On noise- and period-time sensitivity in high order repetitive control

Maarten Steinbuch  Siep Weiland  Johan Van den Eerenbeemt  Tarunraj Singh

Abstract—Repetitive control is useful if periodic disturbances act on a control system. Perfect (asymptotic) disturbance rejection is achieved if the period-time is exactly known. The improved disturbance rejection at the periodic frequency and its harmonics is achieved at the expense of a degraded system sensitivity at intermediate frequencies. A convex optimization problem is defined for the design of high-order repetitive controllers, where a trade-off can be made between robustness for changes in the period-time and for reduction of the error spectrum between the harmonic frequencies. The high order repetitive control algorithms are successfully applied in experiments with the tracking control of a CD-player system.

I. INTRODUCTION

Control systems subject to periodic disturbances may well benefit from the use of repetitive control [12], [4]. Repetitive controllers employ the internal model principle [7] and consist of a periodic signal generator, enabling perfect (asymptotic) rejection of periodic disturbances. One of the drawbacks of repetitive control is the requirement of exact knowledge of the period-time of the external signals. This means that in practical applications, either the period-time is required to be constant, or an accurate measurement of the periodicity is necessary. Another drawback is due to the Bode Sensitivity Integral: the perfect reduction at the harmonic frequencies is counteracted by amplification of noise at intermediate frequencies. Recently, approaches have been reported in the literature to cope with the drawbacks of standard repetitive control. These make use of high order periodic signal generators. In [11] an approach is presented to design a high-order repetitive controller such that robustness for period changes is obtained. In [2] a constraint optimization problem is solved such that a desensitizing effect is obtained for non-repetitive signals.

Approaches to make iterative learning schemes robustly stable with respect to iteration dependent disturbances and uncertainties have been reported in [3], [6]. In this paper we will generalize the results of [2] and show that it can be cast into a convex optimization problem. We will also show that, using appropriate weighting functions, the same problem formulation can yield both the results of [11] and the ones in [2].

In Section 2 we will introduce the structure of high order repetitive controllers, and show how stability can be guaranteed. In Section 3 the new optimization problem will be formulated. In Section 4 an application to a CD player mechanism will be shown. Main results will be summarized in the form of conclusions in Section 5.

II. HIGH ORDER REPETITIVE CONTROL

Consider the general repetitive control system, shown in Figure 1. The repetitive controller is shown in the figure as the device $M(z)$, which includes a memory loop or delay line [11]. In high order repetitive control, the total delay of the memory loop is extended to an integer multiple $p$ of $N$ samples by connecting multiple delays in series in a structure as shown in Figure 2. Note that in this block diagram a robustness filter $Q$ and a learning filter $L$ are incorporated. Their respective delays are compensated in the structure. Normally, the filter $Q$ is designed as a linear phase filter with $q$ samples delay, and the filter $L$ is designed using, for instance, the ZPETC algorithm as proposed in [13], with a phase delay of $l$ samples. Due to the use of multiple delay lines, the control signal can be computed as a weighted sum of the signals of one, two, and more periods ago.

The relation between the input $e$ and the output $z$ of the high order repetitive controller of Figure 2 is described by the transfer function

$$M(z) = \frac{L(z)Q(z)W(z)z^{-(N-l-q)}}{1 - Q(z)W(z)z^{-(N-q)}}$$

(1)

where $W$ is the gain adjusting or high order repetitive function, given by

$$W(z) = \sum_{i=1}^{p} w_i z^{-(i-1)N}$$

(2)

with $\sum_{i=1}^{p} w_i = 1$, $0 \leq w_i \leq 1$ and $|W(z)| \leq 1$. 

