TWO TECHNIQUES FOR MEASURING HIGHER ORDER SINUSOIDAL INPUT DESCRIBING FUNCTIONS.

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Abstract
For high precision motion systems, modeling and control design specifically oriented at friction effects is instrumental. The Sinusoidal Input Describing Function theory represents an approximative mathematical framework for analyzing nonlinear system behavior. This theory however limits the description of the nonlinear system behavior to a quasi linear amplitude dependent relation between sinusoidal excitation and sinusoidal response. In this paper an extension to Higher Order Describing Functions is realized by introducing the concept of the harmonics generator. The resulting Higher Order Sinusoidal Input Describing Functions (HOSIDFs) relate the magnitude and phase of the higher harmonics of the periodic response of the system to the magnitude and phase of a sinusoidal excitation. Based on this extension two techniques to measure HOSIDFs are presented. The first technique is FFT based. The second technique is based on IQ (in-phase/quadrature-phase) demodulation. In a simulation the measurement techniques have been tested by comparing the simulation results to analytically derived results from a known (backlash) non-linearity. In a subsequent practical case study both techniques are used to measure the changes in dynamic behavior as function of drive level due to friction in an electric motor. Both methods prove successful for measuring HOSIDFs.

Key words
Frequency domain analysis, nonlinear systems, harmonic distortion, describing function, system identification.

1 Introduction
In the analysis and synthesis of dynamic systems, frequency domain based concepts like the Frequency Response Function presume linear system behavior. If the system is non-linear, the Frequency Response Function describes a linearized version of the system behavior in a working point or limited operating range. Every real life system is non-linear although the implications are not always noticeable in the operating range. As the required control performance of mechanical systems increases, non-linear behavior becomes of interest due to its adverse influence on system performance. Hence, the negative effects of friction on the dynamics of the machine have to be taken into account. Often, servo controllers are used to reduce the position errors caused by friction. The complexity of these controllers varies from a basic PID action to sophisticated model based compensation schemes, combining advanced friction models with digital signal processing [Armstrong-Hélouvry, Dupont, Canudas de Wit, 1994] and [Hensen, van de Molengraft and Steinbuch, 2002]. Increasing demands on positioning performance also call for a steady advance in the synthesis techniques of controllers. It is evident that the influence of nonlinear system behavior has to be taken into account both for state dependent control actions like gain scheduling as well as for the models used in model based control. Some approaches addressed the describing function analysis [Gelb and Vander Velde, 1968]. [Taylor, 1999] to replace a nonlinear element with a quasi linear descriptor which gain is a function of input amplitude. Chua et al. [Chua and Ng, 1979] presented a unified study on frequency domain analysis of nonlinear systems based on the Volterra functional series. Boyd et al. [Boyd, Tang and Chua, 1983] and Chua et al. [Chua and Liao, 1989] developed measurement procedures to determine higher order Volterra kernel transforms. The Volterra structure however is not able to model hysteresis, dead-zone and backlash. Several authors have studied the effects of nonlinear distortions on frequency response functions measured using multisine test signals. Schoukens et al. [Schoukens, Dobrowiecki and Pintelon, 1998], [Pintelon and Schoukens, 2001] introduced the concept of the Related Linear Dynamic System. Evans at al. [Evans, Rees and Jones, 1994] and Solomou et al. [Solomou, Rees and Chiras, 2004]
classified the nonlinear distortions into harmonic and interharmonic contributions. They showed that the harmonic contributions only depend on the number of input frequencies and the order of the nonlinearity. In this paper we extend the well-known procedures from frequency response analysis of linear systems, towards a class of static non-linear dynamic systems, with harmonic responses with amplitude dependent behavior. Here we introduce a nonlinear element, the virtual harmonics generator as a bridge between the frequency domain analysis of linear systems and the frequency domain analysis of this class of nonlinear systems. Using this non-linear element we show that the concept of the Describing Function can be extended to Higher Order Describing Functions. The paper begins with the definition of the harmonics generator block. Subsequently a brief review of the Sinusoidal Input Describing Function theory is given as an introduction to the concept of Higher Order Describing Functions. In section 4 we will introduce two measurement techniques to measure the Higher Order Sinusoidal Input Describing Functions and in section 5 a simulation is presented. In section 6 experimental results of a motion system case study will be shown. Finally, the main results will be discussed in the form of conclusions in section 7.

