{ q \in \mathbb{R}^n | \varphi(q) \geq 0 \}.$$

in which the generalized coordinates $q$ are bound to stay:

$$\forall t \in \mathbb{R}, \ q(t) \in \Phi.$$

When $q$ is in the interior of the domain $\Phi$, there is no interaction between the walking robot and its environment. On the other hand, when the system lies on the boundary of $\Phi$, the robot and its environment interact, what generates contacts forces $r + f$. Concerning the normal forces, this fact can be described through the inclusion

$$-r \in N(q) \tag{7}$$

involving the normal cone $N(q)$ of $\Phi$ at $q$ as defined in [4] and as illustrated in figure 1.

Now, we can observe that when the system reaches the boundary of $\Phi$ with a velocity $\dot{q}^-$ directed outside of this domain, it won’t be able to continue its movement with a velocity $\dot{q}^+ = \dot{q}^-$ and still stay in $\Phi$ (Fig. 1). A discontinuity of the velocity will have to occur then, corresponding to an impact between contacting rigid bodies. This can be described by the fact that the velocity $\dot{q}^+$ has to belong any time to the tangent cone $T(q)$ of $\Phi$ at $q$ as defined in [4] and as illustrated...
in figure 1. Note that the velocity after this impact $\dot{q}^+$ can be related to the velocity before the impact $\dot{q}^-$ by modelling this impulsive behavior through a contact law. We are not going to precise here this contact law, we will only use the fact that the impact is a dissipative phenomenon, which implies that the kinetic energy $K(q, \dot{q})$ satisfies

$$K(q, \dot{q}^+) \leq K(q, \dot{q}^-).$$

(8)

For a more in-depth presentation of these concepts and equations which may have subtle implications, the interested reader should definitely refer to [9] or [2, 3].

![Diagram of tangent cones $T(q)$ and normal cones $N(q)$ on the boundary of the domain $\Phi$, and example of a trajectory $q(t) \in \Phi$ that reaches this boundary with a velocity $\dot{q}^- \notin T(q)$.]

3.2 Nonsmooth Lagrangian dynamical systems

Classically, solutions to the dynamics of Lagrangian systems such as (1) lead to smooth motions with locally absolutely continuous velocities. But we have seen in section 3.1 that discontinuities of the velocities may occur when the coordinates of such systems are constrained to stay inside closed sets. These classical differential equations must therefore be turned into measure differential equations [9]

$$M(q) \ddot{q} + N(q, \dot{q}) \dot{q} dt + g(q) dt = u dt + dr + f dt,$$

(9)

where the reaction forces are now represented by an abstract measure $dr$ which may not be Lebesgue-integrable. This way, the measure acceleration $\dot{d}q$ may not be Lebesgue-integrable either so that the velocity may not be locally absolutely continuous anymore but only with locally bounded variation, $\dot{q} \in \text{lbv}(\mathbb{R}, \mathbb{R}^n)$ [9]. Functions with locally bounded variation have left and right limits at every instant, and we have for every compact subinterval $[\sigma, \tau] \subset \mathbb{R}$

$$\int_{[\sigma, \tau]} d\dot{q} = \dot{q}^+(\tau) - \dot{q}^-(\sigma).$$

A function $f$ has a locally bounded variation on $\mathbb{R}$ if its variation on any compact interval $[t_0, t_n]$ is finite:

$$\text{Var}(f; [t_0, t_n]) = \sup_{t_0 \leq t_1 \leq \ldots \leq t_n} \sum_{i=1}^{n} ||f(t_i) - f(t_{i-1})|| < +\infty.$$
Rather than for this definition, it is for their properties that functions with bounded variations are useful. Notably, functions with locally bounded variation can be decomposed into the sum of a continuous function and a countable set of discontinuous step functions [8]. In specific cases, as when the definition of the dynamics (9) is piecewise analytic, its solutions can be shown to be piecewise continuous with possibly infinitely (countably) many discontinuities [1]. In this case, it is possible to focus distinctly on each continuous piece and each discontinuity as in the framework of hybrid systems [6]. But this is usually done through an ordering of the discontinuities strictly increasing with time, what is problematic when having to go through accumulations of impacts. The framework of nonsmooth analysis appears therefore as more appropriate for the analysis of impacting systems, even though the calculus rules for functions with bounded variation require some care.

3.3 Some Lyapunov stability theory

The Lyapunov stability theory is usually presented for dynamical systems with states that vary continuously with time [5], [17], Fillipov systems for example [10], but we have seen that in the case of nonsmooth mechanics, the velocity and thus the state may present discontinuities. Lyapunov stability theory is hopefully not strictly bound to continuity properties, and some results can still be derived for discontinuous dynamical systems both in the usual framework of hybrid systems [16] and in the framework of nonsmooth analysis [2].

In the following we will prefer the latter for the reasons mentioned in section 3.2, for which we proposed in [2] the following corollary of a Lagrange-Dirichlet theorem:

**Corollary 1.** Consider a nonsmooth Lagrangian dynamical system experiencing external forces composed of normal contact forces and other Lebesgues-integrable forces \( F \). If these Lebesgues-integrable forces derive from a coercive \( C^1 \) potential function \( P(q) \) with a dissipative term \( h \):

\[
F = -\frac{dP}{dq}(q) + h, \quad \text{with} \quad q^T h \leq 0,
\]

then the set \( S = \{ \text{Arg min}_\phi P(q) \} \times \{0\} \) is Lyapunov stable.

In our case, the Lebesgues-integrable forces \( F \) acting on the dynamics (9) are composed of the gravity forces, the actuation, and the friction forces,

\[
F = -g(q) + u + f.
\]

With \( G(q) \) the potential of the gravity forces, if we replace the control law \( u \) by its expression (6), \( F \) can be expressed as the derivative of potential functions plus dissipative terms

\[
F = -\frac{dG}{dq}(q) - \frac{d\tilde{P}}{dq}(q) - \begin{bmatrix} T \\ 0 \end{bmatrix} \dot{q}_1 + f,
\]

deriving therefore from the potential \( P(q) = G(q) + \tilde{P}(q_1) \), and we can conclude through corollary 1 on the stability of the set \( S = \{ \text{Arg min}_\phi P(q) \} \times \{0\} \) without making any assumption on the state of the contacts.

Now, as shown in [2, 3], this set corresponds to the equilibrium positions of the system. The control law (6) has been designed so that the desired position \( q_d \) is an equilibrium position. We can assume under mild conditions [3] that it is the only one and it is therefore stable.

3.4 Why such a “simple” control law

First of all, note that the control action (6) can’t compensate the impulsive behaviors of the contacts. Now, we have seen in section 3.1 that these impulsive behaviors originating from the discontinuities of the velocity are related to the kinetic energy through relation (8). It is therefore natural to use the energy of the system for the stability analysis, leading to the Lagrange-Dirichlet theorem for nonsmooth Lagrangian dynamical systems [2] and its corollary 1 that is used here. Since the control law (6) has been designed by potential shaping, the energy of the system appears
as a natural candidate for its stability analysis. For the same reason, the choice of a control law trying to compensate completely the system dynamics such as a computed torque \([11, 13]\), appears to be not very judicious because this energy can’t be used any longer for the stability analysis. Finally, it is not possible to compensate completely the external forces in the case of walking machines because of the underactuation (2) what explains the difference with the control law proposed in \([3]\).

4 Conclusion

In this study we aimed at analysing the stability of a regulation of the position of a walking robot without any assumption on the state of the contacts. We thus proposed to work in the framework of nonsmooth dynamical systems since it provides a general formulation of the dynamics that does not depend on the contact state. Classical stability results cannot be applied in this framework, so we needed to derive in \([2]\) a Lyapunov stability theorem for Lagrangian dynamical systems with non-permanent contacts. Based on this theorem, we then proved the stability of a simple control law that realizes the regulation of the position and contact forces of a walking robot. Through this study, we were thus able to propose for the first time a complete stability analysis of the regulation of the position of a walking robot.

References

Robustness analysis of passivity-based controllers for complementarity Lagrangian systems

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Abstract. In this paper we study the robustness of tracking controllers for Lagrangian systems subject to frictionless unilateral constraints. The stability analysis incorporates the hybrid and nonsmooth dynamical features of the overall system. This work provides details on the robustness of such controllers with respect to the modelling errors of the dynamic, the uncertainty on the constraint position, and with respect to the measurement noise.

1 Introduction

The focus of this paper is the tracking control of a class of nonsmooth fully actuated Lagrangian systems subject to frictionless unilateral constraints on the position. Such systems may a priori evolve in three different phases of motion: i) A free motion phase, ii) A permanently constrained phase with a non-zero contact force, iii) A transition phase whose goal is to stabilize the system on some constraint surface. The controller used in this paper was fully detailed in [1]. The present paper gives numerical results on the robustness of two passivity based controllers.

1.1 Dynamics

Let \( X \in \mathbb{R}^n \) denote the vector of generalized coordinates. The systems we study in this paper are complementarity Lagrangian systems. The dynamics is:

\[
\begin{align*}
M(X)\ddot{X} + C(X, \dot{X})\dot{X} + G(X) &= u + \nabla F(X)\lambda_X \\
F(X) &\geq 0, \quad F(X)^T\lambda_X = 0, \quad \lambda_X \geq 0 \\
\text{Collision rule}
\end{align*}
\]

where \( M(X) \) is the positive definite inertia matrix, \( F(X) \in \mathbb{R}^m \) represents the distance to the constraints, \( \lambda_X \in \mathbb{R}^m \) are the Lagrangian multipliers associated to the constraints, \( u \in \mathbb{R}^n \) is the vector of generalized torque inputs, \( C(X, \dot{X}) \) is the matrix of Coriolis and centripetal forces, \( G(X) \) contains conservative forces. \( \nabla \) denotes the Euclidean gradient.

The impact times will be denoted generically as \( t_k \) in the following. The admissible domain \( \Phi \) is a closed domain in the configuration space where the system can evolve, i.e. \( \Phi = \{X | F(X) \geq 0\} \). The boundary of \( \Phi \) is denoted as \( \partial \Phi \). A collision rule is needed to integrate the system in (1) and to render the set \( \Phi \) invariant. In this work, it is chosen as in [6]:

\[
\begin{align*}
\dot{X}(t_k^+) &= -e_n\dot{X}(t_k^-) + (1 + e_n) \arg \min_{z \in T_\Phi(X(t_k^-))} z - \dot{X}(t_k^-) \\
\frac{1}{2} &\left[ z - \dot{X}(t_k^-) \right]^T M(X(t_k)) \left[ z - \dot{X}(t_k^-) \right]
\end{align*}
\]

where \( \dot{X}(t_k^+) \) is the post impact velocity, \( \dot{X}(t_k^-) \) is the pre-impact velocity, \( T_\Phi(X(t)) \) the tangent cone to the set \( \Phi \) at \( X(t) \) and \( e_n \) is the restitution coefficient, \( e_n \in [0, 1] \).

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1.2 Cyclic task

In this paper we restrict ourselves to a specific task, or trajectory: a succession of free and constrained phases $\Omega_{k_{cyc}}$. During the transition between a free and a constrained phase, the dynamic system passes into a transition phase $I_{k_{cyc}}$. In the time domain one gets a representation as:

$$\mathbb{R}^+ = \Omega_0 \cup I_0 \cup \Omega_1 \cup \Omega_2 \cup I_1 \cup \ldots \cup \Omega_{2k_{cyc}} \cup I_{k_{cyc}} \cup \ldots$$

where $\Omega_{2k_{cyc}}$ denotes the time intervals associated to free-motion phases and $\Omega_{2k_{cyc}+1}$ those for constrained-motion phases. The order of the phases is important but the initial phase may be $\Omega_0$ or $I_0$ or $\Omega_1$. Transition between constrained-motion and free motion does not spawn to a specific phase because there is no discontinuity of the state vector.

As explained in [1] the control strategy and stability analysis have to cope with the fact that the time of the first impact, and the time of detachment are unknown. Then four different trajectories are used in the analysis: On the first hand, the signal $X^\star_d(\cdot)$ in the control input (Fig. 2(a)). This trajectory imposes impacts when the tracking error is not zero (imposing impacts improves the robustness if the constraint position knowledge is bad). The trajectory $X^\star_d(\cdot)$ tends to the tangential approach (noted $X^{i,nc}(\cdot)$ on Fig. 2(b)) when $k_{cyc} \to \infty$. On the other hand, the signal $X_d(\cdot)$ enters the Lyapunov function (Fig. 2(c)); this trajectory differs from $X^\star_d(\cdot)$ because between $B$ and $C$ the point $(q^\star_d, \dot{q}^\star_d)$ is not reachable, then $X_d(\cdot)$ is set on the surface $\partial\Phi$ after the first impact of each cycle. The trajectory $X_d(\cdot)$ tends to $X^{i,c}(\cdot)$ (see Fig. 2(d)) which is the impactless trajectory of the system when tracking is perfect.

This is the major discrepancy compared to unconstrained motion control in which all four trajectories are the same, usually denoted as $X_d(\cdot)$.

We see here the hybrid aspect of the convergence analysis. On one hand, we need to guarantee the continuous convergence ($X(\cdot) \to X_g(\cdot)$), and on the other hand the discrete convergence ($X_d(\cdot) \to X^{i,c}(\cdot)$) over the cycles $k_{cyc}$. From the user point of view, the tracking error is $X(\cdot) - X^{i,c}(\cdot)$ (see sec. 4 for illustration on a particular example).

The cycles $\Omega_{2k_{cyc}} \cup I_{k_{cyc}} \cup \Omega_{2k_{cyc}+1}$ duration is not arbitrary set since it depends on phases $I_{k_{cyc}}$ duration, which in turn may depend on control and physical parameters (see an illustration in sec. 4).

2 Stability framework

The stability criterion used in this paper is an extension of the Lyapunov second method adapted to closed loop mechanical system with unilateral constraints and has been proposed in [1] [2]
Fig. 2: The closed-loop desired trajectories

Definition 1 (Ω-weakly stable system). The closed-loop system is Ω-weakly stable if for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\|x(0)\| \leq \delta(\epsilon) \Rightarrow \|x(t)\| \leq \epsilon$ for all $t \geq 0$, $t \in \Omega = \cup_{k_{\infty} \geq 0} \Omega_{k_{\infty}}$. Asymptotic weak stability holds if in addition $x(t) \to 0$ as $t \to +\infty$, $t \in \Omega$. Practical weak stability holds if there is a ball centered at $x = 0$, with radius $R > 0$, and such that $x(t) \in B(0, R)$ for all $t \geq T$; $T < +\infty$, $t \in \Omega$, $R < +\infty$.

Definition 2 (Strongly stable system). The system is said strongly stable if:

(i) it is Ω-weakly stable,
(ii) on phases $I_{k_{\infty}}$, $P_{\Sigma_{T}}$ is Lyapunov stable with Lyapunov function $V_{\Sigma_{T}}$, and
(iii) the sequence $\{t_{k}\}_{k \in \mathbb{N}}$ has a finite accumulation point $t_{\infty} < +\infty$.

In [1], two claims are presented which are useful to prove the stability of Lagrangian systems with respect to the definitions 1 and 2.

3 Tracking controller framework

In this section we briefly develop the tracking controller strategy used in this paper. A more elaborate description is available in [1] with a complete stability analysis. In this paper we focus on the simulations and robustness aspects.
3.1 Controller Structure

To make the controller design easier the dynamical equations (1) are considered in the generalized coordinates introduced in [5]. After transformation in the new coordinates $q = [q_1, q_2]^T$, $q_1 = [q_1^1, \ldots, q_1^m]^T$, $q = Q(X) \in \mathbb{R}^n$, the dynamical system is as follows:

$$
\begin{align*}
M_{11}(q)\ddot{q}_1 + M_{12}(q)\ddot{q}_2 + C_1(q, \dot{q})\dot{q}_1 + g_1(q) = T_1(q)U + \lambda \\
M_{21}(q)\ddot{q}_1 + M_{22}(q)\ddot{q}_2 + C_2(q, \dot{q})\dot{q}_2 + g_2(q) = T_2(q)U \\
q_1^i \geq 0, \quad q_1^i \lambda_i = 0, \quad \lambda_i \geq 0, \quad 1 \leq i \leq m \\
\text{Collision rule}
\end{align*}
$$

(4)

The vector $q_1$ denotes the constraint coordinate and $q_2$ denotes the free coordinate. The controller developed in this paper uses three different low-level control laws for each phase $\Omega_{2k_{cy}'}$, $\Omega_{2k_{cy}'+1}$ and $I_{k_{cy}}$:

$$
T(q)U = \begin{cases} 
U_{nc} = U_{nc}(q, q_1^*, \dot{q}, \dot{q}_2^*, \ddot{q}, \ddot{q}_d^*) \\
U_t = U_{nc}(q, q_1^*, \dot{q}_d^*, \dot{q}_d^*, \ddot{q}, \ddot{q}_d^*) & \text{before the first impact} \\
U_t = U_{nc}(q, q_1^*, 0, 0, \dot{q}, 0) & \text{after the first impact} \\
U_c = U_{nc} - P_d + K_f(P_q - P_d)
\end{cases}
$$

(5)

where $-P_d + K_f(P_q - P_d)$ is the force/position feedback which is added during the $\Omega_{2k_{cy}'+1}$ phases. A supervisor switches between these three control laws. The overall structure of the control strategy is shown on Fig. 3. The transition control law corresponds to the non-constraint one with $q_d^*(t)$ frozen ($q_d^* = \dot{q}_d^* = 0$), and the constraint law correspond to the sum of non-constraint control and a control in force. The asymptotic stability of this scheme makes the system land on the constraint surfaces tangentially after enough cycles of constraints/free motions (one cycle = $\Omega_{2k_{cy}'} \cup I_{k_{cy}} \cup \Omega_{2k_{cy}'+1}$). Asymptotically the transitions between free motion phases and permanently constraint phases are done without any collision.

As observed in the introduction, a control strategy which consists of attaining the surface $\partial \Phi$ tangentially and without incorporating impacts in the stability analysis, cannot work in practice due to its lack of robustness. In view of this, the control law for the transition phase is defined in order:

- To make the system hit the constraint surface (and then dissipate energy during impacts) if the tracking error is not zero.
any velocity jump at $t_k$ makes the asymptotic stability quite difficult to obtain if velocities are subject to discontinuities: any velocity jump at $t_k$ implies $\sigma V(t_k) > 0$ when $V \equiv 0$. Hence if the transition phase is constructed with impacts, one has to find a manner to get $V(t'_f)$ in order to force the system to remain on the desired trajectory $X_d(\cdot)$ (here $q_d(\cdot)$). This is not obvious in general and defining $q_d^*(\cdot)$ as done below is a way to get the result.

3.2 Design of the desired trajectory

During the transition phase the control signal $q_d^*(t)$ is defined as follows (see Fig. 4 for $q_{1d}^*(\cdot)$, where $A, A', B', B$ and $C$ correspond to Fig. 2). Let us define:

- $k_{f_k}$ is chosen by the designer as the start of the transition phase,
- $t_0$ is the time corresponding to $q_{1d}^*(t_0) = 0$,
- $t_\infty$ corresponds to the first impact,
- $t_\infty$ corresponds to the finite accumulation point of the sequence $\{t_k\}_{k \geq 0}$,
- $t'_f$ is the end of the transition phase,
- $\tau_1$ is such that $q_{1d}^* (\tau_1) = -\alpha V(t_1)$ and $q_{1d}^* (\tau_1) = 0$.
- $\Omega_{k_{f_k}+1} = [\begin{array}{c} k_{f_k} \n k_{f_k} \n \end{array}]$.

On $[\tau_1, \tau_0]$, we impose that $q_d^*(t)$ is twice differentiable, and $q_{1d}^*(t)$ decreases towards $-\alpha V(t_1)$ on $[\tau_0, \tau_1]$. In order to cope with the coupling between $q_1$ and $q_2$, the signal $q_{2d}^*(t) \in C^2(\mathbb{R}^+)$ is frozen during the transition phase, i.e.: $q_{2d}^*(t) = q_{2d}^*$, $q_{2d}^*(t) = 0$ on $[\tau_0, \tau_1]$. On $(t_0, t_f)$, we define $q_d$ and $q_d^*$ as: $q_d = (0, q_{2d}^*)^T$, $q_d^* = (-\alpha V(t_1), q_{2d}^*)^T$.

On $[\tau_1, \tau_2]$ we set $q_d = [0, q_{2d}(t)]^T$. The purpose of $q_d^*$ is to create a “virtual” potential force which stabilizes the system on $\partial \Phi$ even if the position of the constraint is uncertain. Consequently the fixed point $(q_d, \dot{q}d)$ of the complementarity system is used in the expression of the Lyapunov function $(\tilde{q} = q - q_d)$, whereas the unreachable fixed point $q_d^*$ is used in the control law. In summary, after the first impact at $t_0$, $q_{1d}(\cdot)$ is set to zero while in case $\tau_1 > t_0$, $q_{1d}(\cdot)$ is set to $-\alpha V(t_1)$ (in other words $U$ switches as indicated in (5)). Since $q_{1d}(t_0) \neq 0$ and $q_{1d}(t_0) \neq 0$ in general, the trajectory $q_{1d}(\cdot)$ behaves like in a sort of plastic collision ($\epsilon_n = 0$). With respect to Fig. 2, one
has $\tau_{1}^{k_{cyc}}$ at $A$, $t_{\infty}$ at $B'$, $t_{0}$ at $A'$, $t_{d}^{k_{cyc}}$ at $C$, and $B$ at $t_{d}^{k_{cyc}}$ (the term $-P_{d} - K_{f}P_{d}$ defines the signal $X_{d}^{c}(t)$ between $B$ and $C$ on Fig. 2). If $V(t_{d}^{k_{cyc}}) = 0$ then $A''$ corresponds to the time $\tau_{1}^{k_{cyc}}$.

The piece of curve $AA'$ on Fig. 2 is normal to $\partial \Phi$. The closed-loop desired trajectory $X_{d}^{c}(\cdot)$ is defined as $q_{d}^{c}(t) = q_{d}^{c}(t)$ on $\Omega_{2k_{cyc}}$, $q_{d}^{c}(t) = q_{d}^{c}(t)$ with $\alpha = 0$ on $I_{k_{cyc}}$, and $q_{d}^{c}(t) = 0$ on $\Omega_{2k_{cyc}+1}$, $q_{d}^{c}(t) = q_{d}^{c}(t)$ on $\mathbb{R}^{+}$. It is impactless.

The choice for $q_{d}^{c}(\cdot)$ is done to get $\sigma_{V}(t_{k}) \leq 0$ on $I_{k_{cyc}}$.

3.3 Two control laws for the nonlinear controller block (on Fig. 3)

The “Non linear controller” block on Fig. 3 can take different formulations. In this section, two schemes are studied: one derived from the Paden-Panja PD+ controller, and one derived from Slotine-Li controller.

**Hybrid Paden-Panja scheme.** The control law used in this first scheme is based on the controller presented in [7], originally designed for free-motion position and velocity global asymptotic tracking. Let us propose:

$$U_{nc} = M(q)\ddot{q}_{d} + C(q, \dot{q})\ddot{q}_{d} + g(q) - \gamma_{1}(q - q_{d}) - \gamma_{2}(\dot{q} - \dot{q}_{d})$$

where $\gamma_{1} > 0$, $\gamma_{2} > 0$. The Lyapunov function associated to this control law is:

$$V(t, \dot{q}, \ddot{q}) = \frac{1}{2} \dot{q}^{T} M(q) \dot{q} + \frac{1}{2} \dot{\gamma}_{1} \ddot{q}^{T} \ddot{q}$$

with $\dot{q}(\cdot) = q(\cdot) - q_{d}(\cdot)$

**Hybrid Slotine-Li scheme.** For the second scheme, the nonlinear controller block (on Fig. 3) is based on the scheme presented in [8]. Let us propose the following:

$$U_{nc} = M(q)\ddot{q}_{r} + C(q, \dot{q})\ddot{q}_{r} + g(q) - \gamma_{1}s$$

where $s = \ddot{q} + \gamma_{2}\ddot{q}$, $s = \ddot{q} + \gamma_{2}\ddot{q}$, $\ddot{q}_{r} = \ddot{q}_{d} - \gamma_{2}\ddot{q}$, and $\ddot{q} = q - q_{d}$, $\gamma_{2} > 0$ and $\gamma_{1} > 0$ are two scalar gains. Let us consider the following positive functions:

$$V_{1}(t, s) = \frac{1}{2} s(t)^{T} M(q(s(t))$$

$$V_{2}(t, s) = \frac{1}{2} s(t)^{T} M(q(s(t)) + \gamma_{2}\ddot{q}(t)^{T} \ddot{q}(t)$$

**Control parameters:** The control parameters which can be tuned by the end user are feedback gains $\gamma_{1}$, $\gamma_{2}$ and $K_{f}$, the gain $\alpha$ and the time $\tau_{0}^{k_{cyc}}$, which define $q_{d}^{*}$.

3.4 Closed-loop stability analysis

The closed-loop dynamical system is now completely defined. This section give result on the stability of the closed-loop with respect to the definition given in section 2.

**Assumption 7** The controller $U_{i}$ in (5) assures that a sequence $\{t_{k}\}_{k \geq 0}$ of impact times exists, with $\lim_{k \to +\infty} t_{k} = t_{\infty} < +\infty$.

**Proposition 1.** Let assumption (7) hold. The system defined by (1) in closed-loop with the controller (5)-(6) and $q_{d}(\cdot)$, $q_{d}^{c}(\cdot)$ defined as in section 3.2, is:

(i) - Asymptotically strongly stable if $q_{d}^{c}(\cdot)$ is designed such that at the first impact time of each phase $I_{k_{cyc}}$ we have:

$$[M_{11}(q(t_{0}))(\dot{q}_{1}(t_{0})^{T} + \dot{q}_{2}(t_{0}^{+})^{T}M_{21}(q(t_{0})))] \dot{q}_{d}(t_{0}^{+}) \leq 0.$$
(ii) - Asymptotically strongly stable if $M_{12} = 0$ and $e_n = 0$.
(iii) - Asymptotically weakly stable if $M_{12} = 0$ and $0 \leq e_n < 1$.

The proof of proposition 1 can be found in [1]. Briefly, the proof shows that the Lyapunov function $V(t)$ in equation (7) decreases on the phase $\Omega_{k_{\text{cycle}}}$. And that one has $V(x(t_{k_{\text{cycle}}}), t_{k_{\text{cycle}}}) \leq V(x(\tau_{k_{\text{cycle}}}), \tau_{k_{\text{cycle}}})$.

**Proposition 2.** Let assumption 7 hold, $e_n \in (0, 1)$ and $q_{1d}$ be defined as in section 3.2. Consider the system defined by (4) in closed-loop with the controller in (5) and (8).

(i) - If the controller $T(q)U$ in (8) assures that $\|\tilde{q}(\tau_{0_{k_{\text{cycle}}}})\| < \epsilon, \epsilon > 0$ for all $k_{\text{cycle}}$ over the cycles, then the system initialized on $\Omega_0$ with $V_2(\tau_{0_{k_{\text{cycle}}}}) \leq 1$ is therefore practically $\Omega$-weakly stable with closed-loop state $x(\cdot) = s(\cdot)$.

(ii) - If the controller $T(q)U$ in (8) assures that $\|\tilde{q}_2(t_{k+1})\| \leq \|\tilde{q}_2(t_k)\|$, for all $t_k$ on $[t_0, t_\infty)$, then the system initialized on $\Omega_0$ with $V_2(\tau_{0_{k_{\text{cycle}}}}) \leq 1$ is therefore practically $\Omega$-weakly stable with closed-loop state $x(\cdot) = [s(\cdot), \tilde{q}(\cdot)]$.

The proof of proposition 2 can be found in [1]. This proof uses the fact that the control law (8) is exponentially decreasing ($\dot{V}_1(t) \leq -\gamma V_1(t)$ outside phase $I_{k_{\text{cycle}}}$).

**4 Simulation**

In this section we show some simulations of the previous control scheme on a planar two-degree of freedom robotic arm (as seen on Fig. 5). The numerical scheme is based on an event driven simulation scheme. The numerical values used for the dynamical model are $l_1 = l_2 = 0.5m$, $I_1 = I_2 = 1kg.m^2$ and $m_1 = m_2 = 1kg$ for respectively the length, the inertia and the mass of the two links of the arm. The restitution coefficient is set to $e_n = 0.7$. The minimization needed for the computation of the impact law is performed using the FSQP algorithm [4]. In the following examples, the system tracks a circle (see Fig. 6). If nothing else is written on the figures, the gains
used for simulations are $\alpha = 100$, $\gamma_1 = 2$, $\gamma_2 = 5$ and $P_e = 3s$ for Paden-Panja scheme ($P_e$ is the desired period of one cycle), and $\alpha = 100$, $\gamma_1 = 10$, $\gamma_2 = 0.5$ and $P_e = 4s$ for Slotine Li scheme.

On Figs. 8 and 9, we compare our control scheme ($\alpha = 100$) and the tangential approach ($\alpha = 0$). Fig. 8 shows that the tangential approach implies longer stabilisation phases even in the perfect case. This demonstrates the influence of $\alpha$ on the duration of phases $I_k^{cyc}$, and consequently on the cycle duration. The Fig. 9 demonstrates the asymptotic convergence of this laws. The desired trajectory $q_{id}$ goes less and less deeper under the constraint as the cycles go on: the term $-\alpha V(t_0^{k+cyc})$ decrease as $k_{cyc}$ increase. The evolution of the Lyapunov function $V(t)$ is displayed on Fig. 10. It can be seen that the system converges asymptotically: the first jump of each cycle of the Lyapunov function decreases over the cycles. The hybrid Slotine-Li scheme provided similar results, so they are not presented here. On the zoom of Fig. 10, we see that the Lyapunov function does not fulfill the requirement of the proposition 1: $V(t_0^k) \geq V(t_0^{k+cyc})$. This is logical since $M_{12} \neq 0$. But the system is still stable because the duration of the phase $\Omega_{2k+cyc+1}$ is long enough to have $V(t_0^3) \leq V(t_0^2)$. On Fig. 7 the orbit of the trajectories can be seen. These orbits converge to a limit cycle. This limit cycle is slightly deformed compared to the desired half circle. This deformation is due to the transition phase $I_{k+cyc}$ (when the $q_{2d}$ trajectories are frozen) and due to the take off phase (when $q_{0d}^2$ and $q_{3d}^2$ signals need to be resynchronized since the take-off time is not known precisely). Finally from the point of view of the end-user, the real tracking error is $X^{2+cyc} - X$ (X tends to $X^{1+cyc}$ asymptotically over cycles). The end user can reduce this error by decreasing the speed used to performed a cycle (as seen on Fig. 18 where the cycle period is set to 16s against 3s for the previous test). Each control law $(U_{nc}, U_c, U_1)$ considered separately possesses good robustness properties (because $U_{nc}$ and $U_c$ are passivity-based, and $U_1$ creates a bouncing-ball dynamics). The problem is: Is this robustness conserved when switching between these 3 controllers as described in sec. 3?
5 Dynamical parameters uncertainties

This section deals with the robustness of this control scheme with respect to uncertainties in the parameters of the control laws (6) and (8). The dynamical model used in the dynamic integration part of the simulator is the same as in the previous section. The model used in the computation of the input torque $T(q)U$ uses mass 30% heavier ($m_1 = m_2 = 1.3$kg). Figs. 11-12 show results for the hybrid Paden-Panja scheme, and Figs. 13-14 show the hybrid Slotine-Li one. These two tests point out that uncertainty in the dynamic model caused tracking error to increase when the reference trajectories vary quickly (as seen on zooms of the Figs. 11-13).

Small variations of the tracking error imply small variations of the Lyapunov function (Figs. 12 and 14), then the system is no more asymptotically stable. Even if the function $V(t)$ become very
small, there are always small impacts. Then some small discontinuities can be seen, at each first impact, on Figs. 12 and 14 ($\sigma_V(t_0)$ is positive).

In conclusion, we can say that this control scheme is robust with respect to the uncertainty on the dynamic model, but errors on the model imply that asymptotic convergence is lost. Secondly, the hybrid Slotine Li scheme is more robust that the hybrid Paden-Panja: on these simulations, the Lyapunov’s function peaks are about $3e^{-4}$ for Slotine Li law against $5e^{-3}$ for Paden Panja scheme.

6 Constraint position uncertainty

In this section, we study the robustness of the controller with respect to the uncertainty on the constraint position. The location of the constraint surface is not known accurately. Two situations may be considered.

1- The estimated position (denoted by $\hat{q}_{1c}$) of the constraint is lower than the real position (denoted by $q_{1c}$), i.e. $\hat{q}_{1c} = q_{1c} + c$ with $c < 0$ (c denotes the error of estimation).