Maarten Steinbuch and Johan Van den Eerenbeemt are with the Dynamics and Control Technology Group, Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, Email: m.steinbuch@tue.nl

Siep Weiland is with the Control Systems Group, Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, Email: s.weiland@tue.nl

Tarunraj Singh is with the Department of Mechanical and Aerospace Engineering, SUNY at Buffalo, Buffalo, NY 14260, U.S.A. Email: tsingh@eng.buffalo.edu
A. Stability

Consider the control system of Figure 1, with \( M(z) \) the high order repetitive controller of Figure 2. The corresponding sensitivity function, relating the input disturbances \( d \) to the tracking error \( e \) is given by

\[
S = \frac{1}{1 + PC(1 + M)}.
\]  

(3)

To analyze the stability of the controlled system, (1) is substituted in (3) to obtain

\[
S = \frac{1}{1 + PC \left( 1 + \frac{LQWz^{-(N-q)}}{1-QWz^{-(N-q)}} \right)} = \frac{1}{1 + PC} \cdot M_S
\]  

(4)

where \( M_S \) is the modifying sensitivity function (or relative sensitivity error transfer function \([2]\)). \( M_S \) is given by

\[
M_S(z) := 1 - QWz^{-(N-q)}
\]  

(5)

\[
\frac{1 - QWz^{-(N-q)}}{1 - QWz^{-(N-q)}(1 - TLz^{+l})}
\]

where \( T \) is the complementary sensitivity

\[
T = \frac{PC}{1 + PC}.
\]

Hence \( M_S \) modifies the standard sensitivity function as a result of the repetitive control action. Using equation (4), the high order repetitive control system of Figure 1 can be transformed into the equivalent error system that is shown in Figure 3.

To simplify the analysis, the following assumptions are made:

- \( q = 0 \) and \( l = 0 \). That is, no delays are introduced by the robustness and learning filters \( Q \) and \( L \).
- \( Q \approx 1 \) for frequencies below the pass band of \( Q \).
- \( LT \approx k_r \). That is, for low frequencies, where the inverse of \( T \) is exactly known, \( k_r \) is known as the learning gain, normally chosen close or equal to one.

Under these assumptions, equation (5) simplifies to

\[
M_S(z) = \frac{1 - W(z)z^{-N}}{1 - W(z)z^{-N}(1 - k_r)}.
\]  

(7)

Furthermore, to make the analysis independent of the period time of the disturbance, the normalized frequency \( \theta = \omega NT_s \) with \( T_s \) the sampling time is introduced. With the substitution \( z = e^{j\omega T_s} \) or \( z = e^{\theta z} \), the modifying sensitivity function in equation (7) becomes a function of the normalized frequency \( \theta \) and is (with some abuse of notation) given by

\[
M_S(\theta) = \frac{1 - W(\theta)e^{-j\theta}}{1 - W(\theta)e^{-j\theta}(1 - k_r)}
\]  

(8)

where

\[
W(\theta) = \sum_{i=1}^{p} w_i e^{-(i-1)\theta}.
\]

is the (normalized) high order repetitive function.

B. Performance

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III. OPTIMIZATION DESIGN PROBLEM

In [5], Inoue proposes the weighting factors in (2) as \( w_i = 1/p \) and shows that this choice minimizes the averaged square of \( |M_S(z)| \) over all frequencies. In [2], the authors propose an ‘evolution strategy’ to minimize the \( H_\infty \) norm of \( M_S \) over all weighting factors \( w_i \), assuming the gain \( k_r \) to be fixed. Here we will extend the approach of [2] in two ways: first we will omit the unnecessary constraint that each weighting factor \( w_i \leq 1 \) (but we retain the condition on the sum!), and secondly we will show that the design optimization problem can be rephrased and solved by a linear programming algorithm. The weighting factors of the high order repetitive function \( W \) defined in (2) are determined in such a way that the infinity norm of the modifying sensitivity function is minimized. That is, we consider the problem

\[
\min_{w_i} \| G(\theta)M_S(\theta) \|_\infty
\]

subject to:

\[
\sum_{i=1}^{p} w_i = 1.
\]

Here, \( G(\theta) \) is a shaping function that is used to determine the effort of the \( H_\infty \) minimization on typical frequencies. The constraint on the weights is required in order to meet the internal model principle: to have high gain at the
harmonics, see also equation (8), its numerator should be zero at these frequencies.