2 Virtual harmonics generator

Consider a stable, non-linear time invariant system. Let \( u(t) = \hat{a}\sin(\omega_0 t + \varphi_0) \) be the input signal. The system response \( y(t) \) is considered to consist exclusively of harmonics of the fundamental frequency \( \omega_0 \) of the input signal \( u(t) \), i.e. we assume that the transient behavior has vanished. Response \( y(t) \) can be written as a summation of harmonics of the input signal \( u(t) \), each with an amplitude and phase, which can depend on the amplitude \( \hat{a} \), phase \( \varphi_0 \) and frequency \( \omega_0 \) of the input signal (figure 1). This system can be modeled as

\[
u(t) = a(t) = \hat{a}\sin(\omega_0 t + \varphi_0) = \sum \frac{A_n}{j\omega_n - \omega_0}\]

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\[
u(t) = a(t) = \hat{a}\sin(\omega_0 t + \varphi_0) = \sum \frac{A_n}{j\omega_n - \omega_0}\]

By defining a separate block for the generation of harmonics in modeling this class of non-linear systems the complexity of the subsequent (non)linear block will be significantly less and (quasi)linear approaches may become feasible depending upon the remaining non-linear behavior. The resulting model structure has strong similarities with a Hammerstein model. This structure however is not a Hammerstein model since the second block is not necessarily linear [Narendra and Gallman, 1966].

3 Higher Order Sinusoidal Input Describing Function

Consider a stable, non-linear time invariant system as described in section 2 with \( u(t) \) as the input signal and \( y(t) \) as system response after the transient behavior has vanished (figure 1). The describing function \( H(\hat{a}, \omega) \) of the system is defined as the complex ratio of the fundamental component \( \hat{g}(t) \) of the system response and the input sinusoid \( u(t) \) (figure 3). The describing func-

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The Fourier coefficients \( a_1 \) and \( b_1 \) are calculated as in (4), (5) with \( T_0 = \frac{2\pi}{\omega_0} \):

\[
a_1 = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} y(t) \cos(\omega_0 t) dt \tag{4}
\]

\[
b_1 = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} y(t) \sin(\omega_0 t) dt \tag{5}
\]

In figure 4 the block representation of the non-linear system with harmonic response is redrawn by separating the harmonics generator from the system. The remaining system can be represented as a parallel connection of subsystems, each relating a harmonic component of the non-linear system output to the corresponding harmonic component of the virtual harmonics generator. The subsystem \( H_1(\bar{a}, \omega) \) is the describing function of the system as defined in (3). This classical describing function can be interpreted as the first element of a set of higher order describing functions \( H_n(\bar{a}, \omega) \). These functions can be defined as the complex ratio of the \( n \)th harmonic component in the output signal to the virtual \( n \)th harmonic signal derived from the excitation signal. This virtual harmonic has equal amplitude as the fundamental sinusoid but its starting-phase is \( n \) times the starting phase of the excitation signal. Like the first order describing function (3), the higher order describing functions are calculated from the corresponding Fourier coefficients (6).

\[
H_n(\bar{a}, \omega) = \frac{A_n(\bar{a}, \omega)e^{j(n(\omega_0 t + \varphi_0) + \varphi_n(\bar{a}, \omega))}}{\bar{a}e^{j\varphi_0}} = \frac{A_n(\bar{a}, \omega)e^{j\varphi_n(\bar{a}, \omega)}}{\bar{a}} = \frac{1}{\bar{a}}(b_n + ja_n) \tag{6}
\]

\( H_n(\bar{a}, \omega) \) can be interpreted as a descriptor of the individual harmonic distortion components in the output of a time invariant non-linear system with harmonic response as function of the amplitude and frequency of the driving sinusoid. In this paper \( H_n(\bar{a}, \omega) \) will be referred to as the Higher Order Sinusoidal Input Describing Function (HOSIDF).