2- The estimated position of the constraint is above the real position. i.e. $c > 0$.

**Case 1 $c < 0$:** In this case the desired trajectories decrease toward $q_{id}(t_1^{k_{r}}) = -\alpha V(t_0^{k_{r}}) - |c|$ instead of $q_{id}(t_1^{k_{r}}) = -\alpha V(t_0^{k_{r}})$). The error $c$ can be incorporated in the term $-\alpha V(t_0^{k_{r}})$ and the stability of the transition phase is not changed. During the constraint phase the controller is:

$$U_c = U_{nc} - (P_d + \gamma_l [c] 0^T) + K_f (P_q - P_d)$$

(10)

The error term $\gamma_l |c|$ is added to the desired force $P_d$ and contributes to keep the contact with the surface during the constrained phase.

The system remains stable but it loses its asymptotic stability: If the tracking is perfect $V(t_0^{k_{r}}) = 0$ and $q_{id}^{\star} = -|c|$, so that the system does not approach the surface tangentially and rebounds occur. An example is depicted on Figs. 15-16.

**Case 2 $c > 0$:** If the tracking is perfect $V(t_0^{k_{r}}) = 0$, the desired trajectory decreases toward $q_{id} = c$ and the system never reaches the constraint. There is no convergence (see Fig. 17 after two cycles). This problem can be solved by monitoring the time of stabilization. If there is no stabilization after an estimated time $\hat{t}_\infty$, the estimated position of the constraint is refreshed as $q_{1c}^{new} = \hat{q}_{1c} - c$. After a finite number of iterations, one gets $\hat{q}_{1c} < 0$. The system is in the previous situation $c < 0$ and the stability is preserved. Figs. 19-20 shows an example of self-adjustment of the estimated position of the constraint. At the beginning of the simulation $\hat{q}_{1c}$ was set to
2cm. The increment of correction $\epsilon$ is set to 1.5cm. Then, at the end of the simulation, one has $q_{1c} = 2 - 2 \times 1.5 = -1$cm. The error of estimation become negative $c = -1$cm.

When tracking is not perfect and $\alpha V(\tau_{k, cyc}) > c$ (like during the first two cycles of simulations on Figs. 17 and 19), the transition phase is able to stabilize the system on the surface $\partial \Phi$. But during the constraint phase, the controller is:

$$U_c = U_{nc} - (P_d - \gamma_1 [c - 0]) + K_f(P_q - P_d)$$

(11)

$P_d$ must be chosen large enough compared to $\gamma_1 c$ to be sure that the system keeps the contact with the surface during the whole constraint phase.

In conclusion we can say that this control scheme is robust with respect to uncertainty on the constraint position if the estimated position is lower than the real (case 1 with $c \leq 0$). In the second case, the controller scheme is robust only if we add to the supervisor a high-level decision law which monitor if contact occur or not. Then decide if this non-attendance of contact is due to constraint position error or due to anything else.

7 Robustness with respect to the measurement noise

This section deals with the robustness of this control scheme with respect to measurement noise on position and velocity. Figs. 21-22 show the Lyapunov function of the two control schemes, respectively the Paden-Panja scheme with white noise of 5% added on $q$ and $\dot{q}$, and the Slotine-Li law with noise level of 40%. These noise levels are the maximum for each law before instability.

Fig. 23 shows the evolution of the Lyapunov function of a simulation of the Paden-Panja scheme at the limit of the stability, with a noise level of 10%. The jumps $\sigma V(\tau_{k, cyc})$ increase over the cycles. This example is not robust because it doesn’t fulfil all requirements of proposition 1 (see remark on Fig. 10 sec. 4).

From these simulations, we conclude that the hybrid Slotine-Li scheme is more robust than the hybrid Paden-Panja one. But on Fig. 24, we see than even if the closed loop system does not
diverge when the noise level is set to 40%, then the tracking control is very bad with such noise. Exponential stability improves robustness of the hybrid Slotine-Li controller.

Another important point to deal with is the magnitude of the force control. If the force controller gains \( k_f \) and reference \( P_d \) are too small, then the noise on the measurement of the position can induce unwanted detachment during the permanently constraint phase.

8 Conclusions

The aim of this paper is to study the robustness of nonsmooth passivity based controllers with respect to errors on the dynamical model parameters, to the constraint position knowledge and to the measurement noise. We study the extension of two laws to the case of nonsmooth dynamic: the Paden Panja PD+ controller the Slotine Li control law. These two laws are robust with respect to dynamical model errors. They are very robust if the constraint position is under-estimated. In the case of constraint estimated above the real position, we present a solution to resolve this difficulty. Finally the hybrid Slotine-Li controller is very robust with respect to measurement noise whereas the hybrid Paden-Panja scheme is much less robust.

It is important to remind that this system is robust with respect to the knowledge of the restitution’s coefficient due to the structure of the controller. Indeed the value of \( e_n \) is never used in the control law.

Globally the hybrid Slotine-Li scheme is more robust than the hybrid Paden Panja scheme: as explained above, this is due to the fact that stability conditions are easier to fulfil for Slotine-Li (proposition 2) than for Paden-Panja scheme (proposition 1). These results are validated on numerical simulations.

References

Disturbance attenuation for a periodically excited
piece-wise linear beam system

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Abstract. In this paper, we consider the problem of disturbance attenuation for a class of piece-wise linear systems. The proposed control design ensures that the closed-loop system is uniformly convergent. Uniform convergence guarantees the existence of a unique globally asymptotically stable steady-state solution for a given periodic disturbance. This property allows to uniquely assess the performance of the controller in terms of disturbance attenuation. Both state-feedback and output-feedback variants of the control design are presented. The effectiveness of the strategy is shown by application to a piece-wise linear beam system.

1 Introduction

The motivation for this work originates from the need to analyse and control the dynamics of complicated engineering constructions including structural elements with piece-wise linear (PWL) restoring characteristics, such as tower cranes, suspension bridges and solar panels on satellites [Heertjes, 1999]. More specifically, the disturbance attenuation problem is an important control problem to be solved to ensure the performance of these systems and to avoid damage to the structures. Since the dynamics of such systems are generally formulated as PWL systems, we will investigate the disturbance attenuation problem for PWL systems. PWL systems are currently receiving a great deal of attention.

In [Johansson and Rantzer, 1998], a new framework was developed, based on piece-wise quadratic Lyapunov functions, to analyse the stability of piece-wise affine (PWA) systems. In [Rantzer and Johansson, 2000] this framework was extended for performance analysis and optimal control. In [Hassibi and Boyd, 1998], a study related to stability analysis and controller design for PWL systems was presented. This study uses common and piece-wise quadratic Lyapunov functions for stability purposes. Here, in the case of a common quadratic Lyapunov function, both the stability analysis and the state-feedback synthesis can be expressed as a convex optimization problem based on constraints in linear matrix inequality (LMI) form. However, it has been pointed out that this is difficult in the case of a piece-wise quadratic Lyapunov function. A solution for this problem has been given in [Feng et al., 2002] and [Rodrigues et al., 2000]. [Feng et al., 2002] presents a $H_\infty$ controller synthesis method based on a piecewise quadratic Lyapunov function that can be cast in the form of solving a set of LMIs using standard LMI solvers. [Rodrigues et al., 2000] shows a method used to design state- and output-feedback controllers with constraints on the smoothness and continuity of the piecewise quadratic Lyapunov function. However, the controller design of [Rodrigues et al., 2000] is restricted, as it is mentioned in [Rodrigues et al., 2000], by two fundamental assumptions: 1) there are no sliding modes at the hyperplane boundaries between regions with different affine dynamics, 2) the examined PWL system and the controller are always in the same region. [Rodrigues and How, 2001] examines the case where the assumptions in [Rodrigues et al., 2000] are violated and presents a general stability analysis of the closed-loop system for that case.

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A common characteristic of the papers [Johansson and Rantzer, 1998], [Rodrigues et al., 2000], [Hassibi and Boyd, 1998] and [Rantzer and Johansson, 2000] is that they guarantee stability of a PWL system for zero inputs. In the papers [Hassibi and Boyd, 1998], [Feng et al., 2002], [Khalil, 2002] and [Rodrigues and How, 2001] it is assumed that given an initial condition for a PWL system, an input signal, and a disturbance, the systems has a unique solution for $t > 0$.

In [Demidovich, 1967] (see also [Pavlov et al., 2004]), the notion of convergence for nonlinear systems with inputs is introduced. A system with this property has a unique globally asymptotically steady-state solution which is determined only by the system input and does not depend on the initial conditions. In [Pavlov, 2004] and [Pavlov et al., 2005], the notion of convergent systems is extended to the notion of (uniformly, exponentially) convergent systems and input-to-state convergent systems (in section 2, further information about these notions is given). Based on the extensions made in [Pavlov, 2004], the design of a controller that renders a non-convergent system convergent, is pursued. Furthermore, in [Pavlov, 2004] the first result on convergence for PWA systems is published.

So far, results related to performance of PWL/PWA systems, in terms of disturbance attenuation, where given among others, in [Rantzer and Johansson, 2000], [Hassibi and Boyd, 1998] and [Feng et al., 2002]. The performance results of these papers, which are based on single or piece-wise quadratic Lyapunov functions, provide an upper bound for the system output by bounding the $L_2$ gain from the system input to the system output. Nevertheless, these results are not very general, since they have been derived under the assumption of zero initial conditions.

In this paper we propose a controller design strategy for a class of bi-modal PWL systems, based on the extended notions of convergence, in order to study the performance of such systems for disturbance attenuation. The convergence property is beneficial in the scope of performance analysis of bi-modal PWL systems, because it ensures that these systems exhibit unique steady-state solutions. Due to the fact that convergence is based on a quadratic Lyapunov function, we can provide an upper bound for the system states in (steady-state) given a bounded input which is similar to the bounds presented in [Rantzer and Johansson, 2000], [Hassibi and Boyd, 1998] and [Feng et al., 2002], for any initial condition. In addition to that, the uniqueness of the system steady-state response allows for a more accurate evaluation of the performance based on computed responses. In this paper, we focus on a specific class of disturbances, namely harmonic disturbances. The motivation for this choice lies in the fact that in engineering practice many disturbances can be approximated by harmonic signals.

More specifically, this paper presents a controller design strategy for a class of bi-modal PWL systems and treats its application to a piece-wise linear model of an experimental beam system. This system consists of a flexible steel beam, which is clamped on two sides and is supported by a one-sided linear spring. Due to the one-sided spring the beam has two different dynamical regimes, which both can be well described as being linear. This system is excited by exogenous periodic disturbances.

The goal of the strategy is the performance of the closed-loop PWL beam system in terms of disturbance attenuation. In order to uniquely define the performance of the closed-loop system it should not have multiple steady-state solutions. This property can be attained by rendering the PWL beam system convergent by means of feedback.

The controller design strategy uses state- and output-feedback control laws in order to render the closed-loop system of the PWL beam convergent. The output-feedback controller is a combination of a model-based switching observer [Juloski et al., 2002] and a state-feedback controller.

The paper structure is as follows. The controller design strategy is introduced in section 2. In sections 3 and 4, state- and output-feedback controllers are designed for a bi-modal PWL system, respectively. A description of the PWL beam system is given in section 5. In section 6, simulation results related to the controller performance are presented. Conclusions and directions for future work are given in section 7.
2 Controller design strategy

We consider the following class of bi-modal time-continuous PWL systems:

\[ \dot{x}(t) = \begin{cases} A_1 x(t) + B w(t) + B_1 u(t) & \text{for } H^T x(t) \leq 0 \\ A_2 x(t) + B w(t) + B_1 u(t) & \text{for } H^T x(t) > 0 \end{cases} \]

where \( x(t) \in \mathbb{R}^n \), \( y(t) \in \mathbb{R}^p \), \( u(t) \in \mathbb{R}^q \) and \( w(t) \in \mathbb{R}^m \) are the state, the output, the control input and the exogenous input of the system, respectively, depending on time \( t \in \mathbb{R} \). The input \( w(t) \) acts as a disturbance on the system and it is considered to be periodic. The matrices \( A_1, A_2 \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, B_1 \in \mathbb{R}^{n \times q}, C \in \mathbb{R}^{p \times n} \) and \( H \in \mathbb{R}^p \). The hyperplane defined by \( \ker H^T \) separates the state space \( \mathbb{R}^n \) in two half-spaces. The considered class of bi-modal PWL systems has identical input matrices \( B, B_1 \) and an identical output matrix \( C \) for both modes.

The goal of the controller design strategy is the disturbance attenuation of such systems for a range of periodic excitations. Disturbance attenuation roughly measures to what extent the amplitude of a periodic disturbance \( w(t) = A \sin \omega t \) is amplified/suppressed in the output or in (each component of) the state \( x(t) \). Obviously, such measure only makes sense if the steady-state response remains bounded and is unique under a periodic excitation.

Due to the fact that PWL systems are nonlinear, they often exhibit multiple steady-state solutions when excited by periodic disturbances. In order to uniquely define the performance of the closed-loop system it should not have multiple steady-state solutions. The present strategy focuses on attaining such property by making PWL systems globally, uniformly convergent. A detailed treatment of convergent systems was given in [Demidovich, 1967] (see also [Pavlov et al., 2004]).

Consider the system

\[ \dot{z} = F(z, w(t)), \]

with state \( z \in \mathbb{R}^d \) and input \( w \in \mathbb{R}^m \), where \( F(z, w) \) is locally Lipschitz in \( z \) and continuous in \( w \). The input \( w(t) \) is a piecewise continuous function of \( t \) defined for all \( t \in \mathbb{R} \).

**Definition 1.** System (2) with given input \( w(t) \) is said to be (uniformly, exponentially) convergent if

1. all solutions \( z(t) \) are well defined for all \( t \in [t_0, +\infty) \) and all initial conditions \( t_0 \in \mathbb{R}, z(t_0) \in \mathbb{R}^m \);
2. there exists a unique solution \( \bar{z}_w(t) \) defined and bounded for all \( t \in (-\infty, +\infty) \);
3. the solution \( \bar{z}_w(t) \) is globally (uniformly, exponentially) asymptotically stable.

If system (2) is convergent for a class of inputs, then for every input from this class it has a unique bounded globally, asymptotically stable, steady-state solution \( \bar{z}_w(t) \).

If the input of a convergent system is periodic with period \( T \), then the corresponding \( \bar{z}_w(t) \) is also periodic with the same period \( T \), see [Pavlov et al., 2004].

In the present work, given the fact that the (convergent) closed-loop system exhibits periodic solutions with period \( T \), we can define performance more specifically by saying that we want to minimize

\[ \max_{s \in [t, t+T]} |x_i(s)|, \text{ for } i = 1, ..., n, \]

over a specific (excitation) frequency range. Herein, \( x_i(s) \) are the state components of system (1).

The problem at hand is to provide a suitable control input \( u(t) \) to system (1) such that, for a given periodic, continuous and bounded input \( w(t) \), 1) the closed-loop system exhibits a unique periodic steady-state solution and 2) the amplification of the bounded input amplitude in (each component of) the closed-loop system states is smaller than the amplification of the bounded input.
amplitude in (each component of) the open-loop system states. Note that we will ensure the first property of the closed-loop system by making it uniformly convergent by means of feedback.

In section 5, the output feedback control design will be based on the input-to-state convergence (ISC) property of the control system. Let us now introduce the ISC property.

Consider the system
\[ \dot{z} = F(z, w, t), \]
t \in \mathbb{R}, z \in \mathbb{R}^d, w \in \mathbb{R}^m, where \( F(z, w, t) \) is piece-wise continuous in \( t \), continuous in \( w \) and locally Lipschitz in \( z \). The input \( w(t) \) is a piecewise continuous function of \( t \).

**Definition 2.** [Pavlov et al., 2004] System (4) is said to be input-to-state convergent (ISC) if it is globally uniformly convergent for a class of piece-wise continuous inputs, and for every input \( w(t) \) taken from this class, the system is input-to-state stable [Khalil, 2002] with respect to the system’s solution \( \tilde{z}_w(t) \), i.e. there exist a KL-function \( \beta(r, s) \) [Khalil, 2002] and a class \( K_\infty \)-function [Khalil, 2002] \( \gamma(\tau) \) such that any solution of this system corresponding to some input \( \tilde{w}(t) := w(t) + \Delta w(t) \) satisfies
\[
|z(t) - \tilde{z}_w(t)| \leqslant \beta(|z(t_0) - \tilde{z}_w(t_0)|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} |\Delta w(\tau)|).
\]

3 State-feedback controller design

In the controller design strategy, a static state-feedback is chosen as the input for the system (1):
\[ u(t) = -Kx(t), \]
where \( u(t) \) is the control action and \( K \in \mathbb{R}^{1 \times n} \) is the controller gain. Consequently, the dynamics of the closed-loop system (1) and (6) can be written as:
\[
\begin{align*}
\dot{x}(t) &= \begin{cases} A_0x(t) + Bw(t) & \text{for } H^Tx(t) \leq 0 \\
A_0x(t) + Bw(t) & \text{for } H^Tx(t) > 0 \end{cases} \quad (7a) \\
y(t) &= Cx(t), \quad (7b)
\end{align*}
\]
where \( A_0 = A_1 - B_1K \) and \( A_0 = A_2 - B_1K \). The closed-loop system described by (7) is also a bi-modal PWL system with an identical input matrix \( B \) and has an identical output matrix \( C \) for both modes. Furthermore, the hyperplane defined by \( \ker H^T \) separates the state-space \( \mathbb{R}^n \) of the closed-loop system in two half-spaces.

The controller design problem can now be formally stated as:

**Problem:** Determine, if possible, the controller gain \( K \) in (6) such that 1) the closed-loop system (7) is globally, uniformly convergent for a class of piece-wise continuous inputs \( w : \mathbb{R}^+ \rightarrow \mathbb{R}^m \) and 2) for a given disturbance \( w(t) \) the maximum absolute value of the state components of (7), \( \max(|x_i|), i = 1, ..., n, \) is lower than the maximum absolute value of the uncontrolled state components \( \max(|x_i|), i = 1, ..., n \).

Note that here we consider a class of bounded periodic disturbances \( w(t) \) and that the uncontrolled system derives from (1) when \( w = 0 \).

The first part of this problem can be solved using a result in [Pavlov, 2004], which states conditions under which system (7) is globally uniformly convergent and ISC for all piece-wise continuous disturbances \( w \):

**Theorem 1.** Consider the state-space \( \mathbb{R}^n \) which is divided into regions \( \Lambda_i, i = 1, ..., l, \) by hyperplanes given by equations of the form \( H_j^T z + h_j = 0 \), for some \( H_j \in \mathbb{R}^n \) and \( h_j \in \mathbb{R}, j = 1, ..., k \). Consider the piece-wise affine system
\[
\dot{z} = A_i z + b_i + Dw(t), \quad \text{for } z \in \Lambda_i, i = 1, ..., l.
\]
Suppose that the right-hand side of (8) is continuous and there exists a positive definite matrix \( Q = Q^T \) such that
\[
QA_i + A_i^T Q < 0, \quad i = 1, ..., l.
\]
(9)

Then the system (8) is globally exponentially convergent and ISC for piecewise continuous bounded inputs.

4 Output-feedback controller design

In general, the entire state of (1) will not be available for feedback. Therefore, the goal of this section is to construct an output-feedback controller that solves the problem stated in the previous section for the system (1).

This output-feedback controller consists of a state-feedback controller as in (6) and a switching model-based observer. This observer recovers the states of the system without any information on which linear dynamics of the system is currently active.

Now, we will propose such observer/controller combination such that the resulting closed-loop system, hereafter called the interconnected system, is globally, uniformly convergent. This will allow once more for a unique performance evaluation.

The choice of the observer/controller combination that renders the interconnected system globally, uniformly convergent is based on a property presented in [Pavlov, 2004]:

**Property 1.** Consider the system

\[
\begin{cases}
\dot{z} = F(z, y, w), \quad z \in \mathbb{R}^d \\
\dot{y} = G(z, y, w), \quad y \in \mathbb{R}^q.
\end{cases}
\]

(11)

Suppose that the z-subsystem is input-to state-convergent with respect to \( y \) and \( w \). Assume that there exists a class KL function \( \beta_y(r, s) \) such that for any piece-wise continuous input \( (w(\cdot), z(\cdot)) \), any solution of the y-subsystem satisfies
\[
|y(t)| \leq \beta_y(|y(t_0)|, t - t_0).
\]

(12)

Then the interconnected system (11) is input-to-state convergent.

In Figure 1 a schematic representation of the interconnected system (11) is depicted.
We consider a switching observer of the following structure

\[
\begin{cases}
\dot{x}(t) = A_1 x(t) + B_1 w(t) + B_1 u(t) + L_1 \Delta y(t), & \text{if } H^T \dot{x} \leq 0 \\
\dot{x}(t) = A_2 x(t) + B_2 w(t) + B_2 u(t) + L_2 \Delta y(t), & \text{if } H^T \dot{x} > 0,
\end{cases}
\]

for the system (1), with \( L_1, L_2 \in \mathbb{R}^{n \times p} \) and \( \dot{x}(t) \in \mathbb{R}^n \). The observer output is \( \hat{y}(t) = C \dot{x}(t) \) and \( \Delta y(t) = y(t) - \hat{y}(t) \). The model output \( y \) is used as observer output injection.

The dynamics of the observer error \( \Delta x(t) = x(t) - \dot{x}(t) \) is described by

\[
\Delta x(t) =
\begin{cases}
(A_1 - L_1 C) \Delta x, & \text{if } H^T x \leq 0 \land H^T \dot{x} \leq 0 \\
(A_2 - L_2 C) \Delta x + \Delta A x, & \text{if } H^T x < 0 \land H^T \dot{x} > 0 \\
(A_1 - L_1 C) \Delta x - \Delta A x, & \text{if } H^T x > 0 \land H^T \dot{x} \leq 0 \\
(A_2 - L_2 C) \Delta x, & \text{if } H^T x > 0 \land H^T \dot{x} > 0,
\end{cases}
\]

where \( \Delta A = A_1 - A_2 \).

In [Juloski et al., 2002] a result is proposed that provides a set of LMI constraints that guarantees global asymptotic stability of the observer error dynamics described in (14). Unfortunately, these constraints are not sufficient in the present case. An extension of this theorem is given in order to provide a set of LMI constraints that guarantees global exponential stability of the observer error.

**Theorem 2.** The observer error dynamics (14) is globally exponentially stable (GES) for all \( x : \mathbb{R}^n \rightarrow \mathbb{R}^n \) (in the sense of Lyapunov), if there exist matrices \( P = P^T > 0, L_1, L_2 \) and constants \( \tau_1, \tau_2 \geq 0, \alpha > 0 \) such that the following set of matrix inequalities is satisfied:

\[
\begin{bmatrix}
(A_2 - L_2 C)^T P + P(A_2 - L_2 C) + \alpha P + \frac{1}{2} \tau_1 H H^T \\
\frac{1}{2} \tau_1 H H^T
\end{bmatrix} \leq 0
\]

(15a)

\[
\begin{bmatrix}
(A_1 - L_1 C)^T P + P(A_1 - L_1 C) + \alpha P + \frac{1}{2} \tau_2 H H^T \\
\frac{1}{2} \tau_2 H H^T
\end{bmatrix} \leq 0.
\]

(15b)
Hence, it can be very efficiently determined whether there exists a quadratic Lyapunov function that proves global exponential stability of the observer error. For the proof of Theorem 2 the reader is referred to Appendix B. Note that $L_1$, $L_2$ are non unique. $L_1$, $L_2$ influence the rate of convergence of the observer error to zero. In case there is measurement noise in the observer output injection, the choice of $L_1$, $L_2$ should be a balance between convergence rate and noise amplification. The inequalities in (15) are nonlinear matrix inequalities in $\{P, L_1, L_2, \lambda_1, \lambda_2\}$, but are linear in $\{P, L_1^T P, L_2^T P, \tau_1, \tau_2\}$. Thus, they can be efficiently solved using linear matrix inequalities solvers (such as the software LMItool for Matlab).

4.2 Input-to-state convergence for the PWL system in closed-loop with the state-feedback controller

Using the control law

$$u(t) = -K \hat{x}(t),$$

in (1a) yields

$$\dot{x}(t) = \begin{cases} A_a x(t) + B w(t) - B_1 K \Delta x(t), & \text{if } H^T x \leq 0 \\ A_b x(t) + B w(t) - B_1 K \Delta x(t), & \text{if } H^T x > 0. \end{cases}$$

(17)

Observing equations (14) and (17), it is straightforward that the corresponding systems constitute an interconnected system as in (11). Using Theorem 1 for (17), we derive the inequalities (10a)-(10c). These inequalities guarantee that system (17) is input-to-state convergent with respect to $w(t)$ and $\Delta x(t)$.

4.3 Global uniform convergence of the interconnected system

By applying Property 1 to the interconnected system, we prove that the interconnected system is globally uniformly convergent. Here, we use that: 1) (17) is ISC with respect to $w(t)$ and $\Delta x(t)$ and 2) (14) is GES. This in fact means that the separation principle holds for the observer/controller combination. Due to the fact that 1) holds, the system state (17) always converges to a unique, bounded steady state solution for every finite initial condition and for bounded inputs $w(t)$ and $\Delta x(t)$. Therefore, the use of the observer (13), for system state reconstruction, has no influence to the stability of the interconnected system. Furthermore, due to the fact that 2) holds, $x_{w, \Delta x}$ will converge to the steady-state solution $x_{w, \Delta x=0}$ ($x_{w, \Delta x=0}$ is the steady-state solution of (17) for $\Delta x = 0$).

5 Application to a piece-wise linear beam system

In this section we introduce a PWL beam system depicted in Figure 2. The developed controller design strategy is applied to this system.

The PWL beam system consists of a steel beam supported at both ends by two leaf springs. The beam is excited by a force $w$ generated by a rotating mass-unbalance, which is mounted at the middle of the beam, see Figure 3. A tacho-controlled motor, that enables a constant rotation speed, drives the mass-unbalance. An actuator applies a control force $u$ to the beam. A second beam, that is clamped at both ends, is located parallel to the first one and represents a one-sided spring. This spring represents a non-smooth nonlinearity in the dynamics of the PWL beam system and as a result the beam system (beam and one-sided spring) has nonlinear and non-smooth dynamics. The restoring characteristic of the one-sided spring is assumed linear; consequently, the beam system can be described as a piece-wise linear system, as shown in the next section.
5.1 Dynamics of the PWL beam system

The dynamics of the PWL beam system can be described by a three-degree-of-freedom (3DOF) model [Doris et al., 2005] of the following form

\[
M \ddot{q} + B_s \dot{q} + K_s q + f_{nl}(q) = h_1 w(t) + h_2 u(t),
\]

where \( h_1 = [1 \ 0 \ 0]^T \), \( h_2 = [0 \ 1 \ 0]^T \) and \( q = [q_{mid} \ q_{act} \ q_\xi]^T \). Herein, \( q_{mid} \) is the displacement of the middle of the beam and \( q_{act} \) is the displacement of the point of the beam at which the actuator is mounted, see in Figure 3. Moreover, \( q_\xi \) reflects the contribution of the first eigenmode of the beam and \( M, B \) and \( K_s \) are the mass, the damping and the stiffness matrices of the 3DOF model, respectively. We apply a periodic (harmonic) excitation force

\[
w(t) = A \sin \omega t,
\]

which is generated by the rotating mass-unbalance at the middle of the beam. Herein, \( \omega \) is the excitation frequency and \( A \) the amplitude of the excitation force. Moreover, \( f_{nl} \) is the restoring force of the one-sided spring:

\[
 f_{nl}(q) = k_{nl} h_1 \min(0, h_1^T q) = k_{nl} h_1 \min(0, q_{mid}),
\]

where \( k_{nl} \) is the stiffness of the spring. The force \( f_{nl} \) acts when there is contact between the middle of the beam and the one-sided spring.

In a state-space formulation, the model takes the form of (1) and by using the observer-based state-feedback (16) it can be written in the form of (17), where \( x = [q^T \ \dot{q}^T]^T \) and \( H = [h_1^T \ 0^T]^T \).
Furthermore,
\[
A_1 = \begin{bmatrix}
0 & I \\
-M^{-1}(K_s + k_{nl} h_1 h_1^T) - M^{-1}B_s \\
\end{bmatrix},
\]
\[
A_2 = \begin{bmatrix}
0 & I \\
-M^{-1}K_s - M^{-1}B_s \\
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
0 \\
M^{-1}h_1 \\
\end{bmatrix},
B_1 = \begin{bmatrix}
0 \\
M^{-1}h_2 \\
\end{bmatrix},
\]
and
\[
0 = [0 0 0]^T.
\]

In the examined case, the output of (1), \( y(t) = Cx(t) \), describes a transversal displacement of a point 1 on the beam, depicted in figure 3. The numerical values of \( M, B_s, K_s, k_{nl} \) and \( C \) are given in Appendix A.

6 Simulation of the PWL beam system

In order to illustrate the effectiveness of the control strategy proposed in sections 2, 3, 4 and 5, simulation results related to the PWL beam are presented.

In the first part of this section, it is shown that the observer error converges to zero exponentially and in the second part, it is shown that the interconnected system consisting of (14) and (17) is globally uniformly convergent. Note that in the examined case, the z- and y-subsystems of (11) are represented by (17) and (14), respectively.

6.1 Global exponential stability of the observer error

In order to design the observer (13) for the interconnected system, the transversal displacement of a properly chosen point on the beam is used as observer output injection. This displacement is the model output \( y(t) = Cx(t) \), see figure 3. The position of this point should be chosen such that the LMIs (15) are feasible. By solving these LMIs the gains \( L_1, L_2 \), that guarantee global asymptotic stability of the observer error, are calculated. The numerical values of these gains are given in the Appendix A.

In figure 4, the observer error states \( \Delta x_4(t) = \hat{\dot{q}}_{\text{mid}}(t) - \dot{q}_{\text{mid}}(t) \), \( \Delta x_5(t) = \hat{\dot{q}}_{\text{act}}(t) - \dot{q}_{\text{act}}(t) \) and \( \Delta x_6(t) = \hat{\dot{q}}(t) - \dot{q}(t) \) and an exponential boundary of the observer error are depicted. This boundary (dashed line) has the form \( 1/\sqrt{\lambda_{\min}(P)} |\Delta x(t_0)| e^{-\alpha t} \) and it is derived from (31). The values for \( P, \alpha \) and \( \Delta x(t_0) \) are given in Appendix A. Based on this figure, the observer error converges to zero exponentially. Therefore, the Property 1 can be applied to the interconnected system.
6.2 Global uniform convergence of the interconnected system and attained disturbance attenuation

In this subsection, we show that 1) the PWL beam system in closed-loop with the observer-based controller exhibits a unique asymptotically stable steady-state solution and 2) the effect of the excitation force \( w \) on the systems response is significantly smaller in the closed-loop system than in the open-loop system. More specifically, we show that the maximum value of the transversal displacement of the points on the beam are significantly smaller when a control force \( u \) is acting on the beam than in the open-loop case.

Numerical computation of the periodic solutions of the open-loop PWL beam system ((1) with \( u = 0 \)) for harmonic disturbances, as in (19), shows that this system is not globally uniformly convergent. Hereto, the collocation method [Doedel et al., 1998] and the path-following procedure [Ascher et al., 1995] are used.

More specifically, in figures 5, 7, and 8, the plots of \( \max(|q_{\text{mid}}|) \), \( \max(|q_{\text{act}}|) \) and \( \max(|q_\xi|) \) for such periodic solutions are depicted for an excitation frequency range of \( 10 - 60 \) [Hz]. \( q_{\text{mid}} \), \( q_{\text{act}} \) and \( q_\xi \) are derived from the open-loop system and they are divided by the the input amplitude \( A \) in order to take a normalized form. Based on these figures, \( q_{\text{mid}} \), \( q_{\text{act}} \) and \( q_\xi \) exhibit two steady-state solutions for excitation frequencies within the frequency range of \( 39 - 56 \) [Hz]. In this frequency range, the dashed line is an unstable harmonic solution and the solid line is a stable \( \frac{1}{2} \) subharmonic solution. Due to the fact that the open-loop system exhibits two steady-state solutions, it is not convergent.

By using numerical analysis for the PWL beam closed-loop system (interconnected system (14) and (17)) for such periodic disturbances, we show that this system is globally uniformly convergent, as guaranteed by the theory. In figures 5, 7, and 8, the plots of \( \max(|q_{\text{mid}}|) \), \( \max(|q_{\text{act}}|) \) and \( \max(|q_\xi|) \) of the closed-loop system are depicted (dash-dotted lines). Based on these figures, \( q_{\text{mid}} \), \( q_{\text{act}} \) and \( q_\xi \) exhibit a unique steady-state solution in the frequency range of \( 10 - 60 \) [Hz]. This fact indicates that the controlled system is convergent and indeed a unique performance assessment in terms of disturbance attenuation can now be performed. For a better understanding of these results also a time response of \( q_{\text{mid}} \) is shown in figure 6. In this figure the time response of \( q_{\text{mid}} \)
Disturbance attenuation for a periodically excited piece-wise linear beam system is depicted for three different initial conditions $x_{0i}$, $i = 1, 2, 3$ (for the numerical values of $x_{0i}$ see Appendix A). The excitation frequency and the force amplitude for the examined case are $f = 45 \text{ Hz}$ and $A = 81 \text{ N}$, respectively. Figure 6 shows that the time response of $q_{mid}$ converges to a unique steady-state solution for different initial conditions.

The comparison of the plots of $\max(|q_{mid}|)$, $\max(|q_{act}|)$ and $\max(|q_\xi|)$ calculated for the open- and closed-loop systems shows that the closed-loop system responses are significantly smaller than those of the open-loop system. Based on this comparison, it is concluded that the effect of the disturbances $w$ to the PWL beam is attenuated due to the control force $u$. Note that especially the nonlinear resonances are suppressed. This can also be noticed in figure 9, where the time response of $q_{mid}$ in steady-state is shown. In this figure the dashed line is the open-loop solution of $q_{mid}$, while the solid line is the closed-loop solution. The excitation frequency for this case is $22 \text{ Hz}$ and the force amplitude is $A = 18 \text{ N}$ (see also the vertical dashed line in Figure 5).

**Remark:** The control gain $K$ is calculated initially by solving LMI (10) using the toolbox LMITool of Matlab. The elements of $K$ derived in this way are in the order of $10^9$. Applying a high gain control in an experimental system may firstly, lead to noise amplification, which is undesirable for the system performance, and secondly, lead to actuator saturation. In addition to that, high control gain implies big control effort for the suppression of the system resonance peaks. Therefore, a more sophisticated way to overcome such high gain controller design is followed. Due to the fact that LMI (10) provides sufficient conditions for convergence, $K$ is not unique. Based on engineering insight, we choose a control gain that adds damping to the nonlinear resonances of the system. In this way, the system resonance peaks are suppressed. By using LMI constraints (10) we check whether the system remains convergent. Based on trial and error technique, we notice that by adding damping in $q_{mid}$, we render the system convergent and reduce the resonance peaks in all system states (see figures 5, 7, and 8). Based on this approach, we achieve small control gain values with respect to the initial ones. These values are in the order of $10^2$. A more constructive way to choose a control gain $K$ is by using an LMI condition that ensures bounds on the control action. The development of such LMI is subject of future work.

**Fig. 5:** Scaled maximum absolute values of the transversal displacement of the middle of the beam, based on the open-loop system (solid line, dashed line) and the interconnected system (14) and (17) (dashed-dotted line).
Fig. 6: The transversal displacement of the middle of the beam, for the interconnected system (14) and (17) and for different initial conditions $x_{0i}$ ($\omega = 2\pi 45$ rad/s and $A = 81$ N).

Fig. 7: Scaled maximum absolute values of the transversal displacement of $q_{act}$, based on the open-loop system (solid line, dashed line) and the interconnected system (14) and (17) (dashed-dotted line).
Disturbance attenuation for a periodically excited piece-wise linear beam system

\[ \max|q_t| \text{ scaled by } A \]

Fig. 8: Scaled maximum absolute values of the transversal displacement of \( q_t \), based on the open-loop system (solid line, dashed line) and the interconnected system (14) and (17) (dashed-dotted line).

\[ q_{\text{mid}} \text{ for } f = 22 \text{ Hz and } A = 18 \text{ N} \]

Fig. 9: The steady-state solution of the transversal displacement of the middle of the beam, based on the closed-loop (solid line) and the open-loop system (dashed line) for \( \omega = 2\pi 22 \text{ rad/s} \) and \( A = 18 \text{ N} \).
7 Conclusions and Future work

The controller design strategy developed in the present work has proven to be suitable for disturbance attenuation of bi-modal piece-wise linear (PWL) systems excited by periodic disturbances.

We propose a convergence-based controller design for disturbance attenuation. More specifically, we use the fact that a nonlinear system has a unique globally asymptotically stable solution when it is uniformly convergent. Convergence has been used in this paper in order to uniquely define the performance of the closed-loop system.

In the present paper, we define disturbance attenuation as the suppression of the vibrations of a PWL system, caused by exogenous periodic disturbances, over a specific frequency range. By performance we indicate the ability of the controller to achieve such disturbance attenuation.

The strategy is applied to a bi-modal PWL beam system. The control laws proposed to render the closed-loop system of the PWL beam convergent and to attain disturbance attenuation are 1) a static state-feedback controller and 2) an output-feedback controller. For the output-feedback controller, a model-based switching observer is used.

The simulation results show that the interconnected system, consisting of the PWL beam in closed-loop with the observer-based controller and the observer, is globally uniformly convergent. In addition, the designed controller has been shown to perform well, since it suppresses all the (nonlinear) resonance peaks of the beam’s transversal vibrations considerably in the presence of periodic disturbances.

Interesting extensions of the present work may include the experimental implementation of the proposed control strategy for PWL systems; especially on the PWL beam system.

References


Appendix A

The matrices $M$, $K_s$, $B_s$, $K$, $C$ and the values of $k_{nl}$ and $x_{01}$, $x_{02}$, $x_{03}$ are

\[
M = \begin{bmatrix} 4.494 & -2.326 & 0.871 \\ -2.326 & 7.618 & 2.229 \\ 0.871 & 2.229 & 2.374 \end{bmatrix},
\]

\[
K_s = 10^6 \begin{bmatrix} 2.528 & -0.345 & 1.026 \\ -0.345 & 1.082 & 0.296 \\ 1.026 & 0.296 & 0.613 \end{bmatrix},
\]

\[
B_s = 10^2 \begin{bmatrix} -0.298 & 2.012 & 0.314 \\ 0.416 & 0.314 & 0.365 \end{bmatrix},
\]

\[
K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 535 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
C = \begin{bmatrix} -0.9579 & 1.2165 & -0.2642 & 0 & 0 & 0 \end{bmatrix},
\]

\[
k_{nl} = 198000 \text{ N/m},
\]

\[
x_{01} = [10^{-3} \ 0 \ 0 \ 0 \ 0 \ 0],
\]

\[
x_{02} = [0 \ 0 \ 0 \ 0 \ 0 \ 0]
\]

\[
x_{03} = 10^{-3} [-0.3 \ -0.3 \ 0.7 \ 2.1 \ 3.7 \ -4.5].
\]

The values of $\Delta x(t_0)$, $L_1$ and $L_2$ are

\[
\Delta x(t_0) = [0.001 \ 0 \ 0 \ 0 \ 0 \ 0],
\]

\[
L_1 = 10^3 [0.0322 \ 0.0468 \ -0.1110 \ -8.9161 \\ 3.3834 \ -5.7828],
\]

\[
L_2 = [0.0329 \ 0.0472 \ -0.1121 \ -8.7315 \\ 3.5947 \ -6.1488].
\]

The values of $P$, $\alpha$ and $|\Delta x(t_0)|_P$ are:

\[
P = 10^{-7} \begin{bmatrix} 2.333 & -2074 & 2.1998 & -0.31 & -1.08 & 0.59 \\ -2074 & 6531 & 14.85.85 & 2.18 & -1.76 & -1.95 \\ 220 & 1486 & 8.64.59 & -0.33 & 1.47 & -0.37 \\ -0.31 & 0.02 & -0.00.33 & 0.01 & -0.02 & 0.00 \\ -1.09 & -1.76 & 0.01.48 & -0.02 & 0.05 & 0.01 \\ 59 & -1.95 & -0.00.37 & 0.00 & 0.01 & 0.01 \end{bmatrix},
\]

\[
\alpha = 100 \text{ and } |\Delta x(t_0)|_P = 1.53 \times 10^{-5}.
\]

Appendix B

Proof of theorem 2

We propose a Lyapunov candidate function $V$ of the following form:

\[
V(\Delta x) = \Delta x^T P \Delta x,
\]

with $P = P^T > 0$.

Based on [Juloski et al., 2002], we can show that if $H^T x \leq 0$ and $H^T (x - \Delta x) \leq 0$ then

\[
\dot{V}(\Delta x) = \Delta x^T ((A_1 - L_1 C)^T P + P(A_1 - L_1 C)) \Delta x,
\]

(22a)
if $H^T x \leq 0$ and $H^T (x - \Delta x) > 0$ then

$$
\dot{V}(\Delta x) = \begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2 C)^T P_+ P \Delta A + P(A_2 - L_2 C) \\ \Delta A^T P \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix}, 
$$

(22b)

if $H^T x > 0$ and $H^T (x - \Delta x) \leq 0$ then

$$
\dot{V}(\Delta x) = \begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} (A_1 - L_1 C)^T P_+ -P \Delta A + P(A_1 - L_1 C) \\ -\Delta A^T P \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix},
$$

(22c)

and if $H^T x > 0$ and $H^T (x - \Delta x) > 0$ then

$$
\dot{V}(\Delta x) = \Delta x^T ((A_2 - L_2 C)^T P + P(A_2 - L_2 C)) \Delta x.
$$

(22d)

Multiplication of $H^T x \leq 0$ and $H^T (x - \Delta x) > 0$ or $H^T x > 0$ and $H^T (x - \Delta x) \leq 0$ leads to:

$$
H^T x \leq 0 \text{ and } H^T (x - \Delta x) > 0 \Rightarrow \\
H^T x H^T (x - \Delta x) \leq 0
$$

(23)

and

$$
H^T x > 0 \text{ and } H^T (x - \Delta x) \leq 0 \Rightarrow \\
H^T x H^T (x - \Delta x) \leq 0.
$$

(24)

We can rewrite the inequality in (23) and (24) as follows:

$$
H^T x H^T (x - \Delta x) \leq 0 \Rightarrow \\
\begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2} H H^T \\ -\frac{1}{2} H H^T & -\frac{1}{2} H H^T \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix} \leq 0
$$

(25)

Moreover $V(\Delta x)$, given by (21) can be written as:

$$
V(\Delta x) = \begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix}.
$$

(26)

It is known that the inequality

$$
\dot{V}(\Delta x) \leq -\alpha V(\Delta x)
$$

(27)

implies global exponential stability of $V(\Delta x)$. Therefore, there exists a $U(t) = U(t_0)e^{-\alpha t}$, with $U(t_0) = \Delta x(t_0)^T P \Delta x(t_0)$ such that:

$$
V(\Delta x(t)) \leq U(t) \Rightarrow \\
\Delta x(t)^T P \Delta x(t) \leq U(t_0)e^{-\alpha t} \Rightarrow \\
\Delta x(t)^T P \Delta x(t) \leq \Delta x(t_0)^T P \Delta x(t_0)e^{-\alpha t} \Rightarrow \\
|\Delta x(t)|_P^2 \leq |\Delta x(t_0)|_P^2 e^{-\alpha t} \Rightarrow \\
|\Delta x(t)|_P \leq |\Delta x(t_0)|_P e^{-\alpha t},
$$

(28)

where $|\Delta x(t)|_P$ is a norm of $\Delta x(t)$ with the form

$$
|\Delta x|_P = \sqrt{\Delta x^T P \Delta x}, \text{ for } \Delta x \in \mathbb{R}^n \text{ and } P = P^T > 0.
$$

(29)
This norm is called the $P$-norm of $\Delta x$.

It is also known that,

$$\lambda_{\text{min}}(P)|\Delta x(t)|^2 \leq |\Delta x(t)|^T P|\Delta x(t)|,$$

(30)

where $\lambda_{\text{min}}(P)$ is the minimum eigenvalue of $P$ and $|\Delta x(t)|$ is the Euclidean norm of $\Delta x(t)$. The combination of (29) and (30) yields

$$\sqrt{\lambda_{\text{min}}(P)}|\Delta x(t)|^2 \leq |\Delta x(t)|^T P|\Delta x(t)| \Rightarrow$$

$$|\Delta x(t)| \leq 1/\sqrt{\lambda_{\text{min}}(P)} |\Delta x(t)|,$$

(31)

Substituting (22) and (26) into (27) yields

$$\Delta x^T((A_1 - L_1 C)^T P + P(A_1 - L_1 C) + \alpha P)\Delta x \leq 0,$$

if $H^T x \leq 0$ and $H^T(x - \Delta x) \leq 0$,

$$\begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2 C)^T P + P \Delta A \\ +P(A_2 - L_2 C) + \alpha P \\ -\Delta A^T P \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix} \leq 0$$

(32b)

if $H^T x \leq 0$ and $H^T(x - \Delta x) > 0$,

$$\begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} (A_1 - L_1 C)^T P + P \Delta A \\ +P(A_1 - L_1 C) + \alpha P \\ -\Delta A^T P \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix} \leq 0$$

(32c)

if $H^T x > 0$ and $H^T(x - \Delta x) \leq 0$ and

$$\Delta x^T((A_2 - L_2 C)^T P + P(A_2 - L_2 C) + \alpha P)\Delta x \leq 0$$

(32d)

if $H^T x > 0$ and $H^T(x - \Delta x) > 0$.

Applying the S-procedure to the sets of inequalities \{(32b), (25)\} and \{(32c), (25)\} the LMI constraints (15a) and (15b) are derived, respectively.
Friction compensation in a controlled one-link robot using a reduced-order observer

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Abstract. Friction compensation in a controlled one-link robot using a reduced-order observer is studied. Since friction is generally velocity-dependent and controlled mechanical systems are often only equipped with position sensors, friction compensation requires velocity estimation. Here, a reduced-order linear observer is used for this purpose. For exact friction compensation, design criteria in terms of the controller and observer parameter settings guaranteeing global exponential stability of the set-point are proposed. Moreover, for non-exact friction compensation it is shown that undercompensation leads to the existence of an equilibrium set and overcompensation leads to limit cycling. These results are obtained both numerically and experimentally.

1 Introduction

The positioning performance of many controlled mechanical systems, such as robots and optical disc drives, is limited by the presence of dry friction [1,9]. For example friction-induced limit cycling is observed by many authors in controlled mechanical systems [2,3,6]. One possible strategy to tackle this problem is model-based friction compensation. In the literature, friction compensation is investigated in both a feedforward (the friction compensation is based on desired variables) and a feedback manner (the friction compensation is based on actual variables) [1,3,7,9]. Here, we will apply a feedback friction compensation strategy to a controlled one-link robot in order to enhance its positioning performance.

In order to implement such a strategy, a model of the friction and knowledge on the variables on which the friction model depends is needed. Based on experiments, a friction model depending on velocity is adopted here. Furthermore, a linear proportional-derivative controller is used. Since only position measurements are available for the one-link robot (and for mechanical systems in general), some form of velocity estimation is required. To this end, numerical differentiation of the position measurements [12] or an observer can be used [5,10]. Here, we opt for an observer, since numerical differentiation is very sensitive to measurement noise.

The combination of dry friction, friction compensation and the observer dynamics can give rise to undesired phenomena, such as limit cycling [10] and the existence of equilibrium sets. The existence of equilibrium sets are due to discontinuities in the friction and friction compensation and can cause a non-zero steady-state positioning error. Consequently, improved insight into the influence of controller and observer design parameters on the existence of these unwanted phenomena is needed. Here, a combination of a reduced-order linear observer and a PD-controller will be used. This combination exhibits only three design parameters (two controller gains and one observer gain), which allows for a simplified analysis of the effect of these parameters on the behaviour of the closed-loop system.

This analysis provides design criteria for the parameters of the controller and observer, which ensure the avoidance of unwanted behaviour such as limit cycling and equilibrium sets, in the case

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of exact friction compensation. These criteria are based on a stability analysis of the set-point. The approach proposed here can be extended towards multi-degree-of-freedom systems. Moreover, the influence of non-exact friction compensation on the positioning performance is investigated numerically as well as experimentally.

The paper is organized as follows. In Section 2, the experimental setup and a corresponding model is introduced, based on experiments. The controller design, observer design and the adopted friction compensation strategy are discussed in Section 3. In Section 4, the dynamic behaviour of the system in case of exact friction compensation is investigated and design criteria for the controller and observer are proposed such that the set-point is globally exponentially stable. Moreover, in Section 5, the effect of non-exact friction compensation on the positioning performance is investigated both numerically and experimentally. Finally, in Section 6 conclusions are presented.

2 Experimental set-up and modelling

The experimental setup involving the one-link robot is depicted schematically in figure 1. The link is driven by a (control)-torque \( u \) supplied by an induction motor. The angular position \( q \) is measured by a position encoder.

The robot is modelled as a single inertia \( J \) (modelling the inertia of the link and the driveline) subject to a viscous friction torque \(-b\dot{q}\), a dry friction torque \(-F_f(\dot{q})\) and a control torque \( u \), which leads to the following model:

\[
J\ddot{q} + b\dot{q} = u - F_f(\dot{q}).
\]  

Using a frequency-domain identification technique, the total inertia of the system is identified to be \( J = 0.026 \) kgm\(^2\)/rad.

In order to identify the dry friction model, break-away experiments are performed to measure the static friction torque and constant velocity experiments are performed to measure the friction torque at non-zero (constant) velocities. A set-valued force law expressed by the following algebraic inclusion is used:

\[
F_f(\dot{q}) \in g(\dot{q})\text{Sign}(\dot{q}),
\]

in which \( g(\dot{q})\text{Sign}(\dot{q}) \) represents the Stribeck curve including the modelling of stiction, with

\[
g(\dot{q}) = F_c + \delta F e^{-(\frac{\dot{q}}{v_s})^n}
\]
Friction compensation in a controlled one-link robot using a reduced-order observer

\[ F_f(\dot{q}) + b \dot{q} \quad [\text{Nm}] \]

Fig. 2: Friction measurements (dots) and friction model (solid line).

<table>
<thead>
<tr>
<th>parameter</th>
<th>$\dot{q} &gt; 0$</th>
<th>$\dot{q} &lt; 0$</th>
<th>mean value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$F_s$ [Nm]</td>
<td>0.5735</td>
<td>0.5123</td>
<td>0.5429</td>
</tr>
<tr>
<td>$F_c$ [Nm]</td>
<td>0.3990</td>
<td>0.3887</td>
<td>0.3939</td>
</tr>
<tr>
<td>$v_s$ [rad/s]</td>
<td>0.0688</td>
<td>0.0817</td>
<td>0.0753</td>
</tr>
<tr>
<td>$b$ [Nms/rad]</td>
<td>0.0828</td>
<td>0.0790</td>
<td>0.0809</td>
</tr>
</tbody>
</table>

Table 1: Friction parameter estimates.

and Sign($x$) the set-valued sign-function:

\[
\text{Sign}(x) = \begin{cases} 
-1 & x < 0 \\
[-1, 1] & x = 0 \\
1 & x > 0 
\end{cases}
\] (4)

Herein, $F_c$ is the Coulomb friction force, $\delta F$ the difference between the static and Coulomb friction force ($\delta F = F_s - F_c$), $v_s$ the Stribeck velocity and $\beta$ the Stribeck shape parameter. The measurement results and the friction model (including both viscous and dry friction) fitted to these data are displayed in figure 2. The resulting friction parameter estimates are given in table 1, where different parameter estimates are obtained for positive and negative velocities indicating an asymmetric friction model. For the remainder of this paper a symmetric friction model will be used, since the asymmetry is not essential in the analysis. The friction parameters used in this symmetric model are the mean values of those for positive and negative velocity, see table 1.

3 Controller design, observer design and friction compensation strategy

In figure 3, the friction compensation strategy incorporating the reduced-order linear observer and a proportional-derivative controller is depicted schematically. The total control action $u$ is composed by the feedback control $u_c$ and the friction compensation $u_{fc}$: $u = u_c + u_{fc}$. Herein,

\[
u_c = n_1(q_r - q) - n_2\dot{q},
\]

(5)

where $n_1, n_2 > 0$ are the proportional gain and the derivative gain, respectively, and $\dot{q}$ is the velocity estimate provided by the observer. Moreover, $q_r$ is the desired reference position, which
will be assumed to equal zero (without loss of generality). Furthermore, the following set-valued friction compensation law is adopted

\[ u_{fc} = rF_f(\hat{q}) \in rg(\hat{q}) \text{Sign}(\hat{q}), \]  

(6)

where \( r \) is a scaling factor of the friction compensation. Clearly, it reflects a feedback compensation strategy where the estimated velocity is provided by an observer. When \( r = 1 \), exact friction compensation is attained and, when \( r \neq 1 \), non-exact friction compensation is attained.

The reduced-order observer is designed as

\[ \dot{\hat{q}} = -\frac{b}{J}\hat{q} + \frac{1}{J}(u - u_{fc}) + L (\hat{q} - \hat{\dot{q}}), \]  

(7)

where \( \hat{q} \) is the observer state (the velocity estimate) and \( L > 0 \) is the observer gain. The observer error is defined as \( e = \dot{q} - \hat{q} \).

From now on, we will adopt the state coordinates \( \mathbf{x} = [q \ \hat{q} \ e]^T \). The dynamics of the closed-loop system, displayed in figure 3, can be formulated in terms of these states by the following differential inclusion:

\[ \begin{align*}
\dot{x}_1 &= x_2 + x_3 \\
\dot{x}_2 &= -\frac{n_1}{J}x_1 - \frac{b + n_2}{J}x_2 + Lx_3 \\
\dot{x}_3 &\in -\frac{b + LJ}{J}x_3 + \frac{1}{J}[rF_f(x_2) - F_f(x_2 + x_3)]
\end{align*} \]  

(8)

The differential inclusion (8) is of Filippov-type and thus obeys Filippov’s solution concept [4]. Consequently, the existence of solutions of system (8) is guaranteed.

## 4 Exact friction compensation

In this section, the behaviour of the closed-loop system is investigated for the case of exact friction compensation, i.e. \( r = 1 \) in (6). First, the existence of an equilibrium set depending on the system (and control) parameters is discussed. Second, the stability of the set-point (the origin) is investigated.

![Fig. 3: Friction compensation strategy.](image-url)
4.1 Equilibria

The equilibria of (8) satisfy the following equations and inclusion:

\[
\begin{align*}
    x_2 &= -x_3 \\
    x_1 &= \frac{LJ + b + n_2}{n_1} x_2,
\end{align*}
\]

\[G(x_2) \in [-F_s, F_s] \tag{9}\]

where

\[G(x) = (b + LJ)x + F_f(x). \tag{10}\]

Let us denote these equilibria by \(x^*\). The origin is always an equilibrium. However, depending on the observer gain \(L\) an equilibrium set exists. In figure 4, the equilibria of the system with exact friction compensation are compared to those of the system with no compensation. In this figure, the effect of the existence the equilibrium set on the steady-state positioning error \(x_1\) is depicted, for \(n_1 = 0\) and \(n_2 = 0\). Clearly, the equilibrium set can induce a non-zero steady-state positioning error. However, friction compensation ensures a large decrease in the size of the equilibrium set. Moreover, in case of exact compensation the equilibrium set shrinks to an isolated equilibrium point for increasing observer gain at some critical value of the observer gain. In order to derive the condition for \(L\) such that a single equilibrium point exist we note that

\[\lim_{x \to 0} G(x) = F_s \quad \text{and} \quad \lim_{x \to 0} G(x) = -F_s.\]

Taking into account the strictly decreasing nature of \(F_f(x)\) for \(x \neq 0\), a sufficient and necessary condition, under which no equilibrium set can exist, is that the function \(G(x)\) is strictly increasing for all \(x \neq 0\) (see inclusion in (9)). This is attained if

\[\frac{\partial}{\partial x} G(x) > 0 \quad \forall x \neq 0\quad \text{and, consequently, if} \quad L > L_c\]

where

\[L_c = \frac{1}{J} (-\lambda - b), \tag{11}\]

\[\lambda = \eta \eta \delta F \quad \text{and} \quad \lambda = \min_{x \in \mathbb{R}\{0\}} \left( \frac{\partial g(x)}{\partial x} \right), \tag{12}\]

and

\[\eta = \begin{cases} 1 & \text{if} \ \beta = 1 \\ \frac{(\beta - 1)e^{-\frac{x}{\beta}}}{\beta} & \text{if} \ \beta > 1 \end{cases}. \tag{13}\]

For the parameters of the model of the one-link robot (using mean values for the friction parameters) the critical observer gain is \(L_c = 73.07\). Note that this value corresponds to the value for which the equilibrium set merges into an isolated equilibrium point in figure 4. The size of the equilibrium set (and thus the maximum steady-state positioning error) can also be influenced by the controller parameters; if \(n_1\) is increased the size of the equilibrium set decreases and if \(n_2\) is increased the size of the equilibrium set increases, see the second equation of (9).

4.2 Stability of the set-point

In order to investigate the stability of the origin of (8), let us study the system in the form of a cascade of a subsystem \(S_I\) and a subsystem \(S_{II}\) as depicted in figure 5. In this figure, \(x_{12} = [x_1 \ x_2]'\) and the system and input matrices of these subsystems are given by

\[A_I = -(\frac{b}{J} + L), \quad B_I = \frac{1}{J}, \quad A_{II} = \begin{bmatrix} 0 & \frac{1}{n_1} \\ -\frac{n_2}{n_1} & -\frac{b + n_2}{n_1} \end{bmatrix}, \quad B_{II} = \begin{bmatrix} 1 \\ L \end{bmatrix}. \tag{14}\]

Note that \(S_I\) describes the observer error dynamics. In order to prove the global exponential stability (GES) of the origin of (8) we adopt the following reasoning. If the following three conditions are fulfilled:
Fig. 4: Extrema for steady-state error in $x_1$ for $n_1 = 0.4$ and $n_2 = 0.02$.

Fig. 5: Cascade representation of the closed-loop system.

(a) $x_3 = 0$ is a globally exponentially stable equilibrium point of system $S_I$ for all $x_2$;
(b) $x_{12} = 0$ is a globally exponentially stable equilibrium point of system $S_{II}$ for zero input $x_3$;
(c) System $S_{II}$ is input-to-state stable (ISS),

then $x = 0$ is a globally exponentially stable equilibrium point of (8). Let us now check whether these conditions are fulfilled.

Firstly, in order to check condition (a), we use a candidate Lyapunov function $V = \frac{1}{2}J x_3^2$ (see [4] and [11] for details on Lyapunov analysis for differential inclusions). Its time-derivative $\dot{V}$ obeys

$$\dot{V} = -(b + LJ)x_3^2 + (F_f(x_2) - F_f(x_2 + x_3))x_3.$$  \hspace{1cm} (15)

In the second term of $\dot{V}$ the discontinuities of both the dry friction torque and the friction compensation design are present. Here, we will estimate this term by realising that the function $F_f(\cdot)$ satisfies the following incremental sector condition:

$$(F_f(x_2) - F_f(x_2 + x_3))x_3 \leq -\lambda x_3^2, \forall x_2, x_3$$  \hspace{1cm} (16)

with $\lambda$ defined by (12). Using (16) in (15) yields

$$\dot{V} \leq -\frac{2}{J}(LJ + b + \lambda)V.$$  \hspace{1cm} (17)
Clearly, for an observer gain satisfying $L > L_c$, with the critical observer gain $L_c$ given by (11), $e = 0$ is a globally exponentially stable equilibrium point of system $S_I$ (independent of $x_2$).

Secondly, conditions (b) and (c) are satisfied since system $S_{II}$ is a LTI-system with a Hurwitz system matrix $A_{II}$ ($A_{II}$ is Hurwitz given the fact that $n_1, n_2 > 0$).

Resuming we can conclude that if $L > L_c$, $x = 0$ is a globally exponentially stable equilibrium point of (8). If $L < L_c$, unwanted behaviour in the form of a non-zero steady-state positioning error, see figure 4, or limit cycling, see figure 6, can occur. The latter figure indicates that $e = 0$ is not stable, which causes a non-zero observer error. This non-zero observer error induces overcompensation leading to limit cycling. For $L > L_c$, the observer error converges fast to zero and stays equal to zero, see figure 7. The observer based friction compensation now performs as desired. Note that we used a switch-model [8] for the numerical integration of (8) to avoid numerical instability at zero velocity.

5 Non-exact-friction compensation

In practice, or in an experimental setup such as that of the one-link robot discussed in this paper, the friction model will never be exact, due to inevitable modelling errors. To study this we introduce a scaled friction compensation law, see figure 3 and equation (6) with $r \neq 1$. Obviously, in practice modelling errors will not be of this form but this type of scaling of the friction compensation law allows to investigate the effects of both undercompensation and overcompensation of the friction. In figure 8, the equilibrium set (in terms of $x_1$) is shown for different values of $r$. In the case of undercompensation ($r < 1$), an equilibrium set will exist irrespective of the value of $L$. The value of $L$, however, influences the magnitude of the equilibrium set. This figure indicates that friction compensation (even in the case of undercompensation) ensures a smaller equilibrium set than exists without compensation, see also figure 4. In the case of the overcompensation ($r > 1$), an equilibrium set only exists for $r$ very close to one; the equilibrium set rapidly shrinks to an isolated (unstable) equilibrium point for increasing $r$.

In figure 9, a bifurcation diagram with bifurcation parameter $r$ is depicted in terms of $x_1$ for a super-critical observer gain $L = 73.5 > L_c$. For the limit cycle, max(abs($x_1$)) over a period of the limit cycle is plotted. This bifurcation diagram clearly shows that an equilibrium set exists when the friction is undercompensated and a stable limit cycle exists in case of overcompensation. A corresponding bifurcation diagram involving experimental results is depicted in figure 10. Herein,
6 Conclusions

A friction compensation strategy for a controlled one-link robot using a reduced-order observer is proposed. Based on experiments, a set-valued friction model is identified to support a model-
based friction compensation approach. Since only position measurements are available and the friction model depends on velocity, a reduced-order observer is used to provide velocity estimates. The combination of the reduced-order observer and a PD-controller exhibits only three design parameters (two controller gains and one observer gain). This allows for a simplified analysis of the effect of these parameters on the behaviour of the closed-loop system.

Both the case of exact friction compensation and non-exact friction compensation are studied. In the case of exact friction compensation, it is shown that the observer gain is critical for the stability of the set-point. If the observer gain is taken larger than this critical value, it is shown the set-point is globally exponentially stable. Moreover, for an observer gain lower than this critical value, an equilibrium set can exist and limit cycling can occur both deteriorating the positioning performance. In the case of non-exact friction compensation, it is shown that undercompensation of the friction leads to the existence of an equilibrium set and that overcompensation leads to limit cycling. These results are obtained both in simulation and experiments. Since the size of the equilibrium set can be influenced by the choice of the controller parameters, it is advisable to opt for undercompensation when exact friction compensation is not possible.

References


Fig. 9: Bifurcation diagram for $n_1 = 0.4, n_2 = 0.02, L = 73.5$ ((I): Stable limit cycle, (II): Equilibrium set and (III): Unstable equilibrium point.)
Fig. 10: Experimental bifurcation diagram with bifurcation parameter $r$.

Part II: Feedback Control Design and Observer Design
for Piece-wise Smooth Systems

Part II of this report contains results regarding controller and observer design for other classes of piece-wise smooth systems. It comprises the following chapters:

- In Chapter 8, an observer design for Luré systems with multi-valued mappings is presented (Authors: Alexander Juloski, Maurice Heemels, Bernard Brogliato).
- In Chapter 9, the concept of feedback linearization is extended towards piecewise linear systems (Authors: Kanat Camlibel, Ilker Ustoglu).
- In Chapter 10, averaging theory is applied to the Zero Average Dynamic control technique in a non-smooth model of a buck converter. (Authors: Fabiola Angulo, Enric Fossas, and Tere Seara).
- In Chapter 11, the concept of convergent systems is used to design state-feedback and output-feedback controllers for continuous piece-wise affine systems (Authors: Alexey Pavlov, Nathan van de Wouw, Henk Nijmeijer).
Observer design for Lur’e systems with multi-valued mappings

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Abstract. In this paper we present an observer design procedure for a class of nonsmooth
dynamical systems, namely systems of Lur’e type with a monotone multivalued mapping in
the feedback path. Examples of such systems include various classes of hybrid systems, such
as piecewise affine systems, relay systems, linear complementarity systems, and electrical
circuits with switching elements. Under the assumption that the observed system is well
behaved, we prove that the proposed observers are well-posed (i.e. that there exists the
unique solution to the observer dynamics), and that the observer asymptotically recovers
the state of the observed system.