4 Measurement techniques for HOSIDFs

As stated in (6), determining HOSIDFs requires a method for measuring the complex ratio of two band-pass filtered signals. Two distinctly different methods are investigated. The first method employs FFT techniques to determine the autospectrum and phase information and operates upon blocks of data. The second method uses IQ demodulation and is sample based [Rader, 1984], [Mitchell, 1989]. In applications where real-time on-line information about HOSIDFs is desirable, sample based processing may turn out to be superior to block based processing with respect to real-time performance. These aspects however are not studied in this paper.

4.1 FFT method

With this measurement technique both the input signal \( u(t) \) and output signal \( y(t) \) (figure 1) are Fourier transformed with a transform size of \( 2m \). The resulting single sided spectra contain \( m+1 \) frequency lines each with 0 Hz in frequency line 0. The frequency spacing is \( \Delta f = \frac{1}{T_b} \) with \( T_b \) the length of the data block. \( T_b \) is chosen a multiple \( p \) times the period \( T_0 = \frac{2\pi}{\omega_0} \) of the excitation signal. This assures that all the power of the excitation signal is concentrated in frequency-line \( p \). The power of the response signal is fully concentrated in the frequency lines \( n \cdot p \) with \( n \in \mathbb{N} \), so leakage is absent. In figure 5 the harmonics generator and the \( k \)th order HOSIDF are highlighted. Let us consider the calculation of the \( k \)th order HOSIDF: According to (6)

\[
H_k(\bar{a}, \omega) = \sum_{k=0}^{k-1} \left( b_n \cos(\omega_0 t + \varphi_n(\bar{a}, \omega)) + j a_n \sin(\omega_0 t + \varphi_n(\bar{a}, \omega)) \right)
\]

Figure 5. Determination of the \( k \)th order HOSIDF using the FFT method.
the measurable input signal $u(t)$. Using (1), (2) the frequency $n\omega_0$, amplitude $\hat{a}$ and phase $n\varphi_0$ of every component $n$ from the output of the harmonics generator can be calculated. In the frequency spectrum of $u(t)$ the frequency line $p$ with complex value $a_p + jb_p$ represents the input signal. The square root of the power in this frequency line is the amplitude $\hat{a}$ (7) and the phase angle of this frequency line equals phase $\varphi_0$ (8).

$$\hat{a} = \sqrt{a_p^2 + b_p^2}$$  \hspace{1cm} (7)

$$\tan \varphi_0 = \frac{-b_p}{a_p}$$  \hspace{1cm} (8)

In the spectrum of the system output signal $y(t)$, the frequency line with number $k \cdot p$ and complex value $a_{kp} + jb_{kp}$ represents the output of the subsystem $H_k(\hat{a}, \omega)$. The square root of the power in this frequency line is the amplitude $A_k(\hat{a}, \omega)$ (9).

$$A_k(\hat{a}, \omega) = \sqrt{a_{kp}^2 + b_{kp}^2}$$  \hspace{1cm} (9)

The phase angle of this frequency line $\varphi_{k_{out}}$ is the sum of the phase of the $k^{th}$ component of the harmonics generator $\varphi_{k_{in}}$ and the system phase $\varphi_k(\hat{a}, \omega)$ (4.1).

$$\varphi_{k_{out}} = \varphi_{k_{in}} + \varphi_k(\hat{a}, \omega) = k\varphi_0 + \varphi_k(\hat{a}, \omega)$$

$$\tan \varphi_{k_{out}} = \frac{-b_{kp}}{a_{kp}}$$  \hspace{1cm} (10)

From (7) and (9) the magnitude of the $k^{th}$ order HOSIDF can be calculated as:

$$|H_k(\hat{a}, \omega)| = \sqrt{\frac{a_{kp}^2 + b_{kp}^2}{a_p^2 + b_p^2}}$$  \hspace{1cm} (11)

The phase $\varphi_k(\hat{a}, \omega)$ of the HOSIDF can be calculated from (8) and (10).