1 Introduction

In this paper the observer design procedure for systems of Lur’e type with maximal monotone
multivalued mapping in the feedback path (see figure 1) is developed. A multivalued or set-valued
mapping is a mapping that assigns a set of possible values to its input argument, and the output
of the mapping can be any value in this set. The requirements that the mapping is maximal and
monotone generalize the usually considered concept of continuous, sector bounded nonlinearity
[14]. Systems of the considered type may arise as a natural consequence of modelling (e.g. models
of friction phenomena, ideal diodes), or the used solution concept (e.g. Filippov solutions [5]).
Examples of systems obtained by interconnecting linear dynamics in a feedback configuration
with maximal monotone mapping, as in figure 1, include various classes of hybrid systems: piece-
wise linear systems [11] (fig. 2a), linear relay systems [7] (figure 2b), linear complementarity systems
[6, 13] (figure 2c), and electric circuits with switching elements (e.g. MOS transistor, characteristic
in fig. 2d).

Two observer structures are proposed in the paper, which are based on rendering the linear part
of the error dynamics strictly positive real (SPR). The considered class of systems and the pro-
posed observers are nonsmooth. To formally analyze and prove their properties tools of nonsmooth
analysis are used. Since the considered systems can be non-Lipschitz, existence and uniqueness
of solutions (i.e. well-posedness) of the system is not guaranteed. Under the natural assumption
that there exists a solution of the observed system, it is proven that there exists a solution of
the proposed observer, and that this solution is unique. Well-posedness of the system is an im-
portant theoretical question, and, from a practical standpoint, if an observer is to be numerically
implemented, well-posedness is necessary to ensure the proper behavior of the implementation.

From the existence of solutions to both the observed system and the observer, the existence of
solutions to the observation error follows. It is further shown that the observer recovers asymp-
totically the state of the observed system (i.e. that the error dynamics is globally asymptotically

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Fig. 1: Lur’e type system with maximal monotone multivalued mapping

stable). These results are applied to a simplified drill-string boring machine model, with the set-valued dry friction with Stribeck effect [9].

Stability of Lur’e type systems with SPR linear part and the discontinuous nonlinearity has been studied in [16], but the problem of existence and uniqueness of solutions for this systems was not considered. Existence and uniqueness of solutions, as well as stability of autonomous Lur’e type systems with maximal monotone nonlinear mappings have been studied in [4]. The main results in this paper generalize results from [4] to the case of systems with external inputs.

An observer design methodology for Lur’e type systems with locally Lipschitz slope restricted nonlinearities was studied before in [1]. However, since nonsmooth and non-Lipschitz nonlinearities are allowed results of [1] are not applicable, and we have to resort to a framework of convex analysis, to establish an observer design procedure for the considered class of systems.

The paper is structured as follows. In the section 2 some basic concepts of convex analysis and differential inclusions are given. The material in this section is taken from [2,3,12]. In section 3 the observer design problem is formally stated. Section 4 contains the main results of the paper. The example of the drill-string system is presented in section 5, and conclusions are presented in section 6.

2 Preliminaries

A mapping \( \rho : X \to Y \), where \( X,Y \subseteq \mathbb{R}^l \), is said to be \textit{multivalued} if it assigns to each element \( x \in X \) a subset \( \rho(x) \subseteq Y \) (which may be empty). The domain of the mapping \( \rho(\cdot) \), dom \( \rho \) is defined as \( \text{dom} \rho = \{ x | x \in X, \rho(x) \neq \emptyset \} \). The mapping \( \rho \) is said to be proper if \( \text{dom} \rho \neq \emptyset \).

A multivalued mapping \( \rho(\cdot) \) is said to be \textit{monotone}, if

\[
\forall x,y \in \text{dom} \rho, \quad \forall p \in \rho(x) \forall q \in \rho(y) \quad \langle p-q,x-y \rangle \geq 0 ,
\]

(1)

where \( \langle \cdot,\cdot \rangle \) denotes the inner product. A multivalued mapping \( \rho(\cdot) \) is said to be \textit{maximally monotone} if for any \( (x,p) \in X \times Y \) the following holds:

\[
\forall y \in \text{dom} \rho, q \in \rho(y) \quad \langle p-q,x-y \rangle \geq 0 \Rightarrow p \in \rho(x)
\]

(2)

In words, (2) means that new values can not be added to the mapping \( \rho \) without violating the monotonicity of the mapping.
A function $\varphi : \mathbb{X} \to \mathbb{R}$, where $\mathbb{X}$ is convex, is said to be convex if for any $x, y \in \text{dom } \varphi$ and any $0 \leq \lambda \leq 1$ the following holds:

$$\varphi(x + (1 - \lambda)y) \leq \lambda \varphi(x) + (1 - \lambda)\varphi(y).$$  \hfill (3)

The subdifferential of the convex function $\varphi$ at $y$, denoted as $\partial \varphi(y)$ is a set of subgradients \cite[chapter 23]{12}, i.e:

$$\partial \varphi(y) = \{ \gamma \in \mathbb{R}^l \mid \forall x, \varphi(x) - \varphi(y) \geq \langle \gamma, (x - y) \rangle \}. \hfill (4)$$

Subdifferentials of proper convex functions are examples of maximal monotone mappings. \cite[Corollary 31.5.2]{12}. For the reverse to hold, i.e. for the convex function $\varphi$ to exist, such that a mapping $\varrho$ is its subdifferential (i.e. $\varrho = \partial \varphi$), mapping $\varrho$ has to be cyclically monotone \cite[theorems 24.8, 24.9]{12}. Cyclic monotonicity (for definition see \cite[chapter 24]{12}) is a stronger requirement than monotonicity, in the sense that all cyclically monotone mappings are in the same time monotone, but the reverse does not hold. For one dimensional mappings, i.e. when $\text{dom } \varrho \subseteq \mathbb{R}$, the concepts of monotonicity and cyclic monotonicity are equivalent.

Differential inclusion (DI) is given by an expression of the form

$$\dot{x} \in F(t, x) \hfill (5)$$
where $F$ is a set-valued mapping, that associates to the state $x$ of the system the set of admissible velocities at time $t$. An absolutely continuous (AC) function $x$ is considered to be the strong solution of the DI (5) if its derivative $\dot{x}$ satisfies (5) almost everywhere. For an exposition on differential inclusions see e.g. [2].

An important result concerning differential inclusions of the form
\[
\dot{x}(t) \in -A(x(t)) + u(t), \quad x(0) \in \text{dom } A
\]
where $A$ is a multivalued mapping and the external input signal $u \in L^1_{\text{loc}}[0, \infty)$. Following [3, section 3.2] we define a continuous function $x$ to be a weak solution to (7) if there exists sequences $u_n \in L^1_{\text{loc}}[0, \infty)$ and $x_n \in C[0, \infty)$ such that $x_n$ is a strong solution to
\[
\dot{x}_n \in -A(x_n(t)) + u_n,
\]
$u_n \to u$ in $L^1_{\text{loc}}[0, \infty)$ sense and $x_n \to x$ uniformly on $[0, \infty)$.

**Proposition 1.** [3, theorem 3.4] For the case when the mapping $A$ in (7) is maximal monotone mapping there exists a unique weak solution $x$ to (7) for every $u \in L^1_{\text{loc}}[0, \infty)$.

A difference between the weak and strong solutions is that the weak solution, while continuous, is not necessarily absolutely continuous. However, the following holds:

**Proposition 2.** [3, proposition 3.2]: For the case when the mapping $A$ in (7) is maximal monotone mapping:

- the strong solution to (7), if it exists, is unique
- any AC function $x$ which is a weak solution to (7) is also the strong solution to (7).

We will frequently use the following fact in the sequel. If the transfer function $G(s) = C(sI - A)^{-1}B$, where $(A, B, C)$ is a minimal representation and $B$ is full rank (i.e. $\text{Ker}\{B\} = \emptyset$), is strictly positive real (SPR) this implies that there exists a $P = P^\top > 0$, $Q = Q^\top > 0$ such that [14, 15]:
\[
\begin{align*}
PA + A^\top P &= -Q \\
B^\top P &= C
\end{align*}
\] (8a)

3 Problem statement

Consider the system whose state space equations are given by the following differential inclusion (see figure 1):
\[
\begin{align*}
\dot{x} &= Ax - Gw + Bu & (9a) \\
w &\in \wp(Hx) & (9b) \\
y &= Cx & (9c)
\end{align*}
\]

where $Hx(0) \in \text{dom } \wp$ and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $G \in \mathbb{R}^{n \times l}$ is full rank, $H \in \mathbb{R}^{l \times n}$ and $C \in \mathbb{R}^{p \times n}$. Mapping $\wp : \mathbb{R}^l \to \mathbb{R}^l$ is assumed to be cyclically monotone, so that there exists a convex lower semicontinuous mapping $\varphi : \mathbb{R}^l \to \mathbb{R}$ such that $\wp = \partial \varphi(\cdot)$. 

Remark 1. Certain multivalued mappings $\varrho(\cdot)$ that are not monotone, can be transformed into the monotone mappings by using loop transformation technique (see for instance [14, section 5.6.2]). An example of such a mapping is given in figure 4, section 5. With loop transformation, a new mapping is defined, as $\tilde{\varrho}(z) = \varrho(z) - \mu z$, where $\mu$ is a vector of appropriate dimensions, chosen so that the mapping $\tilde{\varrho}(z)$ is monotone. Under the assumption that there exists a convex lower semicontinuous function $\varphi$ such that $\tilde{\varrho} = \partial \varphi$, transformed system fits in the description (9). The system matrix $A$ in (9) is then replaced by $\tilde{A} = A + \mu G H$.

We assume that for the system (9) the following holds.

**Assumption 8** System (9) has a strong solution for a given input $u \in L^1_{\text{loc}}([0, \infty))$ and initial condition $x(0)$ where that $H x(0) \in \text{dom} \varrho$.

Note that the assumption 8 defines a particular class of acceptable inputs and solutions for (9). The considered solution concept is standard in the DI literature [2, 3]. For a given system (9) one can check for the existence of solutions using some of the general results that are available in the literature [2]. Results on existence of solutions to particular instances of (9) (e.g. complementarity and relay systems) can also be found in the literature ([6, 10]).

The first proposed observer (“basic” observer scheme) for the system (9) has the following form:

\[
\begin{align*}
\dot{x} &= (A - LC)x - G\hat{w} + Ly + Bu \\
\hat{w} &\in \varrho(H\hat{x}) \\
\hat{y} &= C\hat{x}
\end{align*}
\]

(10a) (10b) (10c)

where $L \in \mathbb{R}^{n \times p}$ and $H\hat{x}(0) \in \text{dom} \varrho(\cdot)$.

The second proposed observer (“extended” observer scheme) has the following form:

\[
\begin{align*}
\dot{x} &= (A - LC)x - G\hat{w} + Ly + Bu \\
\hat{w} &\in \varrho((H - KC)\hat{x} + Ky) \\
\hat{y} &= C\hat{x}
\end{align*}
\]

(11a) (11b) (11c)

where $K \in \mathbb{R}^{l \times p}$ and $y(0)$ and $\hat{x}(0)$ are such that $(H - KC)\hat{x}(0) + Ky(0) \in \text{dom} \varrho(\cdot)$.

The basic observer is a special case of the extended observer with $K = 0$. The reason for treating these two cases separately is that the well-posedness conditions for the two proposed observers (i.e. conditions for the existence and uniqueness of solutions) are somewhat different. Also, the well-posedness proofs are more readable if the cases are treated separately. Stability of the error dynamics will be treated only for the case of the extended observer, as the result for the stability of the basic observer follows immediately.

The problem of observer design consists in finding the gain $L$ ($L$, $K$, respectively) which will guarantee that there exists a unique solution $\hat{x}$ to the observer dynamics on $[0, \infty)$, and that $\hat{x} \to x$ as $t \to \infty$. In the following section we will prove that if $L$ and $K$ are chosen such that the triple $(A - LC, G, H)$ (respectively $(A - LC, G, H - KC)$) is SPR the obtained observers will satisfy the mentioned requirements.

4 Main results

The results on well-posedness of proposed observers are summarized in the following two lemmas:

**Lemma 1.** Consider the system (9) under assumption 8, and the observer (10). If the triple $(A - LC, G, H)$ is SPR, the observer dynamics (10) has a unique weak solution on $[0, \infty)$. 
Proof Since the triple \((A - LC, G, H)\) is SPR there exist \(P, Q\) that satisfy (8). Introduce the change of variables in (10):

\[
z = R\dot{x}
\]

where \(RR = P\). Then, (10) transforms into:

\[
\begin{align*}
\dot{z} &= R(A - LC)R^{-1}z - RG\dot{w} + RBu + RLy \\
\dot{w} &\in \partial \varphi (HR^{-1}z) \\
\dot{y} &= CR^{-1}z
\end{align*}
\]

(13a)-(13c)

Since \(H\dot{x}(0) \in \text{dom} \partial \varphi (\cdot)\), we have \(HR^{-1}z(0) \in \text{dom} \partial \varphi (\cdot)\). Define the function \(f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}\) as \(f(z) = \varphi (HR^{-1}(z))\), which is convex lower semicontinuous. From the SPR condition (8b) it follows that \(H\) has full row rank, and from [12, theorem 23.9] it follows that \(\partial f(z) = R^{-1}H^T \partial \varphi (HR^{-1}z)\). Using this fact and the SPR condition, (13) transforms into:

\[
\dot{z} \in R(A - LC)R^{-1}z - \partial f(z) + RBu + RLy
\]

(14)

where \(z(0) \in \text{dom} f(\cdot)\). From the SPR condition it follows that \(R(A - LC)R^{-1} + R(A - LC^T)R^{-1}\) is negative definite, and since \(f(\cdot)\) is convex lower semicontinuous, \(\partial f(\cdot)\) is a maximal monotone mapping. Therefore, the mapping \(-R(A - LC)R^{-1}z + \partial f(z)\) is maximal monotone [3, lemma 2.4].


In the following lemma we address the questions of well-posedness of the extended observer scheme.

Lemma 2. Consider the system (9), under assumption 8, and the observer (11). If the triple \((A - LC, G, H - KC)\) is SPR, the observer dynamics (11) has a unique weak solution on \([0, \infty)\).

Proof Since the triple \((A - LC, G, H - KC)\) is SPR there exist \(P, Q\) that satisfy (8). Introduce the change of variables:

\[
z = R(\dot{x} + g),
\]

(15)

in (11), where, as before, \(RR = P\) and

\[
g = (H - KC)^T ((H - KC)(H - KC)^T)^{-1}Ky.
\]

(16)

From the SPR condition it follows that \(H - KC\) has full row rank, and hence the inverse in (16) exists. Note that the part of the expression on the right hand side of (16) is a pseudo-inverse of \(H - KC\). In the same way as in the proof of lemma (1), (11) transforms into:

\[
\dot{z} \in R(A - LC)R^{-1}z - \partial f(z) + RBu + RLy + R\dot{g}
\]

(17)

where \(z(0) \in \text{dom} f(\cdot)\) The multivalued mapping \(-R(A - LC)R^{-1}z + \partial f(z)\) is maximal monotone, and by assumption we have \(u \in L^1_{\text{loc}}[0, \infty), y \in L^1_{\text{loc}}[0, \infty]\). Moreover, \(y\) is AC and it follows that \(\dot{y} \in L^1_{\text{loc}}[0, \infty]\). By virtue of 1, (17) and hence (11) posses a unique weak solution.

From lemmas 1 and 2 it follows that both proposed observers have, at least, weak solutions. To ensure the existence of strong solutions more stringent assumptions have to be imposed on the original system and proposed observers. For instance, if we consider again the inclusion (17), sufficient conditions for existence of strong solutions are [3, theorem 3.6]:

\[-u \in L^2_{\text{loc}}[0, \infty), \dot{y} \in L^2_{\text{loc}}[0, \infty)\]
there exists a convex lower semicontinuous mapping $\xi$, such that:

$$ R(A - LC)R^{-1}z - \partial f(z) = \partial \xi(z), $$

(18)

which requires that the mapping $R(A - LC)R^{-1}z - \partial f(z)$ has to be cyclically monotone [12, theorem 24.8]. Note that this is equivalent to the requirement that the linear map $z \mapsto R(A - LC)R^{-1}z$ is cyclically monotone, which is equivalent to $R(A - LC)R^{-1}$ being symmetric positive semi-definite [12, chapter 24]. Hence, in our case the additional design requirement would be the symmetry of the matrix $A - LC$.

We see the issue of weak vs. strong solutions as largely technical, as the only additional requirement that needs to be imposed on weak solutions to get strong solutions is absolute continuity, which is needed to ensure that the solution is differentiable almost everywhere. Therefore, we make the following assumption.

**Assumption 9** Weak solutions for observers (10) and (11) are AC (and thus, weak solutions are strong solutions by 2).

For the extended observer (11) the observation error $e := x - \hat{x}$ dynamics can be formed as:

\begin{align*}
\dot{e} &= (A - LC)e - G(w - \hat{w}) \tag{19a} \\
w &\in \partial \varphi(Hx) \tag{19b} \\
\dot{w} &\in \partial \varphi(H\dot{x} + K(y - \hat{y})) \tag{19c}
\end{align*}

The following theorem states the main result of the paper.

**Theorem 1.** Consider the observed system (9) under assumption 8, the extended observer (11) where the triple $(A - LC, G, H - KC)$ is SPR, under assumption 9, and the observation error dynamics (19). Then $e = 0$ is the unique fixed point of the observation error dynamics (19). Moreover, the fixed point $e = 0$ is globally exponentially stable.

**Proof** Note that $e = 0$ is indeed a fixed point of (19). For $e = 0$, $x = \hat{x}$, and since the arguments of the $\varphi(\cdot)$ mapping in (19b),(19c) are the same it follows that $0 \in w - \hat{w}$, and hence $e = 0$ is the solution of the generalized equation

$$(A - LC)e \in G(w - \hat{w}).$$

(20)

From $(A - LC)e \in G(w - \hat{w})$ it follows that $P(A - LC)e \in PG(w - \hat{w})$, and, using SPR condition (8b) $e^\top P(A - LC)e = ((H - KC)e)^\top (w - \hat{w})$. From (8a) it follows that $e^\top P(A - LC)e \leq 0$, and form the monotonicity condition (1) for $\varphi(\cdot)$ it follows that $((H - KC)e)^\top (w - \hat{w}) \geq 0$. Clearly, $e = 0$ is the only solution of the generalized equation (20).

To show that the unique fixed point $e = 0$ is globally exponentially stable consider the Lyapunov function $V = \frac{1}{2}e^\top Pe$. Since by assumption $8 \times$ is AC, and by assumption $9 \hat{x}$ is AC it follows that $e$ is also AC, and $\dot{e}$ exists almost everywhere. Hence, $V$ is also AC, and the derivative $\dot{V}$ exists almost everywhere. $\dot{V}$ satisfies:

\begin{align*}
\dot{V} &= e^\top \dot{P}e \\
&= e^\top P((A - LC)e - G(w - \hat{w})) \\
&= -e^\top Qe - e^\top (H - KC)^\top (w - \hat{w}) \leq -e^\top Qe
\end{align*}

(21a)

From $V(t) \leq V(0) - \int_0^t e^\top Qe \, d\tau$ it follows that the AC function of time $V$ is nonincreasing, and \(\frac{1}{2} \lambda_{\min}(P) e^\top e \leq V(0) - \int_0^t \lambda_{\min}(Q) e^\top e \, d\tau\) where $\lambda_{\min}(\cdot)$ denotes minimal eigenvalue. From Gronwall's lemma:

$$\frac{1}{2} \lambda_{\min}(P) e^\top(t) e(t) \leq V(0) \exp\left(-2 \frac{\lambda_{\min}(Q)}{\lambda_{\min}(P)} t\right).$$

(22)
This proves the exponential convergence of the observation error.

The remaining issue is computing the gains $L$ and $K$ so that the triple $(A - LC, G, H - KC)$ has SPR property. This can be achieved by solving the matrix inequality:

$$
(A - LC)^\top P + P(A - LC) < 0
$$

$$
G^\top P = H - KC.
$$

Inequality (23) is linear matrix inequality in $P, K, L^\top P$. For necessary and sufficient conditions for the existence of solutions for (23), see for instance, [1] and references therein.

5 Example

A simplified scheme of the deep see drilling equipment is depicted in figure 3. The assembly consists of the drilling tool (depicted by a small disc), rotary table (big disc) which acts as a reservoir of kinetic energy, DC motor, and a drill string, which is used to transmit the energy from the surface to the drilling tool.

![Drilling assembly with a string](image)

An experimental setup mimicking the drilling equipment was realized by Mihaílovic et al. [9]. It was shown that the dynamics of the experimental setup can be accurately described by the following model:

\begin{align}
\dot{x}_1 &= x_2 - x_3 \\
\dot{x}_2 &= \frac{k_m}{J_u} u - \frac{k_g}{J_u} x_1 - \frac{1}{J_u} T_{fu}(x_2) \\
\dot{x}_3 &= \frac{k_g}{J_l} x_1 - \frac{1}{J_l} T_{fr}(x_3)
\end{align}

(24a)

(24b)

(24c)
where $x_1$ is the difference in angular positions of the discs, $x_2$ is the angular velocity of the upper disc and $x_3$ is the angular velocity of the lower disc. Measured variable is taken to be $y = x_1$.

$T_{fru}(\cdot)$ and $T_{frl}(\cdot)$ denote the friction moments at the upper and the lower disc, respectively. $T_{fru}(\cdot)$ is dominated by the viscous friction, and for simplicity, is here taken to be equal to $b_{up} \cdot x_2$. The friction moment at the lower disc $T_{frl}(\cdot)$ is a dry friction with the Stribeck effect, i.e. negative damping appears at a certain range of angular velocities. To describe this friction torque a set-valued characteristic based on neural networks is used in [9], but for our purposes $T_{frl}(\cdot)$ can be approximated by the set-valued friction law, as depicted in figure 4. Note that this approximation is not restrictive, as the observer with the same observer gains, designed for the system (24) with a more complex set-valued friction law is still guaranteed to converge, as long as the maximal negative damping in the used friction law is bigger or equal then the negative damping $\mu$ in the friction law depicted in figure 4.

![Fig. 4: Dry friction characteristic](image)

Numerical values of the parameters in (24) are determined to be: $k_m = 3.5693 \frac{Nm}{V}$, $J_t = 0.4765 \frac{kgm^2}{rad}$, $J_t = 0.0326 \frac{kgm^2}{rad}$, $k_\theta = 0.078 \frac{Nm}{rad}$, $b_{up} = 1.9667 \frac{kgm}{rad}$, $T_{static} = 0.01663Nm$, $T_{kinetic} = 0.0868Nm$, $\mu = -0.0053 \frac{kgm}{rad}$, $\omega_{hi} = -\omega_{low} = 15 \frac{rad}{s}$.

The friction mapping proposed in figure 4 is not monotone, but by using the technique described in remark 1, can be transformed to a monotone one. New friction mapping is defined as $\tilde{T}_{frl}(\omega) = T_{frl}(\omega) - \mu \omega$, and the system matrix $\tilde{A}$ is replaced by $\tilde{A} = A + \mu GH$.

Observer design of the form (11) for system (24) entails finding gains $L$ and $K$ such that the triple $(\tilde{A} - LC, G, H - KC)$ is SPR. The following values for $L$ and $K$ are found: $L = \begin{bmatrix} 2.6883 & -0.0794 & 3.6931 \end{bmatrix}^T$ and $K = 1.0552$.

To implement the observer one requires a solution of the differential inclusion (11). In this case it is possible to use some of the dedicated techniques for simulating friction, e.g. the technique based on switched friction models [8]. The simulation results are depicted in the figure (5). Constant input voltage is applied (i.e. a constant torque is applied to the upper disc), but due to the negative damping in the friction law 4 slip-stick oscillations in the angular velocity of the second disc $x_3$ occur. During this oscillations the velocity of the third disc alternates between 0 (stick phase), and a positive value (slip phase). Designed observer is able to provide the correct estimate of the state.
Fig. 5: Responses of the system (solid) and the observer (dashed): $x_1$ (upper), $x_2$ (in the middle), $x_3$ (lower) under the constant input voltage

6 Conclusions

In this paper observer design for Lur’e type systems with maximal monotone multivalued mappings in the feedback path is considered. In contrast with the previous work on nonlinear observer design, the considered class of systems is nonsmooth and the standard theory does not apply. Existence and uniqueness of solutions is not a priori guaranteed.

Two observer structures are proposed, together with the constructive design method. The approach taken in the paper is based on rendering the linear part of the observation error dynamics SPR, by choosing appropriate observer gains. Under the natural assumption that the observed system has a solution, and that the control input belongs to certain admissible class it is shown that there exists a unique solution for the estimated state, and that the observer recovers the state of the original system asymptotically. The relevance and applicability of the presented results is demonstrated on the example of the drilling system.

Future work will investigate the observer design for more general solution concepts (e.g. solution concepts that allow for state jumps), and the applicability of the proposed observers for certainty equivalence state feedback controller design. Also, further investigation of numerical methods and the development of software tools for computing solutions for DIs is of great practical interest.
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References

Feedback linearization of piecewise linear systems*

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Abstract. One of the classical problems of nonlinear systems and control theory is feedback linearization. Its obvious motivation is that one can utilize linear control theory if the nonlinear system at hand is linearizable by feedback. This problem is well-understood for the smooth nonlinear systems. In the present paper, we investigate feedback linearizability of a class of piecewise linear, and hence nonsmooth, systems.

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1 Introduction

Feedback linearization is a topic that is well-studied in the context of smooth nonlinear systems (see e.g. [14, 15, 17]). Its obvious motivation stems from the fact that the linear systems theory can be employed for both analysis and synthesis for feedback linearizable systems. As far as the authors knowledge, the non-smooth case has not been studied yet in the depth in the literature.

This paper aims at investigating feedback linearization problem for a class of piecewise linear systems that are called conewise linear systems. Basically, these systems consist of a number of linear dynamics that are active on certain cones in the state-space. The papers [3–5] studied the controllability properties of these systems in an increasing level of generality and provided algebraic necessary and sufficient conditions. In this paper, we show that these results on the controllability make it possible to solve the feedback linearization problem for this class of systems.

The organization of the paper is as follows. The next section introduces the notations and the conventions in force throughout the paper. This is followed by a quick review of the feedback linearization problem in the context of smooth nonlinear systems. The main results of the system are presented in two subsequent sections. The first one is devoted to the relatively easier case of bimodal piecewise linear systems. This section gives a flavor of the proof of the presented results and will be followed by a section on conewise linear systems. Finally, the conclusions section closes the paper.

2 Notation

The symbol $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ $n$-tuples of real numbers, $\mathbb{R}^{n \times m}$ $n \times m$ real matrices.

For a matrix $A \in \mathbb{R}^{n \times m}$, $A^T$ stands for its transpose, ker $A$ for its kernel, i.e. the set $\{ x \in \mathbb{R}^m : Ax = 0 \}$, im $A$ for its image, i.e. the set $\{ y \in \mathbb{R}^n : y = Ax \text{ for some } x \in \mathbb{R}^m \}$.

The notation $\langle A \mid \text{im} B \rangle$ denotes the linear space $\text{im} B + A \text{im} B + \cdots + A^{n-1} \text{im} B$ where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. We say that the pair $(A, B)$ is controllable if the relation $\langle A \mid \text{im} B \rangle = \mathbb{R}^n$ holds.

The set of all locally integrable functions are denoted by $L^1_{\text{loc}}$. Sometimes we say that a function is sufficiently smooth meaning that the function is sufficiently many times continuously differentiable.

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A set $Y$ is said to be a cone if $\alpha x \in Y$ for all $x \in Y$ and $\alpha \geq 0$. For a nonempty set $Y$ (not necessarily a cone), we define its dual cone as the set $\{ x \mid x^Ty \geq 0 \text{ for all } y \in Y \}$. It is denoted by $Y^*$.

3 Feedback linearization of nonlinear systems

The problem of rendering a nonlinear system to a linear system by means of a state feedback is called feedback linearization problem. The motivation of this problem comes from the obvious fact that one can employ linear systems theory for the analysis and the synthesis of feedback linearizable nonlinear systems.

To formulate the problem precisely, consider the single-input nonlinear systems of the form

$$\dot{x} = f(x) + g(x)u$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ are sufficiently smooth. We say that the system (1) is feedback linearizable if there exist sufficiently smooth functions $\alpha : \mathbb{R}^n \to \mathbb{R}^n$, $\beta : \mathbb{R}^n \to \mathbb{R}^n$, and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that the feedback transformation

$$u = \alpha(x) + \beta(x)v$$

and the state transformation

$$z = \phi(x)$$

render the system (1) to the linear system

$$\dot{z} = Az + bv$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and the pair $(A, b)$ is controllable.

The solutions of the feedback linearization problem and its variations are among the classical results of the nonlinear system theory (see e.g. [14,15]). It is well-known that the problem is closely related to the notion of relative degree.

Consider the single-input nonlinear system (1) together with the single-output

$$y = h(x)$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a sufficiently smooth function. The system (1) and (5) is said to have relative degree $k$ at $x_0$ if

$$L_g L_f^i h(\bar{x}) = 0$$

for all $\bar{x} \in \mathbb{R}^n$ and $i = 0, 1, \ldots, k - 2$, and

$$L_g L_f^{k-1}(x_0) \neq 0.$$  

Here $L_f h(\bar{x})$ denotes the Lie derivative of $h$ with respect to the vector field $f$ at the point $\bar{x}$, i.e.

$$L_f h(x) = \sum_{i=1}^n \frac{\partial h}{\partial x_i}(\bar{x}) f_i(\bar{x}).$$

In the linear case, i.e. when $f(x) = Ax$, $g(x) = b$, and $h(x) = c^T x$, one gets

$$L_g L_f^i h(\bar{x}) = c^T A^i b.$$  

Hence, the usual definition of relative degree for linear systems is compatible with the above definition.

The following theorem presents necessary and sufficient conditions for the solvability of the feedback linearization problem.
Feedback linearization of piecewise linear systems

**Theorem 1** (Thm. 9.8 of [15]). The system \( (1) \) is feedback linearizable if and only if there exists a sufficiently smooth function \( h \) such that the system \( (1) \) and \( (5) \) has relative degree \( n \) at a point \( x_0 \).

This paper aims at studying a special class of nonlinear systems: piecewise linear systems. As the smoothness assumptions are not satisfied by these systems, one cannot directly apply the available results.

**4 Bimodal piecewise linear systems**

For the moment, we focus on the bimodal piecewise linear systems given by

\[
\dot{x} = \begin{cases} 
    A_1x + bu & \text{if } c^T x \leq 0 \\
    A_2x + bu & \text{if } c^T x \geq 0
\end{cases} \tag{8}
\]

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the input, \( A_1, A_2 \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n \), and \( 0 \neq c \in \mathbb{R}^n \). We assume that the overall vector field is continuous across the hypersurface \( \ker c^T = \{ x \mid c^T x = 0 \} \), i.e.,

\[
c^T x = 0 \Rightarrow x = A_2x. \tag{9}
\]

Equivalently,

\[
A_2 - A_1 = ec^T \tag{10}
\]

for some \( n \)-vector \( e \).

The right hand side of (8) is Lipschitz continuous in the variable \( x \). Hence, it follows from the theory of ordinary equations that for each initial state \( x_0 \in \mathbb{R}^n \) and input \( u \in L^1_{\text{loc}} \) there exists a unique absolutely continuous function \( x \) satisfying (8) almost everywhere.

**4.1 Controllability**

In what follows we will discuss the controllability of the piecewise linear systems as it play an important role in our treatment.

The fundamental concept of controllability in the state space framework was introduced by [13]. The system (8) is said to be completely controllable if given any pair of states \( (x_0, x_f) \) there exists an input \( u \in L^1_{\text{loc}} \) such that the solution of (8) with \( x(0) = x_0 \) satisfies \( x(\tau) = x_f \), for some \( \tau > 0 \).

For (finite-dimensional) linear systems, the notion of controllability is well-understood. In this case, algebraic necessary and conditions have been provided by Kalman, Popov, Belevitch, and Hautus. We refer to [12, 17] for the historical remarks. When it comes to piecewise linear systems, however, necessary and sufficient conditions are hard to come by. In [1], the authors showed that the controllability problem is undecidable even for very simple piecewise linear systems. Nevertheless, the bimodal systems of the form (8) possess a certain structure which can be exploited in order to derive necessary and sufficient conditions.

**Theorem 2** (Thm. 5 of [4]). Let \( e \) be as in (10). The bimodal piecewise linear system (8) is completely controllable if and only if the following conditions hold.