### 4.2 IQ demodulation method

An alternative to the FFT method is the IQ demodulation method [Rader, 1984], [Mitchell, 1989]. The signals $u(t)$ and $y(t)$ from which the HOSIDFs are to be calculated consist only of the excitation sinusoid with frequency $\omega_0$ respectively its harmonics $n\omega_0$ (figure 1). So their frequency spectra can be considered a collection of narrowband signals with bandwidth much less than $\omega_0$. The magnitude and phase of these spectral components can be determined using IQ demodulation. In figure 6 IQ demodulation of the $k^{th}$ harmonic of the output signal is explained in more detail. The system output signal $y(t)$ is multiplied with $2\sin(k\omega_0)$ and $2\cos(k\omega_0)$ in two separate branches. These multiplications result in the generation of two new signals, each consisting of the sum and difference frequencies of the original signal and the oscillator signals (12), (13). These new signals are 90° apart.

$$2\sin(k\omega_0 t) \sum_{n=1}^{\infty} A_n(\hat{a}, \omega) \sin(n\omega_0 t + n\varphi_0 + \varphi_n(\hat{a}, \omega)) =$$

$$\sum_{n=1}^{\infty} A_n(\hat{a}, \omega) \cos(\omega_0 t (n - k) + n\varphi_0 + \varphi_n(\hat{a}, \omega)) -$$

$$\sum_{n=1}^{\infty} A_n(\hat{a}, \omega) \cos(\omega_0 t (n + k) + n\varphi_0 + \varphi_n(\hat{a}, \omega))$$  \hspace{1cm} (12)

$$2\cos(k\omega_0 t) \sum_{n=1}^{\infty} A_n(\hat{a}, \omega) \sin(n\omega_0 t + n\varphi_0 + \varphi_n(\hat{a}, \omega)) =$$

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$$\sum_{n=1}^{\infty} A_n(\hat{a}, \omega) \sin(\omega_0 t (n + k) + n\varphi_0 + \varphi_n(\hat{a}, \omega))$$  \hspace{1cm} (13)

After low-pass filtering with a cut-off frequency $\omega_{LP} \ll \omega_0$ the remaining signals representing the $k^{th}$ harmonic are $A_k(\hat{a}, \omega) \sin(k\varphi_0 + \varphi_k(\hat{a}, \omega))$ called the I-signal (in phase) component and $A_k(\hat{a}, \omega) \cos(k\varphi_0 + \varphi_k(\hat{a}, \omega))$ called the Q-signal (quadrature) component. From the I and Q components the amplitude $A_k(\hat{a}, \omega)$ and phase $\varphi_{k_{out}}$ of the $k^{th}$ harmonic component in the output signal $y(t)$ are computed according to (14), (15).

$$A_k(\hat{a}, \omega) = \sqrt{I_k^2 + Q_k^2}$$  \hspace{1cm} (14)

$$\tan(\varphi_{k_{out}}) = \tan(k\varphi_0 + \varphi_k(\hat{a}, \omega)) = \frac{I_k}{Q_k}$$  \hspace{1cm} (15)

Since the $k^{th}$ component in the output signal $\hat{u}(t)$ can not be measured, the input signal $u(t)$ is used for determining $\hat{a}$ and $\varphi_0$ (figure 7). Like in (12) (15) IQ demodulation of $u(t)$ yields the amplitude $\hat{a}$ (16) and
phase $\varphi_0$ (17).

\[
\dot{a} = \sqrt{I_{in}^2 + Q_{in}^2} \quad (16)
\]

\[
\tan(\varphi_0) = \frac{I_{in}}{Q_{in}} \quad (17)
\]