1. The pair \( (A_1, [b \ e]) \) is controllable,
2. The implication

\[
\begin{bmatrix} w^T & \mu_i \end{bmatrix} \begin{bmatrix} \lambda I - A_i \ b \\ c^T \\ 0 \end{bmatrix} = 0,
\]

\[
\lambda \in \mathbb{R}, \ w \neq 0, \ i = 1, 2 \Rightarrow \mu_1 \mu_2 > 0.
\]

holds.
Remark 1. In case $e \in \text{im } b$ (i.e. $e = fb$ for some real number $f$), the first condition implies the second. To see this, let $w \neq 0$ and $\mu_i, i = 1, 2$ be such that
\[
\begin{bmatrix}
w^T & \mu_i
\end{bmatrix}
\begin{bmatrix}
\lambda I - A_i b & c^T
\end{bmatrix}
= 0
\]
where $\lambda$ is a real number. Since $e \in \text{im } b$, one gets $(\mu_1 - \mu_2)c^T = 0$. As $c$ is a nonzero vector, we get $\mu_1 = \mu_2$. The first condition, i.e. the fact that $(A_1, b)$ is controllable, guarantees that $\mu_1 \neq 0$. To see this, suppose that $\mu_1 = 0$. Then, one has $w^T(\lambda I - A_1) = 0$ and $w^Tb = 0$. Since $(A_1, b)$ is controllable, we get $w = 0$: contradiction! The second condition is satisfied as $\mu_1 = \mu_2 \neq 0$.

4.2 Feedback linearization
The bimodal piecewise linear system (8) is said to be feedback linearizable if there exists a control of the form
\[
u = \begin{cases} k_1^T x + v & \text{if } c^T x \leq 0 \\ k_2^T x + v & \text{if } c^T x \geq 0 \end{cases}
\]
with the property
\[
c^T x = 0 \Rightarrow k_1^T x = k_2^T x
\]
such that the closed loop system is linear, i.e. of the form
\[
\dot{x} = Ax + bu
\]
for some matrix $A$ and vector $b$ with the property that $(A, b)$ is controllable.

Theorem 3. The bimodal piecewise linear system given in (8) is feedback linearizable if, and only if, $e \in \text{im } b$ and $(A_1, b)$ is controllable.

Proof: Suppose that $e \in \text{im } b$ and $(A_1, b)$ is controllable. Let $e = fb$ where $f \in \mathbb{R}$. Consider the feedback transformation
\[
u = \begin{cases} k_1^T x & \text{if } c^T x \leq 0 \\ k_2^T x & \text{if } c^T x \geq 0 \end{cases}
\]
where $k_1$ is arbitrary and $k_2^T = k_1^T + fc^T$. Note that the condition (12) is satisfied. Straightforward calculations show that the system (8) takes the form
\[
\dot{x} = (A_1 + bk_1^T)x + bv
\]
in the closed loop. Since $(A_1, b)$ is controllable, so is $(A_1 + bk_1^T, b)$. Thus, we can conclude that the system (8) is feedback linearizable in view of Remark 1. Suppose, now, that the system (8) is feedback linearizable, i.e. there exists a feedback transformation of the form (11) which render the system into the form
\[
\dot{x} = Ax + bu
\]
for some matrix $A \in \mathbb{R}^{n \times n}$ where $(A, b)$ is controllable. This would mean that
\[
A = A_1 + bk_1^T = A_2 + bk_2^T.
\]
Hence, one gets
\[
A_1 - A_2 = bk_1^T - bk_2^T
\]
\[
ce^T c = f c^T
\]
for some $f \in \mathbb{R}$. The last equation is satisfied only if $e = fb$, i.e., $e \in \text{im } b$. Then, it is enough to show that the pair $(A_1, b)$ is controllable in order to conclude the proof. Note that $A = A_1 + bk_1^T$ for some $k_1 \in \mathbb{R}^n$. Since $(A, b)$ is controllable, so is $(A_1, b)$.

One can relate Theorem 1 to Theorem 3 in the following way.
Theorem 4. There exists a vector $h \in \mathbb{R}^n$ such that the relative degree of the systems
\begin{align*}
\dot{x} &= A_i x + b u \\
y &= h^T x
\end{align*}
for $i = 1, 2$ is $n$ if, and only if, the bimodal system (8) is feedback linearizable.

5 Conewise linear systems

We can extend the above results to a class of piecewise linear systems which consist of a number of linear dynamics that are active on some cones in the input-state space. More specifically, they are systems of the form
\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t) + f(Cx(t)) \\
Cx(t) &\in \mathcal{Y}
\end{align}
where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $\mathcal{Y} \subseteq \mathbb{R}^p$ is a cone, and $f$ is a conewise linear function on $\mathcal{Y}$, i.e. there exist an integer $r$ and cones $\mathcal{Y}_i$, and matrices $M_i \in \mathbb{R}^{n \times p}$ for $i = 1, 2, \ldots, r$ such that
\begin{align}
\bigcup_{i=1}^r \mathcal{Y}_i &= \mathcal{Y}, \\
f(y) &= M_i^T y \text{ if } y \in \mathcal{Y}_i.
\end{align}
These systems will be called conewise linear systems (CLS). Throughout the paper, we assume that
\begin{enumerate}
\item $A_1$, the cones $\mathcal{Y}_i$ are closed, convex, and solid, and
\item $A_2$, the cones $M_i^T \mathcal{Y}_i$ are closed.
\end{enumerate}
For polyhedral cones, Assumption $A_1$ implies $A_2$. However, this does not happen in general (see e.g. [9, Example 2.2.8]).

The simplest examples of CLSs, except the trivial case of linear systems, are the bimodal piecewise linear systems, i.e. systems of the form (8). To fit the system (8) into the framework of CLS (17), one can take $A = A_1$, $B = b$, $C = c^T$, $r = 2$, $\mathcal{Y}_1 = -\mathbb{R}_+$, $\mathcal{Y}_2 = \mathbb{R}_+$, and $M_1 = 0$, $M_2 = e$.

An interesting example of CLSs arise in the context of linear complementarity systems. Consider the linear system
\begin{align}
\dot{x} &= Ax + Bu + Ez \\
w &= Cx + Du + Fz
\end{align}
where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $(z, w) \in \mathbb{R}^{p \times p}$. When the external variables $(z, w)$ satisfy the so-called complementarity relations
\begin{align}
0 \leq z \perp w \geq 0
\end{align}
the overall system (19) is called a linear complementarity system (LCS). A wealth of examples, from various areas of engineering as well as operations research, of these piecewise linear (hybrid) systems can be found in [7,16,20]. For the work on the analysis of general LCSs, we refer to [2,6,10,11,18,19]. A special case of interest emerges when all the principal minors of the matrix $F$ are positive. Such matrices are called $P$-matrices in the literature of the mathematical programming. It is well-known (see for instance [8, Thm. 3.1.6 and Thm. 3.3.7]) that every positive definite matrix is in this class. $P$-matrices enjoy several interesting properties. One of the most well-known is in the context of linear complementarity problem, i.e. the problem of finding an $p$-vector $z$ satisfying
\begin{align}
0 \leq z \perp q + Fz \geq 0.
\end{align}
for a given \( p \)-vector \( q \) and a \( p \times p \) matrix \( F \). It is denoted by \( \text{LCP}(q, F) \). When the matrix \( F \) is a \( P \)-matrix, \( \text{LCP}(q, F) \) admits a unique solution for any \( q \in \mathbb{R}^p \). This is due to a well-known theorem (see [8, Thm. 3.3.7]) of mathematical programming. Moreover, for each \( q \) there exists an index set \( \alpha \subseteq \{1, 2, \ldots, p\} \) such that

1. \(- (F_{\alpha \alpha})^{-1} q_\alpha \geq 0 \) and \( q_\alpha - F_{\alpha \alpha} (F_{\alpha \alpha})^{-1} q_\alpha \geq 0 \),
2. the unique solution \( z \) of the \( \text{LCP}(q, F) \) is given by \( z_\alpha = - (F_{\alpha \alpha})^{-1} q_\alpha \) and \( z_\alpha^c = 0 \)

where \( \alpha^c \) denotes the set \( \{1, 2, \ldots, p\} \setminus \alpha \). This shows that the mapping \( q \mapsto z \) is a conewise linear function. In fact, one can come up \( 2^p \) (i.e. \( r = 2^p \)) (polyhedral) cones \( \mathcal{Y}_i \) and matrices \( M^i \) satisfying \( A_1 - A_2 \) by using the above relations.

### 5.1 Controllability

Controllability properties of the CLSs are addressed in [5].

**Theorem 5 ([5]).** Consider the CLS (17). Suppose that the transfer matrix \( C(sI - A)^{-1} B \) is invertible as a rational matrix. The CLS (17) is completely controllable if, and only if,

1. the relation
   \[
   \sum_{i=1}^{r} (A + M^i C | \text{im} B) = \mathbb{R}^n
   \] (21)
   is satisfied and
2. the implication
   \[
   \begin{align*}
   \lambda & \in \mathbb{R}, z \in \mathbb{R}^n, \text{ and } w_i \in \mathbb{R}^m \\
   [z^T w_i^T] \begin{bmatrix} \lambda I - A - M^i C - B \\ C \\ 0 \end{bmatrix} &= 0 \text{ and} \\
   w_i & \in \mathcal{Y}_i^* \text{ for all } i = 1, 2, \ldots, r \Rightarrow z = 0
   \end{align*}
   \] holds.

**Remark 2.** Similar to the bimodal case, the first condition implies the second if \( \mathcal{Y} = \mathbb{R}^p \) and \( \text{im} M^i C \subseteq \text{im} B \) for all \( i = 1, 2, \ldots, r \). To see this, \( \lambda \in \mathbb{R}, z \in \mathbb{R}^n, \) and \( w_i \in \mathcal{Y}_i^* \) for \( i = 1, 2, \ldots, r \) be such that

\[
[z^T w_i^T] \begin{bmatrix} \lambda I - A - M^i C - B \\ C \\ 0 \end{bmatrix} = 0
\] (22)

holds. Since \( z^T B = 0 \) and \( \text{im} M^i C \subseteq \text{im} B \), one gets

\[
z^T (\lambda I - A) + w_i^T C = 0
\] (23)

for all \( i = 1, 2, \ldots, r \). Since \( \mathcal{Y} = \mathbb{R}^p \), for each \( x \in \mathbb{R}^n \) there exists \( i \in \{1, 2, \ldots, r\} \) such that \( Cx \in \mathcal{Y}_i \). By right-multiplying (23) by \( x \), we get \( z^T (\lambda I - A) x \geq 0 \) as \( w_i \in \mathcal{Y}_i^* \). This means that \( z^T (\lambda I - A) = 0 \). Consequently, \( z \in (A + M^i C | \text{im} B)^{-1} \) for all \( i = 1, 2, \ldots, r \). Thus, we get \( z = 0 \) from the first condition.

### 5.2 Feedback linearization

The CLS (17) is said to be feedback linearizable if there exists a feedback transformation of the form

\[
u = K_i x + v \text{ if } Cx \in \mathcal{Y}_i
\] (24)
where $K_i \in \mathbb{R}^{m \times n}$ and $K_i x = K_j x$ if $x \in \mathcal{Y}_i \cap \mathcal{Y}_j$, such that the closed loop system is linear, i.e.

$$\dot{x} = A' x + B u$$

(25)

for some matrices $A' \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ with $(A', B)$ is controllable.

The following theorem presents necessary and sufficient conditions for the feedback linearizability of CLSs.

**Theorem 6.** Consider a CLS of the form (17) with $\mathcal{Y} = \mathbb{R}^p$. Assume that $M^1 = 0$. Then, (17) is feedback linearizable if, and only if, $\text{im} M' C \subseteq \text{im} B$ for all $i = 1, 2, \ldots, r$ and $\sum_{i=1}^r (A + M' C \mid \text{im} B) = \mathbb{R}^n$.

**Proof:** For the ‘if’ part, suppose that $\text{im} M' C \subseteq \text{im} B$ for all $i = 1, 2, \ldots, r$ and $\sum_{i=1}^r (A + M' C \mid \text{im} B) = \mathbb{R}^n$. Note that the CLS takes the form

$$\dot{x} = (A + M' C + BK_i)x + Bu$$

(26)

with the application of the feedback law (24). Since $\text{im} M' C \subseteq \text{im} B$, one can choose $K_i$ such that $M' C + BK_i = 0$. Then, the CLS (26) becomes a linear system of the form

$$\dot{x} = Ax + Bv.$$ 

(27)

To conclude the proof, it is enough to show that the pair $(A, B)$ is controllable. Note that $(A + M' C \mid \text{im} B) = (A + M' C + BK_i \mid \text{im} B) = (A \mid \text{im} B)$. Since $\sum_{i=1}^r (A + M' C \mid \text{im} B) = \mathbb{R}^n$, one gets $(A \mid \text{im} B) = \mathbb{R}^n$, i.e. $(A, B)$ is controllable.

For the ‘only if’ part, suppose that the CLS (17) is feedback linearizable. This means that there exists a feedback law of the form (24) such that the closed loop system is of the form (25) where $(A', B)$ is controllable. Note that the feedback law (24) yields

$$\dot{x} = (A + M' C + BK_i)x + Bu$$

(28)

in the closed loop. Since the cones $\mathcal{Y}_i$ are all solid, one gets

$$A + M' C + BK_i = A'$$

(29)

for all $i = 1, 2, \ldots, r$. Therefore, one gets

$$M' C + BK_i = M' C + BK_j$$

(30)

for all $i, j \in \{1, 2, \ldots, r\}$. Since $M^1 = 0$ by the hypothesis, we reach $M' C = B(K_1 - K_i)$. In turn, this results in $\text{im} M' C \subseteq \text{im} B$. Note that $(A + M' C \mid \text{im} B) = (A + M' C + BK_i \mid \text{im} B) = (A' \mid \text{im} B)$ for all $i$. Since $(A', B)$ is controllable, we get $\sum_{i=1}^r (A + M' C \mid \text{im} B) = \mathbb{R}^n$.

Note that the structure of (17) reveals that the assumption $M^1 = 0$ can be made without loss of generality.

**6 Conclusions**

We studied the problem of feedback linearization for a class of piecewise linear systems for which the state space partitioned into cones and on each of these cones the dynamics is linear. Earlier work on the controllability of these systems led us to state necessary and sufficient conditions for the solvability of the feedback linearization problem. We also made connections between the results on the smooth nonlinear case and on the nonsmooth piecewise linear case. To extend the results of this paper to more general classes of piecewise linear systems is one of the possibilities for future work.
References

Applied perturbation theory: power converters regulation

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Abstract. In this paper we present the main results obtained by applying average theory to Zero Average Dynamic control technique in a buck converter. In particular we have obtained bounds for output error and sliding surface. The PWM with centered and lateral pulse configurations were analyzed. The analytical results have confirmed the numerical and experimental results already obtained in previous publications. At the end of section II we have generalized the theory, for any second order system with relative grade 2 controlled with this technique.

1 Introduction

Switching sources are devices used in the implementation of power converters. As a consequence of the switching action, chattering, high order harmonic distortion and nonlinear phenomena appear. The latter can be dealt with control techniques [1], while chattering and harmonic distortion, inherent to switching, can be reduced, but not avoided, using fixed switching frequency. To achieve this reduction, some techniques have been reported in the literature: adaptive hysteresis band [11]- [9], signal injection with a selected frequency [9], [14], [13], [7] and [8], zero average current in each iteration (ZACE) [3] and recently zero average error dynamics in each iteration (ZAD). ZAD control schemes, recently proposed in [5], conjugates the advantages of fixed frequency implementations and the inherent robustness of sliding control modes. It is based on an appropriate design of the sliding surface that guarantees the fulfilment of the specifications and on a specific design of a duty cycle in such a way that the sliding surface average in each PWM-period is zero. A comparative study of this algorithm with respect to some of the previously reported can be found in [2], while in [10] this ZAD technique was applied to a linear converter showing good numeric and experimental results. However, the fundamentals explaining this behaviour are not yet reported.

The aim of this paper is to put these basis in the frame of average theory [12], [6] (perturbation theory in mathematics). To be precise, centered and lateral ZAD-PWM schemes will be considered and steady-state maximum values for the error and the sliding surface in a sampling period will be computed as well.

The paper is organized as follows: Section 2 is devoted to introducing the ZAD-PWD schemes for a buck converter, as well as some generalities on average theory. Bounds for the error and the sliding surface, both for the lateral and centered PWM are computed in Section 3. These results are generalised to second order linear systems in Section 4. Conclusions are collected in the last Section.

2 Preliminaries

Figure 1 depicts a simplified model of a buck power converter.

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This system can be modelled by the linear system

\[
\frac{d}{dt} \begin{pmatrix} v \\ i \end{pmatrix} = \begin{pmatrix} -\frac{R}{LC} & \frac{L}{C} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{V}{T} \end{pmatrix} W
\]

where the voltage in the capacitor \( v \) and the current in the inductor \( i \) are state-variables. The control signal \( W \) takes discrete values in the set \{−1, 1\} depending on the switch position. The independent variable time is noted as \( r \) and the sampling time as \( T_c \). Let us define the dimensionless variables \( x_1 = \frac{v}{V} \), \( x_2 = \frac{1}{V\sqrt{LC}}i \) and \( \tau = \frac{r}{\sqrt{LC}} \). The sampling time in the new variables is \( T = T_c / \sqrt{LC} \). The dynamics can be written as

\[
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\gamma & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u
\]

where \( u = W \) is the control input which is implemented through a Pulse Width Modulator. The output of the system, that is, the signal to be controlled is \( y = x_1 \).

Pulse width modulation (PWM) is a powerful technique for controlling analog circuits. PWM is employed in a wide variety of applications, ranging from measurement and communications to power control and conversion. The pulse can be lateral (LPWM), starting just at the sampling time, or centered (CPWM). For full-bridge power converters, the CPWM is defined as

\[
u = \begin{cases} 
1 & \text{if } kT \leq t < (k + d/2)T \\
-1 & \text{if } (k + d/2)T < t < (k + (d/2))T \\
1 & \text{if } (k + 1 - (d/2))T \leq t \leq (k + 1)T
\end{cases}
\]

while the LPWM is defined as

\[
u = \begin{cases} 
1 & \text{if } kT \leq t < (k + d)T \\
-1 & \text{if } (k + d)T < t < (k + 1)T
\end{cases}
\]

d is referred to as the duty cycle.

ZAD-PWM control schemes provide a duty cycle such that the output performs Zero Average Dynamics (ZAD). An sliding surface \( s(x) \) has been defined for the Buck converter. The reason is twofold, first the output \( y = x_1 \) has relative degree 2 and second, to get robustness. As in [4] \( s(x) \) is defined by

\[
s(x) = (x_1 - v_{ref}) + k_s(\dot{x}_1 - \dot{v}_{ref})
\]
where $x_1$ is the variable to control and $k_s$ is the time-constant associated to the error dynamics. Thus,

$$\int_{kT}^{(k+1)T} s(x(\tau))d\tau = 0. \tag{5}$$

$s(x)$ has zero average in each PWM period. The exact calculation of the specific duty cycle $d$ requires solving a transcendental equation. This is a problem for on-line implementations which is overcome by assuming a piecewise linear sliding surface $s(x)$. With this assumption, the equilibrium duty cycle $d^*$ can be approximated by $d^* \approx 0.5(1 + \nu_{ref})$.

As Carpita did his work in the frame of sliding mode, $s(x)$ is also named sliding surface. Properly, the discrete dynamics should be qualified as quasi-sliding because of $s(x)$ samples lie on $s(x) > 0$ and on $s(x) < 0$ consecutively.

Let us rewrite system dynamics in the more suitable variables $e = x_1 - \nu_{ref}$ and $s = e + k_s \dot{e}$, being thereafter $e = T$ the PWM period and $t = \tau / \varepsilon$ the independent variable,

$$\begin{align*}
\dot{\varepsilon} &= \varepsilon \left( \frac{1}{\tau} - k_s \frac{1}{\xi_s} - \gamma \right) \left( \frac{\varepsilon}{s} \right) \\
&\quad + k_s \varepsilon \left( \frac{1}{u - \nu_{ref}} \right) \\
\dot{s} &= \frac{\varepsilon}{\xi_s} k_s \left( \frac{1}{u - \nu_{ref}} \right)
\end{align*} \tag{6}$$

This system reads as $\dot{x} = \varepsilon f(x, t)$ as usual in averaging (perturbation) methods. Taking $x = (e, s)$ equation (6) can be written in a compact form as $\dot{x} = \varepsilon A x + k_s \varepsilon \hat{u}$ where

$$\hat{u} = \begin{pmatrix} 0 \\ u - \nu_{ref} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \frac{-1}{\tau} & \frac{1}{\xi_s} \\ \gamma & \frac{1}{\xi_s} - k_s \frac{1}{\xi_s} - \gamma \end{pmatrix}$$

being $u$ defined by equation (2) or equation (3), depending on the pulse generation scheme (CPWM or LPWM). The control technique selected according to equation (5) guarantees $\langle x_2 \rangle = 0$ and $\langle \dot{u}_2 \rangle := \langle u - \nu_{ref} \rangle = 0$.

**Remark** Prior to define some changes of variables let us consider a generic system $\dot{x} = \varepsilon A x + \varepsilon \hat{u}$ where $\hat{u}$ has jump discontinuities. First, assume that $\hat{u}$, is $C^\infty$ in $[0, 1]$, except at a finite number of points $p_i \in [0, 1]$. Second, let $x(t)$ be a continuous solution of the differential equation $\dot{x} = \varepsilon A x + \varepsilon \hat{u}$. It is continuous in all $[0, 1]$ and $C^\infty$ in $[0, 1]$ except at the points $p_i$. Third, if we define $\overline{x} = \int_0^t \hat{u}$, this is also $C^\infty$ in all $[0, 1]$ except at the points $p_i$, where it is only continuous. Let us define $y = x - \varepsilon \overline{x}$; $y$ is $C^1$ everywhere except at the points $p_i$, where, in principle, it is continuous. This change of variables can be done in each subinterval $(p_i, p_{i+1})$ and extended everywhere thanks to the continuity of $\overline{x}$ at $p_i$. Finally, one obtain, for any $t \in (p_i, p_{i+1})$,

$$\dot{y} = \varepsilon A x = \varepsilon A y + \varepsilon^2 A \overline{x}$$

The function $f(t, y) = \varepsilon A y + \varepsilon^2 \overline{x}$ is continuous with respect to $t$ and $C^1$ with respect to $y$. The ODE existence and uniqueness theorem, states that all the solutions of the equation $\dot{y} = \varepsilon A y + \varepsilon^2 A \overline{x}$ are $C^1$. Moreover, due to the fact that $\dot{y} = \varepsilon A x$ for any $t \in (p_i, p_{i+1})$, and that $x(t)$ is a continuous function everywhere, one obtain that $y(t)$ is a $C^1$ function everywhere as a direct result. This allows to perform the proposed change of variables, which is common in averaging theory, also at the points $p_i$, where a priori was not well defined.

It is important to remark that the validity of the method is based on the explicit computations that can be done because the system we are dealing with is a linear system of ordinary differential equations.

Let us come back to our case. In order to average the system, let us define the change of variables $y = x - \varepsilon U$, where $U = k_s \int_0^t \hat{u} d\tau$. Since $\langle \hat{u} \rangle = 0$, $U$ is periodic. It is straightforward to obtain,

$$\dot{y} = \varepsilon A y + \varepsilon^2 A U \tag{7}$$
Fig. 2: $\hat{u}$ and two first integrals. In (a) the CPWM case and in (b) the LPWM case

For the next change of variables, let us note that the function $\int_0^1 A\hat{u}dt$ can not presumed to be periodic, as Figure 2 shows. Then let $\hat{U} = AU$, $a = \int_0^1 \hat{u}dt$ be the mean of $\hat{U} = AU$ and $\bar{U} = \int_0^1 (\hat{U} - a)dr$, which is indeed periodic.

Again, let us define a new change of variables, namely

$$z = y - \varepsilon^2 U.$$  \hfill (8)

Then,

$$\dot{z} = \dot{y} - \varepsilon^2 \hat{U}$$

This equation is also well defined in switching instants because $U$ is $C^1$. Hence

$$\dot{z} = \dot{y} - \varepsilon^2 (AU - a)$$  \hfill (9)

Replacing (7) and (8) in (9) yields

$$\dot{z} = \varepsilon A z + \varepsilon^2 a + \varepsilon^3 A U$$  \hfill (10)

Neglecting $O(\varepsilon^3)$ terms in equation (10), we get $\dot{z} = \varepsilon A z + \varepsilon^2 a$. $z^* = -\varepsilon A^{-1} a$ is an equilibrium point of the equation. Now, let us define $w = z - z^*$. Then,

$$\dot{w} = \varepsilon A w + \varepsilon^3 \hat{U}$$

where $\hat{U} = AU$.

The general solution is

$$w(t) = e^{\varepsilon A t} w(0) + \varepsilon^3 \int_0^t e^{\varepsilon A (t-\sigma)} \hat{U}(\sigma) d\sigma$$  \hfill (11)

As we are interested in the steady-state periodic solution, $w(1) = w(0)$. Hence,

$$w(0) = (I - e^{\varepsilon A})^{-1} \varepsilon^3 \int_0^1 e^{\varepsilon A (1-\sigma)} \hat{U}(\sigma) d\sigma$$  \hfill (12)
where I is the identity matrix. Finally, the periodic solution in the original variables \( x \) of the system is

\[
x(t) = \varepsilon U(t) + \varepsilon^2 \dot{U}(t) + w(t) - \varepsilon A^{-1}a
\]

where \( w(t) \) is given by equations (11) and (12).

As \( U \) and \( \dot{U} \) only depend on the input control signal and the constant matrix, their integrals are known. Then, in order to bound \( x(t) \), we will proceed in obtaining bounds for each component of the variable \( w(t) \), namely the state transition matrix, \((I - \varepsilon A)^{-1}\) and \( \int_0^1 e^{\varepsilon A(1-\sigma)} \hat{U}(\sigma) d\sigma \).

Although it can appear to be, results developed up to now are not only for a particular example, as the following lemma shows.

**Lemma**

Any unitary gain, second order system with relative degree 2, can be put in the form of equation (1).

**Proof.** Let

\[
Y(s) = \frac{\alpha_1}{s^2 + \alpha_2 s + \alpha_1} U(s)
\]

be the transfer function of a general second linear order system with relative degree 2.

The system can be written as

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\alpha_1 & -\alpha_2
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + \begin{bmatrix}
0 \\
\alpha_1
\end{bmatrix} u \\
y = \begin{bmatrix}
1 & 0
\end{bmatrix} z
\]

The output is the first component of the new state vector. Let \( \tau = \frac{1}{\sqrt{\alpha_1}} t \) be the current independent variable and \( t \) the new independent variable. We have

\[
\frac{dz_1}{dt} = \frac{1}{\sqrt{\alpha_1}} z_2 \Delta t = w_2
\]

and

\[
\frac{dw_2}{dt} = \frac{1}{\alpha_1} \frac{dz_2}{d\tau} = -z_1 - \frac{\alpha_2}{\sqrt{\alpha_1}} w_2 + u
\]

Now, defining \( w_1 = z_1 \) the dynamic system reads as

\[
\begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 & -\frac{\alpha_2}{\sqrt{\alpha_1}}
\end{bmatrix} \begin{bmatrix}
w_1 \\
w_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u \\
y = \begin{bmatrix}
1 & 0
\end{bmatrix} w
\]

Finally, let us define \( \gamma = \frac{\alpha_2}{\sqrt{\alpha_1}} \), \( P = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \) and \( x = Pw \). Then (1) is a minimal realization of the transfer function given by equation (14). Therefore, the methodology developed in the paper is a general tool.

### 3 Bounds

Since the expression for \( \hat{U} \) differs from lateral or centered pulses, the analysis is made separately, however the common terms admit a unique analysis. In addition, because the differences in magnitude between variables \( e \) and \( s \) have to be taken into account, it is better to analyze each one separately. This will be made in the following subsections.
3.1 State transition matrix

The state transition matrix can be written in a compact form as 

e^{A_1} = \begin{pmatrix} e_{11}(t) & e_{12}(t) \\ e_{21}(t) & e_{22}(t) \end{pmatrix}

where

\begin{align*}
    e_{11}(t) &= e^{-\frac{\gamma}{2} \alpha t} \left( \frac{\gamma/2 - 1/k_s}{\alpha} \sin (\alpha \epsilon t) + \cos (\alpha \epsilon t) \right) \\
    e_{12}(t) &= e^{-\frac{\gamma}{2} \alpha t} \frac{1}{\alpha k_s} \sin (\alpha \epsilon t) \\
    e_{21}(t) &= e^{-\frac{\gamma}{2} \alpha t} \left( \frac{\gamma - k_s - 1/k_s}{\alpha} \sin (\alpha \epsilon t) \right) \\
    e_{22}(t) &= e^{-\frac{\gamma}{2} \alpha t} \left( -\frac{\gamma/2 - 1/k_s}{\alpha} \sin (\alpha \epsilon t) + \cos (\alpha \epsilon t) \right)
\end{align*}

and \( \alpha = \sqrt{1 - \frac{2}{\gamma^2}} \). Expanding these coefficients in a Taylor series up to first order and evaluating its maximum in the interval \([0, 1]\) yields to

\begin{align*}
    \max |e_{11}(t)| &\leq \left| \frac{\gamma}{2} - \frac{1}{k_s} \right| \epsilon + 1 := \epsilon_{11} \\
    \max |e_{12}(t)| &\leq \frac{1}{k_s} \epsilon := \epsilon_{12} \\
    \max |e_{21}(t)| &\leq \left| \gamma - k_s - \frac{1}{k_s} \right| \epsilon := \epsilon_{21} \\
    \max |e_{22}(t)| &\leq \left| \frac{\gamma}{2} - \frac{1}{k_s} \right| \epsilon + 1 := \epsilon_{22}
\end{align*}

3.2 Inverse of \((I - e^{A_1})\)

\((I - e^{A_1})^{-1}\) can be expressed as

\[
(I - e^{A_1})^{-1} = \frac{1}{\det (I - e^{A_1})} \begin{pmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{pmatrix} \tag{15}
\]

The bounds of each one of the terms associated to the state transition and to the adjoint matrices are: \( m_{11}(t) = 1 - e_{22}(t), m_{12}(t) = e_{12}(t), m_{21}(t) = e_{21}(t), m_{22}(t) = 1 - e_{11}(t) \) and \( \det (I - e^{A_1}) = 1 - 2e^{-\frac{\gamma}{2} \alpha t} \cos (\alpha \epsilon t) + e^{-\gamma \epsilon t} \). Since for the determinant order 1 \((O(1))\) and order \( \epsilon t \) \((O(\epsilon t))\) terms are cancelled, second order Taylor series expansion are required. Hence,

\[
\cos (\alpha \epsilon t) = 1 - \frac{1}{2} (\alpha \epsilon t)^2 + O ((\alpha \epsilon t)^3)
\]

\[
= 1 - \frac{1}{2} (\alpha \epsilon t)^2 + \epsilon_{cd}
\]

\[
e^{-\frac{\gamma}{2} \alpha t} = 1 - \frac{\gamma \epsilon t}{2} + \frac{1}{2} \left( \frac{\gamma \epsilon t}{2} \right)^2 + O \left( \left( \frac{\gamma \epsilon t}{2} \right)^3 \right)
\]

\[
= 1 - \frac{\gamma \epsilon t}{2} + \frac{1}{2} \left( \frac{\gamma \epsilon t}{2} \right)^2 + \epsilon_{gmd}
\]

\[
e^{-\gamma \epsilon t} = 1 - \gamma \epsilon t + \frac{1}{2} (\gamma \epsilon t)^2 + O ((\gamma \epsilon t)^3)
\]

\[
= 1 - \gamma \epsilon t + \frac{1}{2} (\gamma \epsilon t)^2 + \epsilon_{gd}
\]

where \( \epsilon_{cd}, \epsilon_{gmd} \) and \( \epsilon_{gd} \) correspond to higher order error terms. From Taylor theorem, \( |\epsilon_{cd}| \leq \frac{\alpha (\epsilon t)^3}{2} \), \( |\epsilon_{gmd}| \leq \frac{1}{6} \left( \frac{\gamma \epsilon t}{2} \right)^3 \) and \( |\epsilon_{gd}| \leq \frac{(\gamma \epsilon t)^3}{6} \), and the determinant is reduced to

\[
\det (I - e^{A_1}) = (e^{2t^2} + E)
\]
where \( E = -2e^{-\frac{2}{\gamma}t}\epsilon_{ct} - 2\cos(\alpha\epsilon t)\epsilon_{gm} + \epsilon_{gt} + \frac{1}{8}\gamma^2\alpha^2\epsilon^4t^4 - \frac{1}{2}\gamma\alpha^2\epsilon^3t^3. \)