From (14) and (16) the magnitude of the $k^{th}$ order HOSIDF can be calculated as:

\[
|H_k(\dot{a}, \omega)| = \frac{\sqrt{I_k^2 + Q_k^2}}{\sqrt{I_{in}^2 + Q_{in}^2}} \quad (18)
\]

With (15) and (17) its phase can be calculated. The IQ demodulators for both signals $u(t)$ and $y(t)$ are identical except for the internal oscillator frequency. As a result of this symmetry the filter characteristics do not bias the results. Any asymmetry in the channels of the measuring instrument however will show up in the measured HOSIDF. In real life situations calibration and compensation of channel symmetry errors might prove necessary both for the FFT method and the IQ method.

5 Simulation experiment

In this theoretical case study the proposed measurement techniques are tested in a simulation experiment to get a better understanding of their performance under extreme variations in the excitation amplitude with respect to dynamic range and selectivity. First the HOSIDFs of a known non-linear system will be derived analytically. These expressions serve as the reference for subsequent simulations. Next time-sequences will be generated simulating the dynamic behavior of this system. Subsequently these signals will be analyzed with the two proposed techniques and the results will be compared to the analytically derived input-output relations.

5.1 Theoretical derivation of the HOSIDFs for backlash

The system under test consists of a massless body $m$, connected to the surroundings by a spring with stiffness $k$ (figure 8). The body experiences a Coulomb friction force $F_c$ of magnitude $b$ if $\dot{y} \neq 0$ (figure 9). Due to this friction in combination with the spring, the body will experience a backlash of $2b$ when excited with a periodic force $F(t) = A \sin(\omega_0 t)$ if $A \geq b$. The non-linear relation between input force $F(t)$ and output displacement $y(t)$ is an odd function and can be expressed analytically as:

\[
y(t) = \begin{cases} 
A - b & \text{if } A(t) < b \\
\frac{A-b}{k} \left( \frac{2A}{\pi} - \frac{2A}{\pi} - \gamma \right) & \text{else} 
\end{cases}
\]

\[
\gamma = \begin{cases} 
\frac{\pi}{2} & \text{if } [\omega_0 t]_{mod} 2\pi \leq \pi - \gamma \\
\frac{3\pi}{2} & \text{if } [\omega_0 t]_{mod} 2\pi \leq 2\pi - \gamma 
\end{cases}
\]

\[
\sin(\omega_0 \gamma) = 1 - \frac{2b}{A} \quad (19)
\]

Figure 10 shows the input-output relation for backlash. Using (4) and (5), $y(t)$ can be decomposed in its Fourier coefficients $a_n$ and $b_n$ (Table 1). With (6)
the HOSIDFs are calculated for frequency $\omega_0$ as function of the excitation amplitude $A$. In figure 11 these magnitude and phase relations are displayed. The top left graph is the magnitude of the first order Describing Function. The top right plot shows the phase. In the subsequent rows the magnitudes and phases of the odd HOSIDFs are shown. An interesting observation is the symmetry of the magnitude plots of the HOSIDFs when $\log \frac{|F|}{|F_0|}$ is chosen as x-axis. In this paper no further attempt is made to explain this symmetry.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi n$ + arcsin$(1 - 2\beta)$ + $2(1 - 2\beta)\sqrt{(1 - \beta)}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{2\sqrt{2}}{3} + \frac{2\sqrt{2}}{3} \beta - 1$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{4\sqrt{2}}{3} + \frac{4\sqrt{2}}{3} \beta - 1$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{4\sqrt{2}}{3} + \frac{4\sqrt{2}}{3} \beta - 1$</td>
</tr>
</tbody>
</table>

Table 1. Fourier coefficients of harmonic responses of backlash

5.2 Simulations

The backlash simulations were carried out using Simulink. The simulations are done in discrete time domain so the simulated time series can be treated as the sampled representation of theoretical continuous time series. The range of the excitation amplitude is $10^{-6} \leq (A - b) \leq 10^4$. Figure 12 shows the HOSIDFs up to the $7^{th}$ order for the staircase excitation when measured with the FFT method. Every excitation level was kept constant during 3 periods of the excitation sinusoid. In the processing of the time series the first period of every excitation level was skipped in order to suppress effects caused by the transition between subsequent excitation levels.

<table>
<thead>
<tr>
<th>FFT</th>
<th>IQ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_s = 2048f_0$; 512$f_0$</td>
<td>$f_s = 512f_0$</td>
</tr>
<tr>
<td>$\Delta f = \frac{f_0}{2}$</td>
<td>Butterworth LP filter</td>
</tr>
<tr>
<td>no leakage</td>
<td>$f_{-3dB} = \frac{f_0}{2}$</td>
</tr>
<tr>
<td>rect window</td>
<td>5th order</td>
</tr>
<tr>
<td>$k = 1$</td>
<td>$k = 1$</td>
</tr>
<tr>
<td>$b = 0.25$</td>
<td>$b = 0.25$</td>
</tr>
</tbody>
</table>

Table 2. Relevant measurement parameters
Table 2 gives the relevant measurement parameters for both measurement techniques. The odd behavior of backlash is clearly reflected in the very low magnitude values of the even HOSIDFs. The noise in the corresponding phase-plots is caused by the numeric resolution of the calculations. Figure 13 shows the magnified gain and phase errors of the FFT method for different sampling frequencies used in the simulation. The influence of the simulation sampling frequency on the results increases with the order number. The influence on the phase errors varies with the magnitude of the HOSIDFs. The influence on the magnitude errors is excitation magnitude independent. From these results may be concluded that the sampling frequencies used for the simulation of the HOSIDFs up to order 7 were sufficiently high. Processing the stepped amplitude time sequences with the IQ method can lead to erroneous results if the transient behavior of the filters is not taken into account. In this part of the simulation each excitation level was kept constant for 12 periods of the excitation sinusoid to allow the lowpass filters to settle. In figure 14 the results after setting of the filters are shown. A major difference with figure 12 are the even orders. The odd characteristics of the system are not visible in the even order order results of the measurements. The dynamic range of the odd HOSIDFs is limited for high excitation levels. Figure 15 show the differences between the theoretical results and the results for the odd orders generated by the IQ method. The errors in the results for both the even and the odd orders are caused by non ideal low-pass filtering of the $I_k$ and $Q_k$ signals (figure 6). The implemented digital filters have finite attenuation at the harmonics of the excitation frequency. Power from the sum and difference signals with frequencies $\omega = (n \pm k)\omega_0$ for $n \neq k$ adds onto the power of the difference signal with frequency $\omega = (n \pm k)\omega_0$ for $n = k$. This problem does not exist in the FFT method with a rectangular weighting function if $l_{\text{window}} = \frac{T_0}{4}$ for $k \in \mathbb{N}$ where $l_{\text{window}}$ is the window length and $f_0$ is the excitation frequency.

For weighting functions like Hanning and Hamming to be applicable $k \geq 2$ (figure 16). In this research no special attention is payed to the optimal tradeoff between the digital filter characteristics and the sampling frequency. From the results it may be concluded that the dynamic range of the FFT method under noise free conditions is only limited by the numerical accuracy. The selectivity is determined by the weighting function and can be perfect if the length of the weighting function is chosen correctly. The dynamic range of the IQ method is limited by the selectivity of the lowpass filter. This results in a bias in the measurements of the even order SIDFs which depends upon the magnitudes of the odd order SIDFs. The dynamic range of all HOSIDFs will be limited by the remaining power after filtering the sum and difference signals with frequency outside the passband of the lowpass filter.

Figure 13. Magnitude and phase errors of the odd HOSIDFs derived with the FFT method for two values of $f_s$ during simulation. Legend: black $f_s = 2048f_0$, grey $f_s = 512f_0$.