The leading terms of the adjoint matrix coefficients are of order \( O(\epsilon t) \), hence in the Taylor series expansion they must be developed until order 1. The following expansions are used,

\[
\cos(\alpha\epsilon t) = 1 + O((\alpha\epsilon t)^2) = 1 + \epsilon_{cn} \\
\sin(\alpha\epsilon t) = \alpha\epsilon t + O((\alpha\epsilon t)^2) = \alpha\epsilon t + \epsilon_{sn} \\
e^{-\frac{2}{\gamma}t} = 1 - \frac{2\gamma t}{2} + O\left(\frac{(\gamma t)^2}{2}\right) = 1 - \frac{2\gamma t}{2} + \epsilon_{gmn}
\]

where \( \epsilon_{cn} \leq \frac{(\alpha\epsilon t)^2}{2} \), \( \epsilon_{sn} \leq \frac{(\alpha\epsilon t)^2}{2} \) and \( \epsilon_{gmn} \leq \frac{1}{2}\left(\frac{\gamma t}{2}\right)^2 \). Then the coefficients \( m_{ij} \) and their bounds in the interval \([0 1]\) can be written as

\[
\begin{align*}
  m_{11} &= \left(\gamma - \frac{1}{k_s}\right)\epsilon t + \epsilon_{m11} \\
  m_{11\max} &\leq \left|\gamma - \frac{1}{k_s}\right|\epsilon + |\epsilon_{m11}| \\
  m_{12} &= \frac{1}{k_s}\epsilon t + \epsilon_{m12} \\
  m_{12\max} &\leq \left|\frac{1}{k_s}\right|\epsilon + |\epsilon_{m12}| \\
  m_{21} &= \left(\gamma - \frac{1}{k_s} - k_s\right)\epsilon t + \epsilon_{m21} \\
  m_{21\max} &\leq |\gamma - \frac{1}{k_s} - k_s|\epsilon + |\epsilon_{m21}| \\
  m_{22} &= \frac{1}{k_s}\epsilon t + \epsilon_{m22} \\
  m_{22\max} &\leq \left|\frac{1}{k_s}\right|\epsilon + |\epsilon_{m22}|
\end{align*}
\]

where

\[
\begin{align*}
  \epsilon_{m11} &= -c_1e^{-\frac{2}{\gamma}t}\epsilon_{sn} - c_1\sin(\alpha\epsilon t)\epsilon_{gm} - e^{-\frac{2}{\gamma}t}\epsilon_{cn} \\
  &- \cos(\alpha\epsilon t)\epsilon_{gm} + c_1\frac{\alpha\gamma}{2}\epsilon^2t^2 \\
  \epsilon_{m12} &= -\frac{1}{\alpha k_s}e^{-\frac{2}{\gamma}t}\epsilon_{sn} - \frac{1}{\alpha k_s}\sin(\alpha\epsilon t)\epsilon_{gm} + \\
  &+ \frac{\gamma}{2k_s}\epsilon^2t^2 \\
  \epsilon_{m21} &= c_2e^{-\frac{2}{\gamma}t}\epsilon_{sn} + c_2\sin(\alpha\epsilon t)\epsilon_{gm} + c_2\frac{\alpha\gamma}{2}\epsilon^2t^2 \\
  \epsilon_{m22} &= c_1e^{-\frac{2}{\gamma}t}\epsilon_{sn} + c_1\sin(\alpha\epsilon t)\epsilon_{gm} - e^{-\frac{2}{\gamma}t}\epsilon_{cn} \\
  &- \cos(\alpha\epsilon t)\epsilon_{gm} - c_1\frac{\alpha\gamma}{2}\epsilon^2t^2 
\end{align*}
\]

where \( c_1 = \frac{1}{\alpha k_s} - \frac{\gamma}{2\alpha} \) and \( c_2 = \frac{k_s + \frac{1}{\alpha} - \gamma}{\alpha} \). Finally, let us take \( t = 1 \) and bound the errors by adding the absolute values of the addends. Exponential and cosine functions are bounded by 1, and sine
function is bounded by the angle, hence the expressions for each error term are

$$
|E| \leq 2 \left| \frac{\alpha^2 \varepsilon^3}{6} + \frac{1}{24} \gamma^3 \varepsilon^3 \right| + \frac{1}{8} \gamma^2 \alpha^2 \varepsilon + \left( \frac{1}{2} \gamma \alpha \right)^2 \varepsilon^3
$$

$$
|\varepsilon_{m11}| \leq \left| c_1 \frac{\alpha^2 \varepsilon^2}{2} \right| + \left| c_1 \alpha \gamma^2 \varepsilon^3 \right| + \left| \frac{\alpha^2 \varepsilon^2}{8} \right| + \left| \frac{\gamma^2 \varepsilon^2}{2} \right|
$$

$$
|\varepsilon_{m12}| \leq \left| \frac{1}{2k_s} \varepsilon^2 \right| + \left| \frac{\gamma^2}{8k_s} \varepsilon^3 \right| + \left| \frac{\gamma \varepsilon^2}{2k_s} \right|
$$

$$
|\varepsilon_{m21}| \leq \left| c_2 \frac{\alpha^2 \varepsilon^2}{2} \right| + \left| c_2 \alpha \gamma^2 \varepsilon^3 \right| + \left| c_2 \alpha \gamma \varepsilon^2 \right|
$$

$$
|\varepsilon_{m22}| \leq \left| c_1 \frac{\alpha^2 \varepsilon^2}{2} \right| + \left| c_1 \alpha \gamma \varepsilon^2 \right| + \left| \frac{\alpha^2 \varepsilon^2}{8} \right| + \left| \frac{\gamma^2 \varepsilon^2}{2} \right|
$$

since constants γ, k_s, and α are positive, these expressions can be simplified to

$$
|E| = \varepsilon^3 |E^*|
$$

$$
|\varepsilon_{m11}| \leq \varepsilon^2 \left( \frac{\alpha^2}{2} (1 + |c_1|) + \gamma^2 \left( 1 + \alpha \varepsilon |c_1| \right) + \frac{\alpha \gamma}{2} |c_1| \right)
$$

$$
|\varepsilon_{m12}| \leq \varepsilon^2 \left( \frac{1}{2k_s} \left( 1 + \gamma + \frac{\gamma^2}{4} \varepsilon \right) \right)
$$

$$
|\varepsilon_{m21}| \leq \varepsilon^2 \left( |c_2| \left( \frac{\alpha^2}{2} + \frac{\alpha \gamma}{2} + \frac{\alpha \gamma^2}{8} \varepsilon \right) \right)
$$

$$
|\varepsilon_{m22}| \leq \varepsilon^2 \left( \frac{\alpha^2}{2} (1 + |c_1|) + \gamma^2 \left( 1 + \alpha \varepsilon |c_1| \right) + \frac{\alpha \gamma}{2} |c_1| \right)
$$

where \(|E^*| \leq \frac{\alpha^2}{2} \gamma^3 + \frac{\alpha \gamma}{2} \varepsilon \left( \frac{4}{3} \varepsilon - 1 \right)\). Then, presuming \(\varepsilon |E^*| < 1\), each term of \((I - \varepsilon \mathbf{A})^{-1}\) can be bounded as

$$
\left| \left( I - \varepsilon \mathbf{k} \right)^{-1} \right|_{11} \leq \frac{1}{\varepsilon^2 - \varepsilon^3 |E^*|} \left( \left| \gamma - \frac{1}{k_s} \varepsilon \right| + |\varepsilon_{m11}| \right)
$$

$$
\left| \left( I - \varepsilon \mathbf{A} \right)^{-1} \right|_{12} \leq \frac{1}{\varepsilon^2 - \varepsilon^3 |E^*|} \left( \left| \frac{1}{k_s} \varepsilon \right| + |\varepsilon_{m12}| \right)
$$

$$
\left| \left( I - \varepsilon \mathbf{A} \right)^{-1} \right|_{21} \leq \frac{1}{\varepsilon^2 - \varepsilon^3 |E^*|} \left( \left| \gamma - k_s \right| + \left| \frac{1}{k_s} \varepsilon \right| + |\varepsilon_{m21}| \right)
$$

$$
\left| \left( I - \varepsilon \mathbf{A} \right)^{-1} \right|_{i,j} \leq \frac{1}{\varepsilon^2 - \varepsilon^3 |E^*|} \left( \left| \frac{1}{k_s} \varepsilon \right| + |\varepsilon_{m22}| \right)
$$

In order to establish the maximum value of \(\mathbf{w}(t)\), we compute now the maximum values taken for each component of vector \(\mathbf{w}(0)\). As

$$
\mathbf{w}(0) = \left( I - \varepsilon \mathbf{A} \right)^{-1} \varepsilon^3 \int_0^1 \varepsilon \mathbf{A}(1 - \sigma) \dot{\mathbf{U}}(\sigma) d\sigma,
$$

and the input \(\dot{\mathbf{U}}\) depends on the pulse scheme, the analysis is made separately depending on the pulse is centered or lateral.
3.3 \( w(t) \) CPWM (Centered pulse)

\( w(t) \) is given by

\[
w(t) = e^{zA}w(0) + e^{3z} \int_{0}^{t} e^{zA(t-\sigma)} \hat{w}(\sigma) \, d\sigma
\]

where

\[
w(0) = (1 - e^{zA})^{-1} e^{3z} \int_{0}^{1} e^{zA(1-\sigma)} \hat{w}(\sigma) \, d\sigma
\]

The aim of this section is to evaluate \( w(0) \) and \( w(t) \) in order to determine the maximum of the error \( e(t) \) and of the sliding surface \( s(x) \) both in steady-state. First, let us compute \( \hat{w} \).

Let be \( \hat{u} = [\hat{u}_1 \hat{u}_2]^T \), where \( \hat{u}_1 = 0 \) and

\[
\hat{u}_2 = \begin{cases} 
1 - v_{ref} & \text{if } 0 \leq t \leq d/2 \\
-1 - v_{ref} & \text{if } d/2 < t \leq 1 - d/2 \\
1 - v_{ref} & \text{if } 1 - d/2 < t \leq 1 
\end{cases}
\]

\( d \) is the time the switch takes value +1 as it is expressed in equation (2). The duty cycle in the stationary state is

\[ d = \frac{1 + v_{ref}}{2}.\]

\( U = k_s \int_0^t \hat{u} \, d\tau \) yields to \( U = k_s [U_{11} \ U_{21}]^T \), where \( U_{11} = 0 \) and \( U_{21} \) is a periodic signal given by:

\[
U_{21} = \begin{cases} 
(1 - v_{ref}) t & \text{if } 0 \leq t \leq d/2 \\
2d - 2 + (1 - v_{ref}) t & \text{if } d/2 < t \leq 1 - d/2 \\
2d - 2 (1 - d) t + \frac{1}{2} (1 - v_{ref}) t^2 & \text{if } 1 - d/2 < t \leq 1 
\end{cases}
\]

Hence \( U \) can be expressed as

\[
U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k_s U_{21} \end{bmatrix}
\]

The average of \( U \) (\( \langle U \rangle \)) is zero, then \( a = A \int_0^t U \, dt = 0 \) and

\[
U = \int_0^t (AU - a) \, d\tau = k_s A \left[ U_{11} \ U_{21} \right]^T,
\]

where \( U_{11} = 0 \) and

\[
U_{21} = \begin{cases} 
\frac{1}{2} (1 - v_{ref}) t^2 & \text{if } 0 \leq t \leq d/2 \\
-\frac{1}{2} d^2 + dt - \frac{1}{2} (1 + v_{ref}) t^2 & \text{if } d/2 < t \leq 1 - d/2 \\
(1 - d) - 2 (1 - d) t + \frac{1}{2} (1 - v_{ref}) t^2 & \text{if } 1 - d/2 < t \leq 1 
\end{cases}
\]

which also is a periodic signal. Finally \( U \) can be expressed in a compact form as

\[
U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_{21} \\ (1 - k_s \gamma) U_{21} \end{bmatrix}
\]

and

\[
\hat{U} = \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} = A U = \begin{bmatrix} -\gamma U_{21} \\ (k_s \gamma^2 - \gamma - k_s) U_{21} \end{bmatrix}
\]
Let us define
\[ I = \int_0^1 e^{\epsilon A (1 - \sigma)} \hat{U}(\sigma) d\sigma \]
which components \( I_i \) fulfil
\[ |I_i| \leq (\gamma \epsilon_{i1} + |k_s \gamma^2 - k_s - \gamma| \epsilon_{i2}) \int_0^1 |U_{21}(\sigma)| d\sigma \]

By integrating the absolute value of input \( U_{21}(\sigma) \) over the interval \([0, 1]\) we have
\[ |I_i| \leq \left( \gamma \epsilon_{i1} + |k_s \gamma^2 - k_s - \gamma| \epsilon_{i2} \right) \left( \frac{1}{17} (1 + v_{ref}) - \frac{1}{17} (1 + v_{ref})^3 + \frac{1}{40} (1 + v_{ref})^3 \right) \]

For the coefficients \( m_{ij}(t) \) defined in equation (15), let the maximum values give by equation (16) (i.e. \( m_{ij\text{ max}} \)). Then, taking into account
\[ \left| \int_0^t U_{21}(\sigma) d\sigma \right| \leq \int_0^t |U_{21}(\sigma)| d\sigma \leq \int_0^1 |U_{21}(\sigma)| d\sigma \]
equations (11) and (12), yield to
\[ |w_i(0)| \leq \epsilon \left[ |m_{i1\text{ max}} I_1| + |m_{i2\text{ max}} I_2| \right] \]
\[ |w_i(t)| \leq |\epsilon_1 w_1(0)| + |\epsilon_2 w_2(0)| + \epsilon^3 |I_i| \]

To conclude, using the estimated bound of \( w(t) \), is easy to find from
\[ x(t) = \epsilon U(t) + \epsilon^2 U(t) + w(t) \]
the maximum values for the error \( e = x_1 \) and for the sliding surface \( s = x_2 \).

3.4 \( w(t) \) LPWM (Lateral pulse)

In this case the pulse generation order is +1 -1. As it was stated before, \( d \) is the time the switch is +1 and it is given by \( d = (1 + v_{ref})/2 \). Proceeding as in the previous subsection, let us define \( \hat{u} = [\hat{u}_1 \hat{u}_2]^T \), where \( \hat{u}_1 = 0 \) and
\[ \hat{u}_2 = \begin{cases} 1 - v_{ref} & \text{if } 0 \leq t \leq d \\ -1 - v_{ref} & \text{if } d < t < 1 \end{cases} \]
the evaluation of \( \hat{U} = k_s \int_0^t \hat{u} d\tau \) yields to
\[ \hat{U} = k_s [U_{11} U_{21}]^T \]
where \( U_{11} = 0 \) and
\[ U_{21} = \begin{cases} (1 - v_{ref}) t & \text{if } 0 \leq t \leq d \\ 2d - (1 + v_{ref}) t & \text{if } d < t < 1 \end{cases} \]
\( \mathcal{U} \) can be written as
\[
\mathcal{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k_s U_{21} \end{bmatrix}
\]
In order to evaluate the mean of \( \mathcal{U} \), in this case
\[
\langle \mathcal{U} \rangle \neq 0 = a
\]
let us compute
\[
a = A \int_0^{1} U dt = k_s A \begin{bmatrix} -d^2 + 2d - \frac{1}{2} - \frac{v_{ref} t}{2} \end{bmatrix}
\]
being \( \beta = -d^2 + 2d - \frac{1}{2} - \frac{v_{ref} t}{2} \).
\[
\mathcal{U} = \int_0^{1} (A \mathcal{U}(\sigma) - a) d\sigma \text{ can be evaluated as}
\]
\[
\mathcal{U} = k_s A \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}
\]
where \( U_{11} = 0 \) and
\[
U_{21} = \begin{cases} \frac{1}{2} (1 - v_{ref}) t^2 - \beta t & \text{if } 0 \leq t \leq d \\ -d^2 + (2d - \beta)t - \frac{1}{2} (1 + v_{ref}) t^2 & \text{if } d < t \leq 1 \end{cases}
\]
Finally \( \mathcal{U} \) is given in a compact form as
\[
\mathcal{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_{21} \\ (1 - k_s \gamma) U_{21} \end{bmatrix}
\]
and
\[
\hat{\mathcal{U}} = \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} = A \mathcal{U} = \begin{bmatrix} -\gamma U_{21} \\ (k_s \gamma^2 - \gamma - k_s) U_{21} \end{bmatrix}
\]
As in the preceding case, the components of the \( I \) function
\[
I = \int_0^{1} e^{A(1-\sigma)} \hat{\mathcal{U}}^2(\sigma) d\sigma
\]
can be bounded by
\[
|I| \leq (\gamma |\epsilon_{e1}| + |k_s \gamma^2 - k_s - \gamma| |\epsilon_{e2}|) \int_0^{1} |U_{21}(\sigma)| d\sigma
\]
Integrating the absolute value of the input \( U_{21}(\sigma) \) on the interval \([0,1]\) yields to
\[
|I| \leq \left( \gamma |\epsilon_{e1}| + |k_s \gamma^2 - k_s - \gamma| |\epsilon_{e2}| \right) \left( \frac{1}{12} (1 + v_{ref}) \right)
\]
\[
- \frac{1}{2} (1 + v_{ref})^2 \left( \frac{1 + v_{ref}}{2} \right)
\]
\[
+ \frac{1}{8} (1 + v_{ref})^3 \left( 2 + \frac{v_{ref}}{2} \right) - \frac{1}{16} (1 + v_{ref})^4
\]
(21)
Finally,
\[
|u_1(0)| \leq \varepsilon \frac{1}{|1 - \varepsilon| \|E^*\|} (|m_{i1\text{ max}} I_1| + |m_{i2\text{ max}} I_2|)
\]
and
\[ |w_1(t)| \leq (|\epsilon_{11} w_1(0)| + |\epsilon_{22} w_2(0)|) + \epsilon^3 |I_1| \]
Using this bound for the \( w(t) \) we found the bound for \( e(t) \) and \( s(t) \) using
\[ x(t) = \epsilon U(t) + \epsilon^2 U(t) + w(t) - \epsilon A^{-1} a \]
where \( e = x_1 \) and \( s = x_2 \). It is worth to note that in both cases the output error is defined by
\[ e(t) = w_1(t) + \epsilon^2 U_1 \]
which implies a very low error.

4 Output and sliding surface error estimation

Let us particularise all of these calculus for \( d = 0.9 \) and the parameter values \( \gamma = 0.35 \), \( v_{ref} = 0.8 \) and \( \epsilon = T = 0.1767 \) correspond to \( R = 20 \Omega \), \( C = 40 \mu F \), \( L = 2 mH \) \( V_{ref} = 32 \) and \( V = 40 \) in the actual system.

4.1 CPWM

In this case \( k_s = 4.5 \) and \( \det(I - \epsilon^\Delta) := |\epsilon^2 - \epsilon^3|E^*| \geq 0.0285 \), \( \epsilon_{11} \leq 1.0083 \), \( \epsilon_{12} \leq 0.0393 \), \( \epsilon_{21} \leq 0.7726 \), \( \epsilon_{22} \leq 1.0083 \), \( I_1 \leq 0.0013 \), \( I_2 \leq 0.0371 \), \( m_{11} \max \leq 0.0392 \), \( m_{12} \max \leq 0.0420 \), \( m_{21} \max \leq 0.8640 \) and \( m_{22} \max \leq 0.0559 \).
Hence
\[ w_1(0) \leq 0.00936 \quad w_2(0) \leq 0.0011 \]
\[ w_1(t) \leq 0.00043 \quad w_2(t) \leq 0.0016 \]

Output error estimation As in this case \( a = 0 \) and the first component of the input vector \( U \) is zero, error dynamics is defined by the first component of \( w(t) \) and \( \epsilon^2 U \), thus
\[ e(t) = w_1(t) + \epsilon^2 U_1 \]
The maximum of \( |U_1| = |U_{21}| \) holds at \( t = 0.5 \) and is 0.0225, then
\[ \max |e(t)| \leq 0.0011 \]
This is equivalent to a maximum error value of 0.14% in steady state when the reference reaches 0.8 value. In Figure 3 the error behavior obtained from a numerical simulation is depicted. Notice that real error value is lower than the estimated. However, the bound is really close.

Sliding surface error estimation In this case the expression is adjusted to analyze the second component as
\[ s(t) = \epsilon U_2 + \epsilon^2 U_2 + w_2(t) \]
The maximum associated to \( |U_2| = k_s |U_{21}| \) holds at \( t = 0.45 \) and is 0.4050; while the maximum of \( |U_2| = (1 - k_s \gamma) |U_{21}| \) holds at \( t = 0.5 \) and is 0.0129, then
\[ \max |s(t)| \leq \epsilon |U_2| + \epsilon^3 |U_2| + w_2(t) = 0.0728 \]
Notice the agreement between the maximum of the piece-wise sliding surface approximation and our result. The leading term in the later inequality, \( \epsilon |U_2| \) and is 0.0716. Simulation results are depicted in Figure 4.
3.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
x 10
−4
|error|
|time|

Fig. 3: behavior of error in sampling interval for CPWM scheme

|time|

|sliding surface|

Fig. 4: Behaviour of sliding surface in a sampling interval for CPWM scheme

4.2 LPWM

Now \( k_s = 0.7068 \), with these values one obtains:

\[
| \det(\mathbb{I} - \varepsilon \mathbf{A}) | \geq 0.0285, \quad \varepsilon_{11} \leq 1.2191, \quad \varepsilon_{12} \leq 0.2500, \quad \varepsilon_{21} \leq 0.3130, \quad \varepsilon_{22} \leq 1.2191. \]

Therefore \( I_1 \leq 0.0083, \quad I_2 \leq 0.0159, \quad m_{11 \text{ max}} \leq 0.2297, \quad m_{12 \text{ max}} \leq 0.2799, \quad m_{21 \text{ max}} \leq 0.3501 \) and \( m_{22 \text{ max}} \leq 0.2915 \).

And the bounds for \( w(0) \) and \( w(t) \) are

\[
\begin{align*}
w_1(0) & \leq 0.00124 \quad w_2(0) \leq 0.00146 \\
w_1(t) & \leq 0.00193 \quad w_2(t) \leq 0.00226
\end{align*}
\]

Output error estimation The general expression to calculate the error and the sliding surface is

\[
\mathbf{x}(t) = \varepsilon \mathcal{U}(t) + \varepsilon^2 \mathbf{U}(t) + w(t) - \varepsilon^{-1} \mathbf{A} \mathbf{a}
\]

Computing \( \mathbf{h} = \mathbf{A}^{-1} \mathbf{a} \) and taking into account that \( \mathbf{a} = k_s \mathbf{A} \begin{bmatrix} 0 \\ \beta \end{bmatrix} \) where \( \beta = -d^2 + 2d - \frac{1}{2} - \frac{v_{ref}}{2} \)

we obtain the components of \( \mathbf{h} \) which are \( h_1 = 0 \) and \( h_2 = k_s \beta \). As \( \mathcal{U}_1 = 0 \), the error \( e(t) \) fulfils

\[
e(t) = \mathbf{w}(t) + \varepsilon^2 \mathbf{U}_1
\]

The maximum value for excitation \( \mathbf{U}_1 \) holds at \( t = 0.95 \) and is 0.0023, hence

\[
\max |e(t)| \leq 0.00201
\]
This is equivalent to a maximum error of 0.26% in steady state for a reference equal to 0.8. The output error behaviour is depicted in Figure (5). Simulations took 500 points per sampling period.

Sliding surface error estimation

In this case we have

\[ s(t) = \varepsilon U_2 + \varepsilon^2 U_2 + w_2(t) - \varepsilon h_2 \]

\[ \leq \varepsilon |U_2 - h_2| + |\varepsilon^2 U_2| + |w_2(t)| \]

The maximum associated to |U_2| holds at \( t = 0.9 \) and is 0.1272, while the maximum of |U_2| holds at \( t = 0.95 \) and is 0.0017. In addition, \( h_2 = 0.0636 \). Hence,

\[ \max |s(t)| \leq 0.0136 \]

Simulation results are depicted in Figure (6).

5 Conclusions

In this work, approximations made in [2, 5, 10] have been supported in the frame of averaging theory. In those papers the sliding surface \( s \), is assumed to be piece-wise linear w.r.t. time. Our
final approximation reads as

$$s(t) = \varepsilon U_2 + \varepsilon^2 U_2 + w_2(t) - \varepsilon h_2,$$

the leading term is of order 1; it is a straight line and it corresponds to the primitive of the input of the system. In addition, it has been analytically bounded.

The bounds depend explicitly on the reference value ($v_{ref}$). As it can be seen, the error increases when reference decreases. These results are coherent with other numerical studies.

References


Convergent piecewise affine systems: analysis and design: continuous case

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Abstract. In this paper convergence properties for piecewise affine (PWA) systems are studied. The notions of exponential, uniform and input-to-state convergence are introduced and studied. For PWA systems with continuous right-hand sides it is shown that the existence of a common quadratic Lyapunov function for the linear parts of the system dynamics in every mode is sufficient for the exponential and input-to-state convergence of the system. For a class of PWA control systems we design (output) feedback controllers that make the closed-loop system input-to-state convergent. The conditions for such controller design are formulated in terms of LMIs. The obtained results can be used for designing observers and (output-feedback) tracking controllers for PWA systems.

1 Introduction

In many control problems it is required that controllers are designed in such a way that all solutions of the corresponding closed-loop system “forget” their initial conditions. Actually, this is one of the main tasks of a feedback to eliminate dependency of solutions on initial conditions. In this case, all solutions converge to some steady-state solution which is determined only by the input of the closed-loop system. This convergence property of a system plays an important role in many (nonlinear) control problems including tracking, synchronization, observer design, the output regulation problem and performance analysis of nonlinear systems see e.g. [2–6] and references therein.

The property of convergence was formalized in the notion of convergent systems and studied first for periodically excited systems in [7] and then for systems with arbitrary excitations in [8], see also [9]. For systems in Lur’e form convergence was investigated in [10]. Similar properties have been studied in [11, 12]. Among recent papers one should mention the works [13–15], in which the authors studied convergence-like properties of dynamical systems using various formalizations, definitions and techniques.

In this paper we study the convergence properties for the class of piecewise affine (PWA) systems. This class of systems attracted a lot of attention over the last years, see e.g. [16], [17] and references therein. This class includes mechanical systems with piecewise linear restoring characteristics, systems with friction, electrical circuits with diodes and other switching characteristics and control systems with switching controllers. In this paper we present conditions for convergence of PWA systems with continuous right-hand sides. The case of PWA systems with discontinuous right-hand sides is considered in the second part of the paper [18]. Most of the known checkable conditions for convergence (or convergence-type properties like incremental stability, contraction, [8, 14, 15]) rely on linearization of the system and therefore they are not applicable to PWA systems, which are non-smooth systems. This fact indicates the novelty of the presented results. Moreover, based on the obtained conditions, we present a new result on observer design for PWA systems and a result on designing output-feedback controllers for PWA systems that make the corresponding closed-loop system convergent.

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The paper is organized as follows. In Section 2 definitions of (uniformly, exponentially, input-
to-state) convergent systems are given and some basic (interconnection) properties of convergent
systems are presented. Sufficient conditions for the exponential and input-to-state convergence
properties for PWA systems with continuous right-hand sides are provided in Section 3. The
problem of designing a controller for a PWA system that makes the corresponding closed-loop
system convergent is addressed in Section 4. Section 5 contains conclusions.

2 Convergent systems

In this section we give definitions of convergent systems. These definitions extend the definition
given in [8]. Consider the system

\[ \dot{x} = f(x, t), \]

where \( x \in \mathbb{R}^n \), \( t \in \mathbb{R} \) and \( f(x, t) \) is locally Lipschitz in \( x \) and piecewise continuous in \( t \).

**Definition 1.** System (1) is said to be

- convergent if there exists a solution \( \bar{x}(t) \) satisfying the following conditions
  1. \( \bar{x}(t) \) is defined and bounded for all \( t \in \mathbb{R} \),
  2. \( \bar{x}(t) \) is globally asymptotically stable.

- uniformly convergent if it is convergent and \( \bar{x}(t) \) is globally uniformly asymptotically stable.

- exponentially convergent if it is convergent and \( \bar{x}(t) \) is globally exponentially stable.

The solution \( \bar{x}(t) \) is called a limit solution. As follows from the definition of convergence, any
solution of a convergent system “forgets” its initial condition and converges to some limit solution
which is independent of the initial condition. In general, the limit solution \( \bar{x}(t) \) may be non-unique.
But for any two limit solutions \( \bar{x}_1(t) \) and \( \bar{x}_2(t) \) it holds that \( |\bar{x}_1(t) - \bar{x}_2(t)| \to 0 \) as \( t \to +\infty \). At
the same time, for uniformly convergent systems the limit solution is unique, as formulated below.

**Property 1 ([1, 5]).** If system (1) is uniformly convergent, then the limit solution \( \bar{x}(t) \) is the only
solution defined and bounded for all \( t \in \mathbb{R} \).

The convergence property is an extension of stability properties of asymptotically stable linear
time-invariant (LTI) systems. Recall that for a piecewise continuous vector-function \( \phi(t) \), which
is defined and bounded for \( t \in \mathbb{R} \), the system \( \dot{x} = Ax + \phi(t) \) with a Hurwitz matrix \( A \) has a
unique solution \( \bar{x}(t) \) which is defined and bounded on \( t \in (-\infty, +\infty) \). It is given by the formula

\[ \bar{x}(t) := \int_{-\infty}^{t} \exp(A(t - s))\phi(s)ds. \]

This solution is globally exponentially stable with the rate of convergence depending only on the matrix \( A \). Thus, an asymptotically stable LTI system excited by a bounded piecewise-continuous function \( \phi(t) \) is globally exponentially convergent.

In the scope of control problems, time dependency of the right-hand side of system (1) is
usually due to some input. This input may represent, for example, a disturbance or a feedforward
control signal. Below we will consider convergence properties for systems with inputs. So, instead
of systems of the form (1), we consider systems

\[ \dot{x} = f(x, w) \]

with state \( x \in \mathbb{R}^n \) and input \( w \in \mathbb{R}^m \). The function \( f(x, w) \) is locally Lipschitz in \( x \) and continuous
in \( w \). In the sequel we will consider the class \( \mathbb{PC}^m \) of piecewise continuous inputs \( w(t) : \mathbb{R} \to \mathbb{R}^m \)
which are bounded for all \( t \in \mathbb{R} \). Below we define the convergence property for systems with inputs.

**Definition 2.** System (2) is said to be (uniformly, exponentially) convergent if it is (uniformly, 
exponentially) convergent for every input \( w \in \mathbb{PC}^m \). In order to emphasize the dependency on the
input \( w(t) \), the limit solution is denoted by \( \bar{x}_{w}(t) \).

The next statement summarizes some properties of uniformly convergent systems excited by
periodic or constant inputs. These properties are natural for linear systems, whereas for nonlinear
systems they, in general, do not hold.
Property 2 (\cite{8}). Suppose system (2) with a given input \(w(t)\) is uniformly convergent. If the input \(w(t)\) is constant, the corresponding limit solution \(\dot{x}_w(t)\) is also constant; if the input \(w(t)\) is periodic with period \(T\), then the corresponding limit solution \(\dot{x}_w(t)\) is also periodic with the same period \(T\).

The next definition extends the uniform convergence property to the input-to-state stability framework.

Definition 3. System (2) is said to be input-to-state convergent (ISC) if it is globally uniformly convergent and for every input \(w \in \mathbb{PC}_m\) system (2) is ISS with respect to the limit solution \(\dot{x}_w(t)\), i.e., there exist a \(KL\)-function \(\beta(r, s)\) and a \(K_\infty\)-function \(\gamma(r)\) such that any solution \(x(t)\) of system (2) corresponding to some input \(\hat{w}(t) := w(t) + \Delta w(t)\) satisfies

\[
|\dot{x}(t) - \dot{x}_w(t)| \leq \beta(|x(t_0) - \dot{x}_w(t_0)|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} |\Delta w(\tau)|).
\]

In general, the functions \(\beta(r, s)\) and \(\gamma(r)\) may depend on the particular input \(w(t)\). If \(\beta(r, s)\) and \(\gamma(r)\) are independent of the input \(w(t)\), then such system is called uniformly input-to-state convergent.

Similar to the conventional ISS property, the property of input-to-state convergence is especially useful for studying convergence properties of interconnected systems. One can easily show that parallel connection of (exponentially, uniformly, input-to-state) convergent systems is again an (exponentially, uniformly, input-to-state) convergent system. Series connection of two input-to-state convergent systems is an input-to-state convergent system, as summarized in the next property.

Property 3 (\cite{1, 5}). Consider the system

\[
\begin{cases}
\dot{x} = f(x, y, w), & x \in \mathbb{R}^n \\
\dot{y} = g(y, w), & y \in \mathbb{R}^q.
\end{cases}
\]

Suppose the \(x\)-subsystem with \((y, w)\) as inputs is input-to-state convergent and the \(y\)-subsystem with \(w\) as input is input-to-state convergent. Then system (4) is input-to-state convergent.