Figure 14. Simulation results using IQ with $f_s = 512f_0$, LP filter 5th order Butterworth $f_{-3dB} = \frac{f_0}{2}$.

Figure 15. Magnitude and phase errors in the odd HOSIDFs generated with the IQ method. $f_s = 512f_0$, LP filter 5th order Butterworth $f_{-3dB} = \frac{f_0}{2}$.

6 Measured HOSIDFs of a mechanical system with friction

To put these measurement techniques into practice the HOSIDFs of a real mechanical system with friction
are measured. The test object is a system, which consists of a 20W electric DC collector motor. The motor is powered by a voltage-to-current converter (figure 17). The input to the system, i.e. the motor current \( I_m \), is measured with a current probe with a sensitivity of 2 A/V. The response signal is angular velocity \( \omega_{\text{out}} \). \( J \) represents the inertia of the motor and \( T \) is the driving torque. A block diagram of the measurement set-up is given in figure 18. For small rotations the angular velocity is measured with a dual fibre laser vibrometer. The angular velocity \( \omega_{\text{out}} \) is approximated by the linear velocity difference between two points spaced 180° on the circumference of the shaft divided by the spacing of the points. The resulting sensitivity is 0.588 rad/s/V. Friction in the bearings and seals will cause a friction-induced resonance [Symens, Al-Bender, Swevers and van Brussel, 2002], [Nuij, 2002], [Hensen, van de Molengraft and Stein-buch, 2002] which resonance-frequency will depend upon the excitation level.

6.1 Measurement of the FRF using white noise excitation

The \( H_1 \) Frequency Response Function \( \omega_{\text{out}} \) was measured with a SigLab 20-42 dynamic signal analyzer [Siglab,2002] providing 90 dB aliasing protection. The resolution was \( \Delta f = 0.313 \) Hz in the frequency range of 0 Hz to 1 kHz. A Hanning weighting function was applied. The excitation signal was band-limited random noise in the same frequency range. The crest factor of the noise was 3. The excitation levels were 1.5 mA\text{RMS}, 6 mA\text{RMS} and 36 mA\text{RMS}. These levels are not high enough to force the system from the stick phase into the slip phase so the system-behavior will be non-linear due to the dominant influence of friction. This situation occurs frequently in accurate point-to-point motion tasks. Figure 19 shows the results after 20 averages per measurement. In the frequency range up to approximately 400 Hz the magnitude plot of the 1.5 mA measurement shows a +1 slope which is consistent with the phase of 90 deg and indicates stiffness dominated behavior. In the plots a resonance is visible with a frequency varying between 540 Hz and 200 Hz. It is caused by the friction induced stiffness in combination with the motor inertia \( J \). Its damping varies significantly as can be seen from the differences in phase gradients. The non-linear system behavior is also clearly reflected in the coherence plot.

6.2 Measurements of the HOSIDFs

To further investigate this non-linear behavior, the HOSIDFs were determined using the measurement techniques described in 4.1 and 4.2. In this case study the HOSIDFs were determined for only one excitation
frequency. Subsequent measurement at different frequencies are required to gather information over a frequency range. The excitation frequency was chosen 320 Hz. The reason for this is that a signal with this frequency excites the system both above and below its friction induced resonance frequency depending upon the instantaneous amplitude of the excitation signal, see the vertical dashed line in figure 19. Other considerations are that 320 Hz is not a multiple of the 50 Hz mains frequency and that the signal can be generated with an integer number of 12.8 kHz samples per period, being one of the sampling frequencies of the SigLab 20-42 dynamic signal analyzer. This assures leakage free results when being processed with the FFT method. The theoretical background for the HOSIDFs as explained in 3 presumes a single sinusoid excitation which implies a piecewise constant amplitude as function of time. As a result one has to wait for the system to settle after every step in the amplitude. In this example measurement, a linear amplitude-time relation is chosen such that the amplitude may be considered quasi constant in relation to the excitation frequency in order to reduce the measurement time. Figure 20 shows the generator signal, the input current signal and the system response. The maximum angular displacement of the system can be calculated from the system response and is approximately 25 μrad. The main parameters used for the FFT method are a block-size of 1600 samples and a sampling frequency of 12.8 kHz so Δf = 8 Hz. Hanning window, no overlap processing. For the IQ method the low-pass filters are 5th order Butterworth with 4 Hz cut-off frequency.

6.3 Results

Figure 21 shows the amplitude dependency of the HOSIDFs measured at the fixed frequency of 320 Hz. The solid line shows the results from the IQ method, the dots indicate the measurements from the block based FFT method. The results of the two measurement techniques do not show the discrepancies as described in 5.2. The lowpass filter in the IQ method offers better than 300 dB suppression of the sum and difference signals with non 0 Hz frequency. In the left column the magnitude plots are presented for the higher order Sinusoidal Input Describing Functions. The right hand column gives the corresponding phase relations. In the magnitude plot of the first order SIDF we can distinguish three regions. From 0 mA to approximately 0.5 mA the system gain is excitation independent. Between 0.5 mA and approximately 2.5 mA a strong excitation level dependency is visible. Above 2.5 mA the gain is independent of the excitation level at a stable 18 dB but the system remains non-linear as can be concluded from the plots of the higher order SIDFs. The gain of the third order SIDF decreases initially until it reaches a minimum at an excitation of 0.5 mA. This is due to the low signal to noise ratios in this region resulting in large uncertainties in the calculations. For increasing excitation its magnitude increases and reaches a maximum of -8 dB at approximately 2.5 mA. Above that excitation level the gain decreases again slightly. The same pattern is visible for the fifth order SIDF, however its maximum of -15 dB is reached at an excitation level of 4.5 mA. The magnitude characteristics of the even orders have a lower value compared to the odd orders. The amplitude dependency is small too. In

![Figure 21](image-url)
The phase values at approximately 2.5 mA, which correspond with the resonance condition, can be expressed as \( \varphi(n)_{res} = (n + 1)45^\circ - 2^{(n+1)-1}90^\circ \). Because of this increase in steepness of the phase gradient as function of the odd order number the use of this higher order phase information can be beneficial in the detection of this sliding resonance. The results clearly show that the concept of HOSIDFs is suitable for analyzing amplitude dependent non-linear system behavior.

7 Conclusion and further research
An extension of the theory of Sinusoidal Input Describing Functions was presented towards Higher Order Sinusoidal Input Describing Functions (HOSIDFs). Hereto the concept of the Virtual Harmonics Generator was introduced. The theory was developed for a class of non-linear system with a periodic response on a sinusoidal excitation. Two measurement methods are described able to measure the HOSIDFs. The first method is FFT based and uses the auto spectrum and phase information to determine the HOSIDFs. The second technique uses IQ demodulation techniques with digital filters and is sample based. In a simulation case-study the HOSIDFs of backlash were derived analytically and compared with simulation results. The FFT based method generated superior results due to ideal filter characteristics. In a second case study a mechanical system with friction was measured. Initial FRF measurements with various levels of random noise excitation revealed a significant input level dependency of the system dynamics. The friction-induced stiffness caused a resonance to vary significantly in frequency and damping. With both measurement techniques the HOSIDFs have been identified successfully and it was demonstrated that the concept of HOSIDFs can be successfully employed to generate valuable information about amplitude dependent non-linear system behavior. Since the HOSIDF is not only a function of amplitude but also of excitation frequency many single frequency measurements have to be done in order to quantify its frequency dependency. Future work will comprise merging the multi-sine excitation techniques [Pintelon and Schoukens, 2001] with the HOSIDF technique described in this paper. The second subject for future work will be the implications of excitation signals with continuously varying amplitude. Implementation of this combined knowledge in hardware like FPGAs will hopefully result in the construction of a new, valuable and practical measurement tool.

References