The next property deals with bidirectionally interconnected input-to-state convergent systems.

Property 4 (\cite{1, 5}). Consider the system

\[
\begin{cases}
\dot{x} = f(x, y, w), & x \in \mathbb{R}^n \\
\dot{y} = g(x, y, w), & y \in \mathbb{R}^q.
\end{cases}
\]

Suppose the \(x\)-subsystem with \((y, w)\) as inputs is input-to-state convergent. Assume that there exists a class \(KL\) function \(\beta_y(r, s)\) such that for any input \((x, w) \in \mathbb{PC}_{n+m}\) any solution of the \(y\)-subsystem satisfies

\[
|\dot{y}(t)| \leq \beta_y(\|y(t_0)\|, t - t_0).
\]

Then the interconnected system (5) is input-to-state convergent.

Remark. Property 4 can be used for establishing the separation principle for input-to-state convergent systems as it will be done in Section 4. In that context system (5) represents a system in closed loop with a state-feedback controller and an observer generating state estimates for this controller. The \(y\)-subsystem corresponds to the observer error dynamics.

Notice that the (uniform) convergence and the input-to-state convergence properties are invariant under smooth coordinate transformations, since all the ingredients in the definitions of these properties (see Definitions 1-3) are invariant under smooth coordinate transformations.
3 Convergent piecewise affine systems

In the previous sections we presented the definitions and basic properties of convergent systems. The next question to be addressed is: how to check whether a system exhibits these convergence properties? For smooth systems this question has been answered in [8], whereas for non-smooth systems this question has been answered only for systems in Lur’e form with one (non-smooth) scalar nonlinearity, see [10]. Piecewise affine systems constitute an important class of non-smooth systems. In this section we provide sufficient conditions for convergence of piecewise-affine systems with continuous right-hand sides.

Consider the state space $\mathbb{R}^n$ divided into polyhedral cells $\Lambda_i$, $i = 1, \ldots, l$, by hyperplanes given by equations of the form $H^T_j z + h_j = 0$, for some $H_j \in \mathbb{R}^n$ and $h_j \in \mathbb{R}$, $j = 1, \ldots, k$. We will consider piecewise-affine systems of the form

$$\dot{x} = A_i x + b_i + Dw, \quad \text{for} \ x \in \Lambda_i, \ i = 1, \ldots, l. \tag{6}$$

Here $A_i \in \mathbb{R}^{n \times n}$ and $b_i \in \mathbb{R}^n$, $i = 1, \ldots, l$, are constant matrices and vectors, respectively. The vector $x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^m$ is the input. The hyperplanes $H^T_j x + h_j = 0$, $j = 1, \ldots, k$, are the switching surfaces. In the sequel we will deal with piecewise affine systems which have continuous right-hand sides. This continuity requirement on the right-hand side of system (6) can be characterized by the following simple algebraic lemma. Its proof can be found, for example, in [5].

**Lemma 1.** Consider system (6). The right-hand side of system (6) is continuous iff the following condition is satisfied: for any two cells $\Lambda_i$ and $\Lambda_j$ having a common boundary $H^T z + h = 0$ the corresponding matrices $A_i$ and $A_j$ and the vectors $b_i$ and $b_j$ satisfy the equalities

$$G_H H^T = A_i - A_j$$

$$G_H h = b_i - b_j, \tag{7}$$

for some vector $G_H \in \mathbb{R}^n$.

The following theorem establishes sufficient conditions for the exponential and input-to-state convergence of system (6).

**Theorem 1.** Consider system (6). Suppose the right-hand side of (6) is continuous and there exists a positive definite matrix $P = P^T > 0$ such that

$$PA_i + A_i^T P < 0, \quad i = 1, \ldots, l. \tag{8}$$

Then system (2) is exponentially convergent and input-to-state convergent.

Before giving the proof of this theorem, we formulate and prove an important technical lemma, which will be used in the proof of the theorem and in further analysis in Section 4. Denote the right-hand side of (6) $f(x, w)$.

**Lemma 2.** Under the conditions of Theorem 1 it holds that

$$(x_1 - x_2)^T P (f(x_1, w) - f(x_2, w)) \leq -\alpha (x_1 - x_2)^T P (x_1 - x_2). \tag{9}$$

for all $x_1, x_2 \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, for some $\alpha > 0$ and for the matrix $P$ satisfying (8).

**Proof.** Since $P$ satisfies LMI (8), there exists a constant $\alpha > 0$ such that

$$PA_i + A_i^T P \leq -2\alpha P, \quad i = 1, \ldots, l. \tag{10}$$

Let us show that this $\alpha$ is the constant for which inequality (9) holds for all $x_1, x_2 \in \mathbb{R}^n$ and all $w \in \mathbb{R}^m$. We will show this in two steps. First, consider the case when both $x_1$ and $x_2$ belong to
the same cell $A_i$ with the dynamics $\dot{x} = A_i x + b_i + Dw$. Then, $f(x_1, w) = A_i x_1 + b_i + Dw$ and $f(x_2, w) = A_i x_2 + b_i + Dw$. Therefore,

\[
(x_1 - x_2)^T P(f(x_1, w) - f(x_2, w)) \\
= (x_1 - x_2)^T P(A_i x_1 - A_i x_2) \\
= \frac{1}{2} (x_1 - x_2)^T (PA_i + A_i^T P)(x_1 - x_2) \\
\leq -\alpha (x_1 - x_2)^T P(x_1 - x_2).
\]

Thus, inequality (9) holds for any pair of points $x_1$ and $x_2$ lying in the same cell $A_i$.

Next, we consider the case of arbitrary $x_1$ and $x_2$. Consider the line segment $[x_1, x_2]$ connecting these two points. Denote $y_1 := x_1$, $y_p := x_2$ and $y_i$, $i = 2, \ldots, p - 1$, the points of intersection of the line segment $[x_1, x_2]$ with the switching surfaces such that any pair of points $y_i$, $y_i + 1$ belongs to the same cell $A_j$ (including its borders), $y_i \neq y_{i+1}$, $i = 1, \ldots, p - 1$, and the sequence $y_1, y_2, \ldots, y_p$ is ordered, see Fig. 1. Denote $e := (x_1 - x_2)/|x_1 - x_2|p$, where $|x|_p := \sqrt{x^T P x}$. Since all points $y_i$, $i = 1, \ldots, p$, lie on the same line segment $[x_1, x_2]$ and they are ordered, then

\[
e = \frac{y_i - y_{i+1}}{|y_i - y_{i+1}|_p}, \quad i = 1, \ldots, p - 1.
\]

Taking this fact into account, we obtain

\[
(x_1 - x_2)^T P(f(x_1, w) - f(x_2, w)) \\
= |x_1 - x_2|p \sum_{i=1}^{p-1} e^T P(f(y_i, w) - f(y_{i+1}, w)) \\
= |x_1 - x_2|p \times \\
\times \sum_{i=1}^{p-1} \frac{(y_i - y_{i+1})^T P(f(y_i, w) - f(y_{i+1}, w))}{|y_i - y_{i+1}|_p}.
\]

Since each pair of points $y_i$ and $y_{i+1}$, $i = 1, \ldots, p - 1$, belongs to a cell with the same dynamics, from the first step of the proof we obtain

\[
(y_i - y_{i+1})^T P(f(y_i, w) - f(y_{i+1}, w)) \\
\leq -\alpha (y_i - y_{i+1})^T P(y_i - y_{i+1}).
\]

![Fig. 1: The line segment $(x_1, x_2)$ intersects the switching planes in the points $y_1, \ldots, y_4$.](image-url)
Substituting this inequality into (13), implies
\[
(x_1 - x_2)^T P (f(x_1, w) - f(x_2, w)) \leq -\alpha |x_1 - x_2|^p \sum_{i=1}^{p-1} |y_i - y_{i+1}|^p.
\] (15)

Since all points \(y_i, i = 1, \ldots, p\), lie on the same line segment \([x_1, x_2]\) and they are ordered,
\[
\sum_{i=1}^{p-1} |y_i - y_{i+1}|^p = |y_1 - y_p|^p = |x_1 - x_2|^p.
\] (16)

This fact together with (15) implies (9). Due to the arbitrary choice of \(x_1, x_2\) and \(w\) we obtain that (9) holds for all \(w \in \mathbb{R}^m\) and all \(x_1, x_2 \in \mathbb{R}^n\). This completes the proof of this lemma. \(\square\)

Now we can prove Theorem 1.

**Proof.** Given the result of Lemma 2, the proof of exponential convergence repeats the proof from [9, 19]. We only need to show that system (2) is input-to-state convergent. Consider some input \(w(t)\) and the corresponding limit solution \(\hat{x}_w(t)\). Let \(x(t)\) be a solution of system (2) corresponding to some input \(\hat{w}(t)\). Denote \(\Delta x := x - \hat{x}_w(t)\) and \(\Delta w := \hat{w} - w(t)\). Then \(\Delta x\) satisfies the equation
\[
\Delta \dot{x} = f(\hat{x}_w(t) + \Delta x, w(t) + \Delta w) - f(\hat{x}_w(t), w(t)).
\] (17)

We will show that system (17) with \(\Delta w\) as input is ISS. Due to the arbitrary choice of \(w(t)\), this fact implies that system (2) is input-to-state convergent.

Consider the function \(V(\Delta x) = \frac{1}{2}(\Delta x)^T P \Delta x\). Its derivative along solutions of system (17) satisfies \(\dot{V} = \Delta x^T P \{f(\hat{x}_w(t) + \Delta x, w(t) + \Delta w(t)) - f(\hat{x}_w(t), w(t))\}\)
\[
= \Delta x^T P \{f(\hat{x}_w(t) + \Delta x, w(t) + \Delta w(t))
- f(\hat{x}_w(t), w(t) + \Delta w(t))\}
+ \Delta x^T P \{f(\hat{x}_w(t), w(t) + \Delta w(t)) - f(\hat{x}_w(t), w(t))\}.
\] (18)

Applying Lemma 2 to the first component in (18), we obtain
\[
\Delta x^T P \{f(\hat{x}_w(t) + \Delta x, w(t) + \Delta w(t))
- f(\hat{x}_w(t), w(t) + \Delta w(t))\} \leq -\alpha |\Delta x|^2_P,
\] (19)
where \(|\Delta x|_P^2 := (\Delta x)^T P \Delta x\). Since \(f(x, w)\), the right-hand side of system (6), is linear in \(w\), the second component in formula (18) equals
\[
\Delta x^T P \{f(\hat{x}_w(t), w(t) + \Delta w(t)) - f(\hat{x}_w(t), w(t))\}
= \Delta x^T P D \Delta w.
\] (20)

Applying the Cauchy inequality to (20), we obtain
\[
|\Delta x^T P D \Delta w| \leq |\Delta x|_P D |\Delta w|_P \leq c |\Delta x|_P |\Delta w|,
\] (21)
where the constant \(c\) depends only on \(D\) and \(P\). After substituting this estimate together with estimates (19) and (20) in formula (18), we obtain
\[
\frac{dV}{dt} \leq -\alpha |\Delta x|^2_P + |\Delta x|_P c |\Delta w|.
\] (22)
From this formula we obtain
\[ \frac{dV}{dt} \leq -\frac{\alpha}{2} |\Delta x|^2, \quad \forall |\Delta x|^p \geq \frac{2}{\alpha} |\Delta w|. \] (23)

By the Lyapunov characterization of the ISS property (see e.g. [20], Theorem 5.2), we obtain that system (17) is input-to-state stable. This completes the proof of the theorem. \(\square\)

Theorem 1 not only allows to check the input-to-state convergence property for a given system, but also serves as a useful tool in designing controllers that make the corresponding closed-loop system convergent. This controller design problem is considered in Section 4.

From the result of Theorem 1 one may conjecture that for a PWA system with a discontinuous right-hand side, the existence of a common quadratic Lyapunov function for the linear parts of the dynamics in all modes is also sufficient for convergence. Yet, this conjecture is not true, as follows from a counterexample presented in the second part of this paper [18]. In [18] we study convergent PWA systems with discontinuous right-hand sides.

4 Controller design for convergent systems

The convergence property is desirable in many control problems because the steady-state dynamics of a convergent system are independent of the initial conditions. In this section we address the problem of how to achieve the convergence property in a piecewise affine control system by means of feedback. Consider the following PWA system

\[
\begin{align*}
\dot{x} &= A_i x + b_i + B u + D w, \quad x \in A_i, \ i = 1, \ldots, l. \\
y &= C x + E v
\end{align*}
\]

(24)

with state \(x \in \mathbb{R}^n\), control \(u \in \mathbb{R}^k\), external input \(w \in \mathbb{R}^m\) and output \(y \in \mathbb{R}^p\). Here \(A_i, b_i, \ i = 1, \ldots, l, B, D, C \) and \(E\) are constant matrices of the appropriate dimensions. As in the previous section, \(A_i\) are polyhedral cells with disjoint interior which together constitute the state space \(\mathbb{R}^n\). In this setting the input \(u\) corresponds to the feedback part of the controller. The external input \(w\) includes external time-dependent inputs such as disturbances and feedforward control signals.

Once the convergence property is achieved by a proper choice of feedback, the feedforward control signals can be used in order to shape the steady-state dynamics of the closed-loop system (see e.g. [5, 21]). We will focus on the problem of finding a feedback that makes the closed-loop system input-to-state convergent and will not address the problem of shaping the steady-state dynamics by means of a feedforward controller.

The following lemma provides conditions under which there exists a state feedback rendering the corresponding closed-loop system input-to-state convergent.

Lemma 3. Consider the system (24). Suppose the right-hand side of (24) is continuous and the LMI

\[
\begin{align*}
P_c &= P^T_c > 0, \\
A_i P_c + P_c A_i^T + B Y + Y^T B^T &< 0, \quad i = 1, \ldots, l,
\end{align*}
\]

(25)
is feasible. Then the system (24) in closed-loop with the controller \(u = K(x+v)\) with \(K := Y P_c^{-1}\) and \((v, w)\) as inputs is input-to-state convergent.

Proof. The closed-loop system has the form

\[
\dot{x} = (A_i + BK)x + b_i + BK v + Dw, \quad x \in A_i,
\]

(26)
i = 1, \ldots, l. Since the right-hand side of system (24) is continuous, then the right-hand side of the closed-loop system (26) is also continuous. Since the LMI (25) is feasible, for the matrix \(K := Y P_c^{-1}\) it holds that

\[
P_c^{-1}(A_i + BK) + (A_i + BK) P_c^{-1} < 0, \quad i = 1, \ldots, l.
\]

Therefore, the closed-loop system (26) satisfies the conditions of Theorem 1 with the matrix \(P := P_c^{-1} > 0\). Hence, system (26) with \((v, w)\) as inputs is input-to-state convergent. \(\square\)
The next lemma shows how to design an observer based on the convergence property.

**Lemma 4.** Consider system (24). Suppose the right-hand side of (24) is continuous and the LMI
\[
P_o = P^T_o > 0, \\
P_o A_i + A_i^T P_o + \mathcal{X} C + C^T \mathcal{X}^T < 0, \quad i = 1, \ldots, l,
\]
is feasible. Then the system
\[
\dot{x} = A_i x + b_i + B u + D w + L (\hat{y} - y), \quad \hat{x} \in A_i, \\
\hat{y} = C \hat{x} + E w, \quad i = 1, \ldots, l,
\]
with \( L := P_o^{-1} \mathcal{X} \), is an observer for system (24) with globally exponentially stable error dynamics. Moreover, the observer error dynamics
\[
\Delta \dot{x} = g(x + \Delta x, u, w) - g(x, u, w),
\]
where \( g(x, u, w) := A_i x + b_i + B u + D w + L (C x + E w) \) for \( x \in A_i, \ i = 1, \ldots, l \), is such that for any bounded \( x(t) \) and \( w(t) \) and any feedback \( u = U(\Delta x, t) \) all solutions of system (29) satisfy
\[
|\Delta x(t)| \leq c e^{-a(t-t_0)}|\Delta x(t_0)|,
\]
where the numbers \( c > 0 \) and \( a > 0 \) are independent of \( x(t), w(t) \) and \( u = U(\Delta x, t) \).

**Proof.** Let us first prove the second part of the lemma. Consider the function \( g(x, u, w) \). After unifying the terms containing \( x \), we obtain \( g(x, u, w) := (A_i + LC)x + b_i + Bu + (D + LE)w \) for \( x \in A_i, \ i = 1, \ldots, l \). Since the right-hand side of system (24) is continuous, then \( g(x, u, w) \) is also a continuous piecewise-affine function. Moreover, since the LMI (27) is feasible, for \( P := P_o \) and \( L := P_o^{-1} \mathcal{X} \) it holds that
\[
P(A_i + LC) + (A_i + LC)^T P < 0, \quad i = 1, \ldots, l.
\]
Applying Lemma 2 to the function \( g(x, u, w) \), we obtain
\[
\Delta x^T P(g(x + \Delta x, u, w) - g(x, u, w)) \leq -a \Delta x^T P \Delta x
\]
for all \( x, \Delta x, u \) and \( w \) and some constant \( a > 0 \) independent of \( x, \Delta x, u \) and \( w \). Consider the function \( V(\Delta x) := 1/2 \Delta x^T P \Delta x \). The derivative of this function along solutions of system (29) satisfies
\[
\frac{dV}{dt} = \Delta x^T P(g(x + \Delta x, u, w) - g(x, u, w)) \leq -2a V(\Delta x).
\]
This inequality, in turn, implies that there exists \( c > 0 \) depending only on the matrix \( P \) such that if \( x(t) \) and \( w(t) \) are defined for all \( t \geq t_0 \) then the solution \( \Delta x(t) \) is also defined for all \( t \geq t_0 \) and satisfies (30). It remains to show that system (28) is an observer for system (24). Denote \( \Delta x := \hat{x} - x(t) \). Since \( x(t) \) is a solution of system (24), \( \Delta x(t) \) satisfies equation (29). By the previous analysis, we obtain that \( \Delta x(t) \) satisfies (30). Therefore, the observation error \( \Delta x \) exponentially tends to zero.

Lemmas 3 and 4 show how to design a state feedback controller that makes the closed-loop system input-to-state convergent and how to design an observer for this system with an exponentially stable error dynamics. In fact, for such controllers and observers one can use the separation principle in order to design an output feedback controller that makes the closed-loop system input-to-state convergent. This statement follows from the next theorem.

**Theorem 2.** Consider the system (24). Suppose the LMIs (25) and (27) are feasible. Denote \( K := \mathcal{P}_o^{-1} \) and \( L := \mathcal{P}_o^{-1} \mathcal{X} \). Then system (24) in closed loop with the controller
\[
\dot{x} = A_i x + b_i + B u + D w + L (\hat{y} - y), \quad \hat{x} \in A_i, \\
u = K \hat{x}, \quad i = 1, \ldots, l,
\]
with \( w \) as an input is input-to-state convergent.
Proof. Denote $\Delta x := \hat{x} - x$. Then in the new coordinates $(x, \Delta x)$ the equations of the closed-loop system are

$$
\begin{align*}
\dot{x} &= (A_i + BK)x + b_i + BK\Delta x + Dw, \quad x \in A_i, \\
\Delta \dot{x} &= g(x + \Delta x, u, w) - g(x, u, w), \\
u &= K(x + \Delta x).
\end{align*}
$$

By the choice of $K$, system (33) with $(\Delta x, w)$ as inputs is input-to-state convergent (see Lemma 3). By the choice of the observer gain $L$, for any inputs $x(t)$, $w(t)$ and for the feedback $u = K(x(t) + \Delta x)$, any solution of system (34), (35) satisfies

$$
|\Delta x(t)| \leq ce^{-a(t-t_0)}|\Delta x(t_0)|,
$$

where the numbers $c > 0$ and $a > 0$ are independent of $x(t)$ and $w(t)$ (see Lemma 4). Applying Property 4, we obtain that the closed-loop system (33)-(35) is input-to-state convergent.

5 Conclusions

In this paper we have studied convergence properties for piecewise affine (PWA) systems. We have introduced the notions of exponential, uniform and input-to-state convergence and studied their basic (interconnection) properties. For PWA systems with continuous right-hand sides it has been shown that the existence of a common quadratic Lyapunov function for the linear parts of the system dynamics in every mode is sufficient for the exponential and input-to-state convergence of the system. Based on this result, for a class of PWA control systems we have designed observers and (output) feedback controllers that make the closed-loop system input-to-state convergent. The conditions for such observer and controller design are formulated in terms of LMIs. The obtained results can be used for designing observers and (output-feedback) tracking controllers for PWA systems.

References

Part III: Control of Bifurcations in Non-smooth Systems

Part III of this report consists of Chapter 12, in which a control strategy for the suppression of limit cycles is proposed using switching control strategy based on non-smooth bifurcation theory (Authors: Fabiola Angulo, Mario di Bernardo, Enric Fossas, Gerard Olivar).
Feedback Control of Limit Cycles: a switching control strategy based on nonsmooth bifurcation theory

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Abstract. In this paper we present a method to control limit cycles in smooth planar systems making use of the theory of nonsmooth bifurcations. By designing an appropriate switching controller, the occurrence of a corner-collision bifurcation is induced on the system and the amplitude and stability properties of the target limit cycle are controlled. The technique is illustrated through a representative example.

1 Introduction

One of the most common sources of instability in applications is the onset of unwanted oscillatory behavior. For instance, recent progress in nonlinear dynamics has shown that so-called Hopf bifurcations can lead to the onset of such oscillatory motion in a variety of different systems. Undesirable stable oscillatory motion has been observed in aircraft systems [1], mechanical devices, control systems and electrical circuits [2], [3]. It has been shown that limit cycles associated to these oscillations are usually locally stable and can, at times, coexist with the desired steady-state behavior.

Classical control techniques can be used to suppress these unwanted oscillations by means of feedback control actions aimed at changing the system dynamics over the entire region of interest [4]. Thus, in the case where two or more different attractors exist, the controller objective is that of eliminating them, taming the system dynamics onto a desired stable equilibrium point. Many authors (see for example [5], [6] and the references therein) have studied the problem of controlling bifurcations within a smooth feedback framework. Examples include the method based on manifold reduction presented in [6] and the use of smooth nonlinear control laws discussed in [7] to tame a limit cycle occurring in a flutter problem.

Recently, it has been proposed that results from bifurcation theory can be used to synthesize ad hoc control strategies for nonlinear systems [8], [9], [10]. The main aim of this paper is to present a novel approach to control limit cycles in planar dynamical systems. Namely, an appropriate switching controller is synthesised by using results from the theory of bifurcation in nonsmooth system. Rather than aiming at changing the entire dynamics of the system of interest, we shall seek to design a controller acting in a local neighborhood of the target limit cycle. (For a review of the theory of nonsmooth bifurcations in dynamical systems we refer the reader to [11], [12]). The analysis is based on the study of the Poincaré map associated to the limit cycle and relies on the theory of corner-collision bifurcations recently presented in [13]. Namely we will show that it is possible to control the amplitude of an oscillatory motion, or even suppress it (if unwanted)

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by means of a switching controller acting in a relatively small neighborhood of the limit cycle. By appropriately selecting the switching manifolds and the control action, it is possible to move the fixed point corresponding to the target limit cycle on the Poincaré map and hence control the cycle itself. In so doing, the control effort is low as control is only activated in a small neighborhood of the cycle. To synthesize the controller, we will proceed in two separate stages. A control law, based on cancellation, is used as a first step towards the synthesis of a controller which instead will not rely on cancellation. To select the features of the limit cycles in a controlled way, we will use the strategy to classify so-called border-collisions (or C-bifurcations) of fixed points of nonsmooth maps recently presented in [14].

Note that we are not designing a controller to change the bifurcation properties of the system, but rather choosing a local control strategy to place the system close to a known bifurcation phenomenon. We will then use our analytical understanding of such phenomenon to achieve the control goal, i.e. suppress or modify the limit cycle of interest. Namely, the controller applied to the system flow will be based on a switching action which is designed by taking into account a nearby nonsmooth bifurcation of the cycle under control and then influence the properties of the associated Poincaré map in order to change its properties according to the classification strategy presented in [14]. In so doing, we will do explicit use of the technique to derive the approximate Poincaré map of the system analytically during the control design stage. It is worth to note that although the method also works to even suppress the limit cycle and obtain an equilibrium point (nonlocal action) our analytical understanding of the bifurcation phenomenon can only explain local changes (the cycle amplitude variation, for example).

The rest of the paper is outlined as follows. In section II the proposed method to control limit cycles is presented. In section III a brief description of corner collision is made. In section IV controller synthesis is explained. In section V an example of application is shown. In the example the system presents a limit cycle and the technique was successful. Finally the conclusions are presented in section VI.

2 Controlling limit cycles through corner collision

Let us consider a general system of the form:

$$\dot{x} = F(x) + B(x)u$$

where $F := (f_1, f_2, \ldots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a sufficiently smooth and differentiable vector field over the region of interest, say $D \subseteq \mathbb{R}^n$. For the sake of clarity, we assume that $B(x)$ is the identity (note that this assumption is not necessary for the strategy presented here to be valid). Also, we suppose that, at some parameter value, the system exhibits a stable limit cycle of period $T$, i.e. $\dot{x}(t) = x(t + T)$. We want to design a feedback controller to suppress such periodic oscillations or, alternatively, to select its characteristics (periodicity, amplitude etc.). Note that while the control action will be applied on the continuous-time system, the aim is to change the properties of its Poincaré map.

For this purpose our aim is to synthesise a controller based on the theory of nonsmooth bifurcations. Namely, we will select a switching feedback controller $u$ in order to vary the main features of the local Poincaré map associated to the limit cycle of interest. This in turn will allow the variation of the properties of such local map and hence the local control of the cycle.

As it will be seen in Sec. 3, locally to a corner collision bifurcation point the Poincaré map can be estimated analytically as a piecewise-linear one. Thus, the main idea is for the controller to put the system close to a corner collision bifurcation event of the cycle of interest with an appropriately defined switching strategy in state-space. In so doing, the controller will switch from one configuration to the other whenever the system trajectories cross the boundaries defining a corner-like switching manifold in phase space. By varying the functional form of the control signal, we will change the properties of the local map and hence the main features of the fixed point corresponding to the cycle of interest.

In so doing we need to:
1. Choose an appropriate Poincaré section and define the Poincaré map for the system under investigation;
2. Synthesize a feedback controller to change the properties of this map, i.e. choose (i) the corner region in phase space and (ii) the controller functional form;
3. Validate the effectiveness of the controller through numerical simulations.

First, we give a brief overview of the theory of corner-collision bifurcations.

3 Corner–Collision: a brief description

In many control systems and electronic switching devices, switching conditions may be governed by several overlapping inequalities. A generic feature of such examples is that the discontinuity boundary has a corner-type singularity formed by the intersection between smooth codimension one surfaces \( \Gamma_1 := \{ x \in \mathbb{R}^n : H_1(x) = 0 \} \) and \( \Gamma_2 := \{ x \in \mathbb{R}^n : H_2(x) = 0 \} \) at a non-zero angle.

The locus of corners \( C \) will in general be a \((n-2)\)-dimensional subset of the phase space \( \mathbb{R}^n \). The passage of a trajectory through a point in \( c \in C \) is a non-smooth bifurcation event because, in a neighborhood of the corner, there are distinct trajectories that do not behave similarly with respect to regions \( S_1 \) and \( S_2 \) on either side of \( \Gamma \) (the border of regions \( S_1 \) and \( S_2 \)), which is a subset of \( \Gamma_1 \cup \Gamma_2 \). If such a corner-colliding trajectory is part of an isolated periodic orbit \( p(t) \), we shall refer to this as a corner-collision grazing bifurcation, or ‘corner collision’ for short (this is the case, for example, of DC/DC buck converters [2]). Figure 1 illustrates the geometry that we are considering.

Here, there are two different regions, namely \( S_1 \) and \( S_2 \). In each zone the system presents a different dynamical behavior described by different vector fields. Whenever the boundary between the two regions is crossed (i.e. whenever the trajectory crosses the corner), the system vector field loses continuity.

We assume that the system of interest is planar and can be described as:

\[
\dot{x} = \begin{cases} 
G(x), & \text{if } x \in S_1, \\
F(x), & \text{if } x \in S_2. 
\end{cases}
\] (2)

As some parameter is varied, a corner-collision can occur where a limit cycle hits the tip of the corner region. Further parameter variations can lead to several different scenarios. To classify the possible scenarios following a corner collision the key issue is to be able to construct the Poincaré normal form map of the cycle undergoing the bifurcation. Recently it was shown that a local map describing the dynamics of the system close to a corner-collision point can be derived by using the concept of discontinuity map [13].

Namely, suppose we want to construct a Poincaré map for the cycle of interest. Then, in the absence of the corner, the map would be defined by considering the system flow from some suitable Poincaré section \( \Sigma \) back to itself. The discontinuity map is a local map that describes the correction that needs to be made to trajectories that pass through region \( S_2 \) close to the corner in order to solve the global Poincaré map. It was shown that such map is locally piecewise linear so that a corner-collision bifurcation of the flow implies a border-collision of the associated fixed point of the map [15].

Next, we calculate the Poincaré section and the discontinuity map.

3.1 The Poincaré Map

As our analysis is concerned with a planar dynamical system, the Poincaré map is one-dimensional. In the following, we will determine the behavior of the fixed point associated to a limit cycle undergoing a corner-collision event. Initially, we calculate the Poincaré map, assuming that the evolution of the whole system in state space is through the flow \( \Phi_F \) associated to the vector...
Fig. 1: Scheme of a border collision bifurcation which destroys a limit cycle
Fig. 2: Scheme of the discontinuity map.

Field $F$. The point at which corner collision occurs will be identified as $x_c$ and corresponds to the intersection of $\Gamma_1$ and $\Gamma_2$. Without loss of generality, we assume that:

$$\Gamma_1 := \{ x \in \mathbb{R}^2 : H_1(x) := -x_2 = 0 \}$$

and that the flow $\Phi_F$ is transversal to $\Gamma_1$. A convenient subset $\Sigma_1 \subset \Gamma_1$ is therefore chosen as a suitable Poincaré section for the flow. We consider also $\Sigma_2 = \{ x \in \Gamma_2 : x_2 \geq 0 \} \subset \Gamma_2$. Clearly, the tip of the corner is located at the point $x_c = (b/m, 0)$.

Say $(x_0^1, 0)$ a fixed point of the map defined on $\Sigma_1$ associated to a limit cycle of the whole flow in $\mathbb{R}^2$. To construct the map from $\Sigma_1$ back to itself, we consider a perturbation of such point $(x_0^1 + \delta, 0)$ for a small $|\delta|$. Note that if $x_0^1 + \delta < b/m$ the limit cycle evolves entirely in region $S_2$ while, if $x_0^1 + \delta > b/m$ then the cycle penetrates the corner for some time.

Let’s first consider the case where the limit cycle lies entirely to the left of the corner (i.e. assume $\delta$ to be such that $x_0^1 + \delta < b/m$). Let $t_{\text{min}} > 0$ be the minimum time for which the system evolves from $(x_0^1 + \delta, 0)$ until it hits $\Sigma_1$ again. Then

$$H_1(\Phi_F((x_0^1 + \delta, 0), t_{\text{min}})) = 0$$

is fulfilled. To leading order, we can then write that the Poincaré map $\Pi_0$ can be written as:

$$\Pi_0 : \Sigma_1 \rightarrow \Sigma_1$$

$$(x_1, 0) \rightarrow (P x_1, 0)$$

where

$$P = \frac{\Phi_F((x_0^1 + \delta, 0), t_{\text{min}})}{x_1^0 + \delta} \quad (6)$$

Now we construct the Poincaré map when the cycle interacts with the corner, i.e. $x_0^1 + \delta > b/m$. We follow the same procedure first discussed in [13]. As it is schematically shown in Fig. 2, to obtain the Poincaré map in this case we need to compose $\Pi_0$ as derived above with the so-called discontinuity map (i.e. a map that makes the appropriate corrections to $\Pi_0$ in order to take into account the fact that the system trajectory is now crossing into region $S_1$).

To obtain such correction, from initial conditions on $\Sigma_1$ we first solve the equations given by $\dot{x} = G(x)$ until $\Sigma_2$ is reached, at a point $(x_f^1, x_f^2)$. This map from $\Sigma_1$ to $\Sigma_2$ will be $\Pi_G$. Then, from $(x_f^1, x_f^2)$ we solve the equations for the reverse flow $\Phi_F(x, -t)$ until we hit again $\Sigma_1$. In this
case the map from $\Sigma_2$ to $\Sigma_1$ will be denoted by $\Pi_{-F}$. The correction that needs to be made to $\Pi_0$, i.e. the discontinuity mapping, is then given by $\Pi_{-F} \circ \Pi_G$.

Finally, to obtain the Poincaré map from $\Sigma_1$ back to itself, in this case, we compose the three maps, i.e:

$$\Pi_0 \circ \Pi_{-F} \circ \Pi_G : \Sigma_1 \to \Sigma_1.$$  

Concretely, let $t_1 > 0$ and $t_2 > 0$ be the minimum times verifying

$$H_2(\Phi_G((x_1,0), t_1)) = 0$$

and

$$H_1(\Phi_{-F}((x_1^f, x_2^f), t_2)) = 0.$$  

Hence we have

$$\Pi_G : \Sigma_1 \to \Sigma_2$$

$$(x_1,0) \to (x_1^f, x_2^f) := \Phi_G((x_1,0), t_1)$$

and

$$\Pi_{-F} : \Sigma_2 \to \Sigma_1$$

$$(x_1^f, x_2^f) \to (x_1', 0) := \Phi_{-F}((x_1^f, x_2^f), t_2)$$

As shown in [13] and discussed below, it is possible to obtain an analytical estimate of the discontinuity mapping by considering a set of appropriate approximations.

### 3.2 The discontinuity map

Equipped with the definition of the map given above, we can now compute the discontinuity map analytically in order to obtain the analytical expression of the global Poincaré map. Following the methodology presented in [13], when the trajectory in $S_2$ collides with the corner zone we solve the equations through a first order approximation of the flow. Hence, if $g_1$ and $g_2$ are the components of the vector field in region $S_1$ we have

$$\Phi_G = x + G^0 t + \text{h.o.t}$$

with $x = (x_1,0)$, and $G^0 = (g_1^0, g_2^0)$ is $G$ evaluated at the corner point. The intersection of the flow with $\Sigma_2$ yields to

$$g_2^0 t - mx_1 - mg_1^0 t + b = 0$$

This implies that

$$t = \frac{mx_1 - b}{g_2^0 - mg_1^0}$$

With this time, the intersection point is

$$\begin{pmatrix} x_1^f \\ x_2^f \end{pmatrix} = \begin{pmatrix} x_1 + g_1^0 \frac{mx_1 - b}{g_2^0 - mg_1^0} \\ g_2^0 \frac{mx_1 - b}{g_2^0 - mg_1^0} \end{pmatrix}$$

Now we proceed to find the time that has to be spent for the original system in reverse time starting at the point $(x_1^f, x_2^f)$ until the surface $\Sigma_1$ is crossed. Doing an expansion to first order of the original flow, being $f_1^0$ and $f_2^0$ the components of the vector field $F$ at the corner point, we have

$$\Phi_F = x^f - F^0 t + \text{h.o.t}$$
Considering the second component only, we have

\[ x^f_2 - f^0_2 t = 0 \]

then, taking into account that

\[ x^f_2 = g^0_2 \frac{m x_1 - b}{g^0_2 - m g^0_1}, \]

the reverse time can be found as

\[ t = \frac{g^0_2}{f^0_2} \frac{m x_1 - b}{g^0_2 - m g^0_1}. \]  

Hence, the discontinuity map is given by

\[ x_1 \to x_1 + \frac{m x_1 - b}{g^0_2 - m g^0_1} \left( f^0_2 g^0_1 - f^0_1 g^0_2 \right) \]  

or equivalently as

\[ x_1 \to x_1 + \frac{m x_1 - b}{g^0_2 - m g^0_1} \left( f^0_2 \frac{g^0_1}{f^0_2} m x_1 - b \frac{f^0_2}{g^0_2} \right) \]

(13)

Note that the condition \( f^0_2 \neq 0 \) is always fulfilled with an appropriate choice of the Poincaré section \( \Sigma_1 \).

The derivation for the case of \( n \)-dimensional nonsmooth systems undergoing corner collisions can be found in [13].

4 Control Synthesis

According to the theory of corner-collisions, as confirmed by the derivations reported above, the Poincaré map of a limit cycle undergoing such a bifurcation is piecewise-linear and dependent on the vector fields inside and outside the corner. The corner is also supposed such that sliding on its
boundaries is not possible. As before, we label $S_1$ the region inside the corner while $S_2$ the region outside of it.

Without loss of generality, we select $u$ as the switching controller defined by:

$$
u = \begin{cases} 0 & \text{if } \mathbf{x} \in S_2 \\ \phi(\mathbf{x}, t) & \text{if } \mathbf{x} \in S_1 \end{cases}$$

(17)

with $S_1 \subseteq \mathbb{R}^2$ being the region (corner) limited by the manifolds defined by $H_1(\mathbf{x})=0$ and $H_2(\mathbf{x})=0$ as depicted in Fig. 1.

With this choice of $u$ the controlled system becomes:

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}(\mathbf{x}) & \text{if } \mathbf{x} \in S_2 \\ \mathbf{F}(\mathbf{x}) + \phi(\mathbf{x}, t) := \mathbf{G}(\mathbf{x}) & \text{otherwise} \end{cases}$$

(18)

The control effort can be calculated in analytical form. Since the RMS value for a periodic signal is given by the $L^2$-norm

$$f_{RMS} = \sqrt{\frac{1}{T} \int_0^T ||f(\tau)||^2 d\tau},$$

we can evaluate the RMS value for the control signal in stationary state once the amplitude of the limit cycle has been changed and the system has stabilized. Taking into account that the signal control acts during a time given by Eq. (10), then the RMS value is

$$u_{RMS} = \sqrt{\frac{1}{T} \int_0^t ||\phi(\mathbf{x}, \tau)||^2 d\tau}$$

(19)

where $t$ is calculated from Eq. (10). Since $\phi(\mathbf{x}, \tau)$ is independent of $\tau$, the expression can be simplified to

$$u_{RMS} = \sqrt{\frac{1}{T} (\phi_1(\mathbf{x})^2 + \phi_2(\mathbf{x})^2) \left( \frac{m x_1 - b}{b^2 - m g_1^0} \right)}$$

(20)

4.1 Step 1: Choosing the switching strategy

In order for the control to be effective we need to select the boundaries of regions $S_1$ and $S_2$, i.e. define the corner in phase space. According to the theory of corner-collision, the corner must be such that (i) sliding or Filippov solutions are not possible on its boundaries; (ii) it penetrates the cycle to be controlled as one of its defining parameters is changed so that at some critical parameter value the target limit cycle undergoes a corner-collision bifurcation.

In order to avoid sliding mode [13] we choose $H_1(\mathbf{x})$ and $H_2(\mathbf{x})$ so that:

$$\langle \nabla H_1(\mathbf{x}), \mathbf{F}(\mathbf{x}) \rangle < 0, \quad \langle \nabla H_1(\mathbf{x}), \mathbf{G}(\mathbf{x}) \rangle < 0,$$

$$\langle \nabla H_2(\mathbf{x}), \mathbf{F}(\mathbf{x}) \rangle > 0, \quad \langle \nabla H_2(\mathbf{x}), \mathbf{G}(\mathbf{x}) \rangle > 0.$$  

(21)

For simplicity we suppose counterclockwise direction of the vector field in neighborhood of corner collision point. We select $H_1$ and $H_2$ as in section II, with $m$ and $b$ being real constants. It is easy to see that it is possible to rescale the system coordinates so that a corner-collision occurs when $b = 0$ at the point $(0, 0)$. Therefore varying $b$ we can move the tip of the corner and hence yield a corner-collision bifurcation. Without loss of generality, we assume therefore that the corner collision bifurcation occurs at the point $(0, 0)$.

From (14), we then have that the local interaction of the cycle with the corner can be described by using the local mapping given by:

$$x_1 \rightarrow \begin{cases} x_1 & \text{if } x_1 \leq b/m \\ x_1 + \frac{m x_1 - b}{g_2^0 - m g_1^0} \left( \frac{f_2^0 g_1^0 - f_1^0 g_2^0}{f_2^0} \right) & \text{if } x_1 > b/m \end{cases}$$

(22)
where $f_0^0$ and $g_0^0$ are the components of the vector field evaluated at the corner collision point $(0,0)$. Note that in order for the mapping to be well defined we must have:

$$g_2^0 - mg_1^0 \neq 0 \quad (23)$$

and

$$f_2^0 \neq 0. \quad (24)$$

Also in order for the control to have an influence on the map properties we must have

$$f_2^0 g_1^0 - f_1^0 g_2^0 \neq 0. \quad (25)$$

These conditions represent an important sets of constraints on the control design. The quantities above can be computed analytically by means of any algebraic manipulation software (see Sec. V for a representative example of such computation). The next step is to choose the control signal $\phi(x,t)$. 

4.2 Choosing $\phi(x,t)$

**Perfect Knowledge of F(x)** In the most general case, a first choice for $\phi(x,t)$ in Eq. (18) can be expressed as

$$\phi(x,t) = -F(x) + K(x). \quad (26)$$

where $K(x)$ is a generic control function to be appropriately chosen. This means that the control signal contains two actions: the first compensates the nonlinear dynamic terms acting on the system; the second, instead, allow us to select the desired dynamics within the corner.

The main disadvantage of this controller is that it relies on perfect knowledge of the system vector field $F(x)$. This is a rather strong assumption that can hardly be satisfied in realistic applications. Thus, in Sec. 4.2, we will show that control can also be achieved successfully by removing the need for a perfect cancellation of the nonlinear dynamics.

In what follows, to illustrate the main idea, we detail the derivation of the controller, starting with the assumption that cancellation of nonlinear dynamic is possible. With this choice of the controller, the closed-loop system is given by:

$$\dot{x} = \begin{cases} F(x) & \text{if } x \in S_2 \\ K(x) := G(x) & \text{otherwise} \end{cases} \quad (27)$$

where $K(x)$ must be chosen so that conditions (23)-(25) are satisfied. Notice that this excludes the case where $K(x)$ is chosen as a purely proportional action. In fact, in this case, $K(0) = 0$ and therefore the control would not affect the map to leading order but introduces higher order effects which are beyond the scope of this paper and will be discussed elsewhere.

In general, from the theory of corner-collision and the expression of the local map (13), we see that $K(x)$ must be chosen so that $K(x_c) \neq 0$ if we want the control to cause first order variations of the map. Thus, we choose:

$$K(x) = c \quad (28)$$

where $c$ is an appropriately selected constant vector. According to (17) the controller is then defined as:

$$u = \begin{cases} 0 & \text{if } x \in S_2 \\ -F(x) + c & \text{if } x \in S_1 \end{cases} \quad (29)$$

In this case $u_{RMS} =$

$$\sqrt{\frac{1}{T} \left( f_1^0 f_2^0 - 2f_1^0 c_1 + f_0^0 f_2^0 + c_2^2 - f_2^0 c_2 \right) \left( \frac{m x_1 - b}{c_2 - m c_1} \right)} \quad (30)$$
Then, fixed $c_1$ and $c_2$ the amplitude of the limit cycle depends on the corner penetration and the point $x_1$ which corresponds to the first coordinate of the limit cycle in stationary state, when it enters $S_1$.

Notice that the control strategy given by (29) is indeed a feedback control strategy. Namely, even if the control action is determined by the addition of two state-independent constants $c_1$ and $c_2$, its switching is determined by state-dependent boundary conditions. As we will show in Sec. V, this results in a sequence of short carefully selected additions of constant perturbations of the vector field which steer the trajectory towards the desired goal.

In this case, each component of $u$ is such that in the region $S_1$ the system evolves according to the following equations

$$
\dot{x}_1 = c_1, \\
\dot{x}_2 = c_2,
$$

where $c_1$ and $c_2$ are two suitably chosen constants. With this choice of the vector field, the system inside the corner will follow the trajectory given by:

$$
x_1 = c_1 t + x_1(0) \\
x_2 = c_2 t + x_2(0)
$$

(32)

It is necessary to take into account that sliding needs to be avoided. According to conditions (21), we must then have

1. $c_1 > 0$, $c_2 > 0$, and $c_2/c_1 > m$ or
2. $c_1 < 0$, $c_2 > 0$.

If we construct the map for this planar system, using Eq. (16) we have

$$
x_1 \rightarrow \{ \\
P x_1 \\
P((1 + qm)x_1 - qb)
$$

if non-crossing

if crossing

(33)

where $q = \frac{f_0^b c_2 - f_0^a c_1}{f_2^b (mc_1 - c_2)}$

Here we observe the explicit dependence on $c_1$ and $c_2$ of the Poincaré map, confirming that by varying the control constants we can effectively change the properties of the map and hence those of the fixed point associated to the limit cycle of interest. Namely the fixed point of the map is

$$
x^*_1 = \frac{Pqb}{P(1 + qm) - 1}
$$

and so the amplitude of the limit cycle can be changed. An example will be discussed in Sec. V.

Now, let us assume that we have no a priori knowledge of the system vector field $F(x)$ for feedback.

**$F(x)$ unknown** We now remove the assumption that the vector field system $F(x)$ is perfectly known. In this case the control signal in a simplified form becomes:

$$
u = \begin{cases} \\
0 & \text{if } x \in S_2 \\
c & \text{if } x \in S_1
\end{cases}
$$

(34)

where $S_1$ is defined as above; and the closed-loop system takes the form:

$$
\dot{x} = \begin{cases} \\
F(x) & \text{if } x \in S_2 \\
F(x) + c := G(x) & \text{otherwise}
\end{cases}
$$

(35)

In this case the RMS value of control effort can be calculated as

$$
u_{RMS} = \sqrt{\frac{1}{T}(c_1^2 + c_2^2) \left( \frac{m x_1 - b}{f_2^b + c_2 - m f_1^b - mc_1} \right)}
$$

(36)
Now, it is possible to consider points near the origin and to proceed to analyze the vector field with the aim of guaranteeing a change in this field, according with the objectives. We consider the Poincaré map in the corner collision zone. In this case according to Eq. (16), we have

\[ x_1 \rightarrow \begin{cases} \frac{P}{P((1 + qm)x_1 - qb)} & \text{if non-crossing} \\ \end{cases} \text{if crossing} \]

(37)

where

\[ q = \frac{f_1^0(f_2^0 + c_2) - f_2^0(f_1^0 + c_1)}{f_2^0(m(f_1^0 + c_1) - (f_2^0 + c_2))} \]

Hence the main features of the Poincaré map are again explicitly dependent on \( c_1 \) and \( c_2 \) and therefore it is possible to change the behavior of the fixed point associated to the limit cycle by appropriately selecting the control action. The effectiveness of the control action presented above will be discussed using a representative example in what follows.

### 4.3 Changing the properties of the local map: Feigin’s strategy

The next step is now to choose the control constants \( c_1 \) and \( c_2 \) in order to vary the properties of the local map associated to the corner-collision of the cycle and hence change its features. The main idea is to use the fact that the corner-collision of the cycle implies a border-collision of the corresponding fixed point of the local map [15]. Thus, controlling the cycle can be achieved by changing the properties of the map in order to control the scenario following a border-collision.

To this aim, we use the strategy for the classification of border-collisions presented in [13]. Namely, according to such strategy, different scenarios are possible at a border-collision which can be classified using the slopes of the map on both sides of its discontinuity boundaries. In particular, if we say \( \alpha \) the slope of the map when non-crossing and \( \beta \) its slope when crossing, according to Feigin’s strategy we have the following three possible simplest scenarios (for a list of all possible scenarios we refer to [13]):

- **Persistence**: the bifurcating fixed point (limit cycle) crosses the boundary, changing continuously into a fixed point (limit cycle) lying on the other side of the boundary which may or may not have the same stability properties if \( \alpha < 1 \) and \( \beta < 1 \), or otherwise \( \alpha > 1 \) and \( \beta > 1 \).
- **Nonsmooth Saddle-Node**: a stable fixed point (limit cycle) collides with an unstable point (cycle), on the boundary and they both disappear if \( \alpha \) and \( \beta \) do not fulfill any of the conditions above.
- **Nonsmooth Period-Doubling**: a two-periodic point of the map \((x_1, x_2)\) characterised by having one iteration on each side of the boundary is involved in the bifurcation scenario. Note that according to a given set of conditions this might either result into a period-two orbit arising from the bifurcation point or being annihilated through it. (This case is obviously impossible in planar cases as flip bifurcations of cycles are not possible in \( \mathbb{R}^2 \).)

Thus by carefully choosing \( c_1 \) and \( c_2 \) in the controller equation, we can change the slope \( \beta \) of the map when crossing and in turns select the scenario following the border-collision. For example, selecting a value for \( \beta \) such that a nonsmooth saddle node occurs at the corner-collision, corresponds to selecting a controller that suppress locally the oscillatory motion in the system. On the other hand, selecting \( \beta \) so that persistence is observed correspond to changing the amplitude of the cycle etc.

We will now better illustrate the strategy by means of a representative example. In what follows we will indicate by \( A \) the fixed point associated to the original limit cycle we want to control and by \( B \) the cycle of the controlled system. We will use \( a \) and \( b \) to indicate unstable cycles and \( \rightarrow \) to indicate the occurrence of a corner-collision.
5 2-D representative example

Next, we show a simple example to illustrate the stages of the control design presented above. We choose the planar normal form of a Hopf bifurcation, described by:

\[
\begin{align*}
\dot{x}_1 &= \varepsilon (x_1 + 1) \left( a - \sqrt{(x_1 + 1)^2 + x_2^2} \right) - x_2 := f_1 \\
\dot{x}_2 &= \varepsilon x_2 \left( a - \sqrt{(x_1 + 1)^2 + x_2^2} \right) + x_1 + 1 := f_2
\end{align*}
\]

(38)

This system exhibits a limit cycle, which is a perfect circle of radius \(a\) centered in \((x_1^* , x_2^*) = (-1, 0)\). The system flow moves in anti-clockwise direction as time increases and hence crosses the line \(\{ x_2 = 0 \}\) upwards.

We choose the region \(S_1\) as the phase space set (corner) bounded by \(H_1(x) := -x_2 = 0\) and \(H_2(x) := x_2 - mx_1 + b = 0\), which satisfies the relations involving \(F(x)\) in Eq. (21). Note that when \(a = 1\) and \(b = 0\) a corner-collision occurs, as the limit cycle of radius 1 hits the tip of the corner defined above at the point \((0, 0)\). Varying the control parameter \(b\) will cause the corner to penetrate the limit cycle and hence change its properties.

In what follows, we suppose without loss of generality, that \(m = 1, a = 1, \varepsilon = 0.1\) and \(b = -0.1\).

5.1 \(F(x)\) perfectly known, \(\phi(x, t) = -F(x) + c\)

In this case, according to the development made in Section 4.2, and in order to fully satisfy Eq. (21) we need to choose \(c_1\) and \(c_2\) such that:

1. \(c_2 > 0, c_1 > 0\) and \(\frac{c_2}{c_1} > 1\) or
2. \(c_2 > 0\) and \(c_1 < 0\).

Taking into account that in the case under investigation \(f_1^0 = 0, f_2^0 = 1, g_1^0 = c_1\) and \(g_2^0 = c_2\), the equation describing the local piecewise-linear map, in a Poincaré section, is

\[
\begin{cases}
P x_1 & \text{if non-crossing} \\
P \left( x_1 - c_1 x_1 - b \right) & \text{if crossing}
\end{cases}
\]

(39)

with \(P = 0.5335\), which is computed with Eq. (6) and \(\delta\) small.

As discussed above, the fixed point of this map associated to the limit cycle undergoing a corner-collision as \(b\) is varied, undergoes a border-collision bifurcation. Hence, we shall seek to control the cycle by varying \(c_1\) and \(c_2\) in order to change the properties of the map and hence affect the nature of the border-collision of its fixed point.

Taking into account the slope of the map at the fixed point, and Feigin conditions [14], we can deduce some interesting results regarding the control design of the system.

The slope of the map at the fixed point (Eq. (39)) is given by \(\alpha = P = 0.5335 < 1\) on one side and

\[
\beta = P \frac{c_2}{c_2 - c_1}
\]

on the other.

Thus, using Feigin’s conditions we can distinguish two cases:

1. \(-1 < \beta < 1\)
2. \(\beta > 1\)

Case I: \(-1 < \beta < 1\) According to Feigin conditions, in this case we have persistence of the fixed point and hence of the cycle. As discussed above, to avoid sliding on the corner boundaries, we must choose either

1. \(c_1 > 0, c_2 > 0\), and \(c_2/c_1 > 1\) or
2. \( c_1 < 0, c_2 > 0, \)

Note that for both cases, \( \beta > 0. \) Thus to have persistence we want \( \beta < 1, \) i.e.

\[
0 < \beta < 1 \Leftrightarrow 0 < P \frac{c_2}{c_2 - c_1} < 1
\]

As \( \beta \) is certainly positive whatever the choice of \( c_1 \) and \( c_2, \) we want to have:

\[
P \frac{c_2}{c_2 - c_1} < 1 \Leftrightarrow P < \frac{c_2 - c_1}{c_2} = 1 - \frac{c_1}{c_2} \Leftrightarrow \frac{c_1}{c_2} < 1 - P = 0.4665. \tag{40}
\]

Thus, the limit cycle will persist with a different amplitude if \( c_1 \) and \( c_2 \) are chosen so that (40) is satisfied. For example, if \( c_1 = 0.2 \) and \( c_2 = 1, \) and thus \( \beta = 0.6669, \) these conditions are fulfilled, and the possibilities according to [14] are: (with Feigin notation)

- \( A \rightarrow b \) Stable fixed point to Unstable fixed point
- \( a \rightarrow b \) Unstable fixed point to Unstable fixed point
- \( A \rightarrow B \) Stable fixed point to Stable fixed point
- \( a \rightarrow B \) Unstable fixed point to Stable fixed point

Since before the bifurcation we have a stable fixed point in the Poincaré map (corresponding to the stable limit cycle in the system), and after the bifurcation, the slope \( \beta \) in this case is such that \( 0 < \beta < 1 \) (which corresponds to a stable fixed point), we can deduce that the bifurcation scenario is

\[
A \rightarrow B
\]

Figure 3 shows the analytical piecewise linear map (only \( P \) is numerically computed through Eq. 6), the numerically computed Poincaré map, and the line \( x_1(k + 1) = x_1(k) \) for \( c_1 = 0.2 \) and \( c_2 = 1, \) and for \( c_1 = -1 \) and \( c_2 = 1. \) It can be observed that the piecewise-linear map is a very good approximation of the numerically computed one, specially at the corner point. By varying \( b, c_1 \) and \( c_2 \) we can move the fixed point of the map and therefore change the amplitude of the limit cycle. Namely the fixed point of the map when trajectories cross the corner can be derived from Eq. (39) as

\[
x^*_1 = \frac{Pc_1b}{Pc_2 + c_1 - c_2}.
\]

Note that the location of \( x^*_1 \) is related to the amplitude of the corresponding cycle in phase space. Thus, varying \( b, c_1 \) and \( c_2 \) one can control locally the amplitude of the cycle. For example, for \( b = 0.1, c_1 = -1, c_2 = 1, \) the limit cycle is shown in Fig. 4(a). The cycle exhibited by the close-loop system has a smaller amplitude. Note that a further exploration of the effects of varying the control parameters on the amplitude of the limit cycle will be reported in Section B for the more realistic case of \( F(x) \) being unknown.

**Case II: \( \beta > 1 \)** Following a similar derivation, we can now select \( c_1 \) and \( c_2 \) in order to have \( \beta > 1. \) In this case, according to Feigin’s condition, we should observe the occurrence of a nonsmooth saddle-node and hence the disappearance of the limit cycle when control is activated. If we choose \( c_1 > 0, c_2 > 0 \) and \( c_2 > c_1, \) we then have that at the border-collision induced by the controller we have:

\[
A, b \rightarrow \emptyset
\]

*(Stable and unstable fixed points merge and disappear)*

Hence the fixed point disappears in a border collision bifurcation. Note that this result is local, thus the cycle is suppressed locally to the corner collision point. The existence of other cycles might
be possible according to the global properties of the map that cannot be studied analytically and must therefore be validated by appropriate simulations.

For example for \( c_2 = 1 \) and \( c_1 = 0.5, \beta = 1.067 \), Fig. 3 shows the analytical piecewise linear map, the numerically computed Poincaré map, and the line \( x_1(k+1) = x_1(k) \).

![Analytical and numerical maps](image)

Fig. 3: Case \( F(x) \) perfectly known. Evolution of fixed point for several constant values. The corner collision point is \((-0.1, 0)\). Other bifurcation scenarios are not possible.

Fig. 4(b) shows the evolution of the continuous system in the case considered here. We see that the control is successful in suppressing the cycle of the open-loop system [16]. As expected, the strategy is local and a larger limit cycle is detected in Fig. 4(b). This is due to the fact that the map shown in Fig. 3 eventually intersects the identity line for larger values of \( x_1 \). Thus, the control is only effective in suppressing the limit cycle in the region of interest. Global control strategies should instead be used to achieve global results.

5.2 \( F(x) \) unknown, \( u = c \)

Now, each component of the control signal in region \( S_1 \) is defined as \( u_i = c_i, i = 1, 2 \). In this case, the controlled system inside the corner is described by

\[
\begin{align*}
\dot{x}_1 &= f_1 + c_1 \\
\dot{x}_2 &= f_2 + c_2 \\
\end{align*}
\]

and knowing that \( f_1^0 = 0, f_2^0 = 1, g_1^0 = c_1 \) and \( g_2^0 = 1 + c_2 \), the Poincaré map is given by

\[
x_1 \rightarrow \begin{cases} 
P x_1 & \text{if non-crossing} \\ 
P \left( x_1 - c_1 (1 - \frac{b}{c_1 - c_2 - 1}) \right) & \text{if crossing} 
\end{cases}
\]

with \( P = \alpha = 0.5335 \).

In this case the slope of the map when crossing is given by

\[
\beta = P \frac{c_2 + 1}{c_2 + 1 - c_1}
\]

Again we need to choose \( c_1 \) and \( c_2 \) in order to avoid sliding. Following derivations similar to those outlined above, we find that \( \beta > 0 \) for all \( c_1 \) and \( c_2 \) satisfying (21). Note that now

\[
x_1^* = \frac{Pc_1 b}{c_1 - c_2 - 1 + P(1 + c_2)}
\]

and so the amplitude of the limit cycle can be controlled with parameters \( b, c_1 \) and \( c_2 \).

Thus as before we can use the controller to achieve two different aims.
Fig. 4: Evolution of the system controlled with $u = -F(x) + c$ with different control parameters. Subfigures (a) and (b) are in the same scale. Subfigure (c) shows the behaviour of the maximum effort control in case (b).
Case I: $\beta < 1$ Some algebra shows that this case can be possible when

$$c_1 < (1 - P)(c_2 + 1).$$

For example with $c_2 = 1$ and $c_1 = -2$ the slope is $\beta = 0.2667$. This means that fixed point in a neighborhood continues existing after the bifurcation. Fig. 5(a) shows the behavior of the continuous-time system with and without control. We see that the limit cycle persists as expected with a minor change of its amplitude. This can be emphasized by selecting different values of $c_1$ and $c_2$ and hence moving further the fixed point of the map.

Case II: $\beta > 1$ Now we have:

$$A, b \rightarrow \emptyset$$

and the limit cycle disappears locally in a border collision bifurcation. This case is possible too. It can be easily checked that the conditions on $c_1$ and $c_2$ are given by

$$(c_2 + 1)(1 - P) < c_1 < c_2 + 1$$

For example with $c_2 = 3$ and $c_1 = 2$ the slope is $\beta = 1.067$. Fig. 5(b) shows the behavior of the continuous-time system. We observe that the target cycle has disappeared locally after a border collision bifurcation; the evolution of the map moving towards another fixed point outside the range where the local description is valid. Such fixed point corresponds to the large-amplitude limit cycle depicted in the Figure. Figure 6 shows the analytical and numerical maps for Cases I and II confirming the excellent agreement between the analysis and the numerics. (Note that, for the sake of brevity, to obtain an analytical approximation of the Poincaré map, we have truncated the discontinuity map to its linear terms.)

It is worth to note that in all the cases, if the slope of the piecewise-linear map is very high, the map based on the linear approximation becomes representative of the system behavior in a relatively small neighborhood of the bifurcation point (see Fig. 7).

Control effort In this subsection, the analytical formulas derived to estimate the control effort, when $F(x)$ is unknown, are compared with the numerical values obtained. Constants $c_1$ and $c_2$ are varied to obtain different lower or bigger amplitude limit cycles. Once the constants have been fixed, the corner penetration is varied to obtain different amplitudes. There is no unified measure of an amplitude of a limit cycle. The original limit cycle is a circle and a natural measure can be its radius. Since the obtained limit cycles (after the control is applied) are approximately circles also, which are centered at point $(−1,0)$, we take $x_1 + 1$ (being $x_1$ the first coordinate when the limit cycle enters $S_1$) as a measure of the amplitude. We have also computed another measure of the amplitude, given by

$$\sqrt{\frac{1}{T} \int_0^T \left((x_1(\tau) + 1)^2 + x_2(\tau)^2\right) d\tau}$$

with equivalent results.

Table I shows the numerically computed RMS value for cycles with amplitudes bigger than the amplitude of the original limit cycle (in this case $c_1 = 2$ and $c_2 = 3$). Also, Fig. 8 shows the relation between amplitude of the limit cycle and the numerical (solid line) and analytical (dots) RMS value when the desired oscillation has an amplitude lower than the original (in this case $c_1 = -2$ and $c_2 = 1$).

As it can be seen from Fig. 8, the limit cycle can disappear (the amplitude is decreased to zero). Thus the method can also control the appearance and disappearance of limit cycles, though this fact is nonlocal and cannot be explained by the theory in this paper. Figure 9 shows different limit cycles when the corner penetration is varied far from the nonsmooth bifurcation.
Fig. 5: Evolution of the system controlled with $u = c$. Subfigures (a) and (b) are in the same scale. Subfigure (c) shows the behaviour of the maximum effort control in case (b).
Fig. 6: Case $F(x)$ unknown. Evolution of the fixed point for several constant values. The corner collision point is $(-0.1, 0)$. Other bifurcation scenarios are not possible.

Fig. 7: Piecewise-linear map with a high slope, $c_1 = 2.8$ and $c_2 = 3$. As it can be seen, far from the corner, the approximation loses validation.

Table 1: Amplitudes of limit cycles and RMS values

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>RMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.47</td>
<td>0.524</td>
</tr>
<tr>
<td>1.61</td>
<td>0.612</td>
</tr>
<tr>
<td>1.69</td>
<td>0.675</td>
</tr>
<tr>
<td>1.79</td>
<td>0.765</td>
</tr>
<tr>
<td>1.86</td>
<td>0.837</td>
</tr>
<tr>
<td>1.91</td>
<td>0.920</td>
</tr>
</tbody>
</table>
Fig. 8: Relation between amplitude of the limit cycle and RMS value when the desired oscillation has an amplitude lower than the original. The solid line corresponds to numerically computed values, while the points are computed with the analytical formula deduced in Section IV.

6 Conclusions

In this paper we have shown that it is possible to synthesize a switching control law to suppress or change the main features of a target limit cycle in planar smooth dynamical system. Other authors [5], [6] have studied the bifurcation control problem from a smooth feedback framework. Our approach is different since switching (and thus nonsmooth) control laws are proposed. In so doing, the theory of non-smooth bifurcations was explicitly used in the design process. Namely, by appropriately selecting the control constants and the switching manifolds, it is possible, to change the properties of the Poincaré map associated to the cycle of interest. The resulting control action is acting on the system in a relatively small neighborhood of the corner-collision point and hence guarantees the achievement of the control goal with a minimal control expenditure. We wish to emphasize that rather than being a technique for the control of bifurcations in nonlinear systems, the strategy presented here aims at exploiting the theory of non-smooth bifurcations for control system design.

Ongoing research is aimed at further exploring the ideas presented in this paper and establish formal links between the controller gains and the properties of $\Omega$-limit set of the closed-loop system. Also, the extra degrees of freedom corresponding to the control law parameters $c_1$ and $c_2$ (chosen here by using the additional constraint of avoiding sliding) can be further exploited to obtain, for example, a given slope $\beta$ of the map when crossing, or to have solutions satisfying certain performance criteria. Future work will investigate this further and will also be concerned with the experimental validation of this control strategy.

References

Fig. 9: Different values for the corner penetration give different meaningful amplitude reductions, even suppressing the limit cycle. (a) The corner penetration is set to $-0.6$. A small reduction in the amplitude of the limit cycle is obtained. (b) The corner penetration is set to $-0.99$. A considerable reduction of the amplitude is observed. (c) The corner penetration is set to $-1$, and the limit cycle turns into an equilibrium point. Also, the control effort is reduced to zero at the stationary point.