To analyze this problem, consider the case where the learning gain $k_r = 1$. In this case, the modifying sensitivity function (8) simplifies to $M_S(\theta) = 1 - W(\theta)e^{-j\theta}$ and the optimization criterion becomes:

$$\min_{w_i} \| G(\theta)(1 - W(\theta)e^{-j\theta}) \|_\infty$$

where $W(\theta) = \sum_{i=1}^p w_i e^{-j(\theta-i)\phi}$ is subject to the constraint $\sum_{i=1}^p w_i = 1$.

Observe that in this case, for any given $G$ and fixed $\theta$, the function $G(\theta)M_S(\theta)$ is affine in the weighting parameters $w_i$. That is, we can write

$$G(\theta)M_S(\theta) = G(\theta) - \sum_{i=1}^p w_i G_i(\theta)$$

where $G_i(\theta) = G(\theta)e^{-j\phi}$. Hence, the optimization amounts to solving

$$\min_{w_i} \sup_{0 \leq \theta \leq 2\pi} |G(\theta) - \sum_{i=1}^p w_i G_i(\theta)|$$

subject to $\sum_{i=1}^p w_i = 1$.

and its computationally tractable approximation is

$$\min_{w_i} \max_{\theta \in \Theta} |G(\theta) - \sum_{i=1}^p w_i G_i(\theta)|$$

subject to $\sum_{i=1}^p w_i = 1$.

where $\Theta = \{\theta_1, \ldots, \theta_K\}$ is a (uniform) finite grid of the interval $[0, \pi]$. Equivalently, we wish to solve

$$\min_{w_i, t} \left\{ t \mid |G(\theta) - \sum_{i=1}^p w_i G_i(\theta)| \leq t, \text{ for all } \theta \in \Theta \right\}$$

where $n$ is an integer, $\phi = \frac{\pi}{n}$, and $p_n(z) := \max_{i=1, \ldots, n} |a \cos(i\phi) + b \sin(i\phi)|$ is the polyhedral norm consisting of the maximum of absolute values of $n$ linear forms of $a$ and $b$. The optimization (10) is then effectively approximated within accuracy $1 - \cos(\pi/2n)$ by

$$\min_{w_i, t} \left\{ t \mid p_n(G(\theta) - \sum_{i=1}^p w_i G_i(\theta)) \leq t, \text{ for all } \theta \in \Theta \right\}$$

This is a linear programming problem. See, e.g., [1] for details. This can be solved in Matlab using the linprog routine, achieving an arbitrary high accuracy of the minimal value of (10).

In the remainder of this section, two cases are distinguished considering the frequency range where low system sensitivity is desirable.

A. Robustness for changes in the period time

To achieve robustness for changes in the period time, the magnitude of the modifying sensitivity function is forced to be small in the frequency region close to the periodic frequency, i.e., for frequencies $\theta \in [0, \theta_1]$. This can be attained by choosing the shaping function $G(\theta)$ as:

$$G(\theta) = \begin{cases} 1 & \text{if } 0 \leq \theta < \theta_1, \\ 0 & \text{if } \theta_1 \leq \theta \leq \pi. \end{cases}$$

A graphical interpretation is shown in Figure 4.

![Fig. 4. Minimization of the modifying sensitivity function](image)

With $k_r = 1$, $\theta_1 = \pi/5$, a polyhedral approximation order $n = 2$ and various orders $p$ of the high order repetitive function $W$, the following optimal sets of weighting factors $W_{opt}(p, k_r) = (w_1^*, \ldots, w_p^*)$ for the corresponding linear programming problem are obtained:

$$W_{opt}(2, 1) = (1.85, -0.85) \approx (2, -1)$$
$$W_{opt}(3, 1) = (2.93, -2.93, 1) \approx (3, -3, 1)$$
$$W_{opt}(4, 1) = (4.01, -6.01, 4.02, -1.02) \approx (4, -6, 4, -1)$$
$$W_{opt}(5, 1) = (4.97, -9.96, 10.09, -5.16, 1.07) \approx (5, -10, 10, -5, 1).$$

(11)
It can be observed from (11) that the obtained weightings are equal to the analytically derived ones in [11], see also [8].

The results are shown in Figure 5. Indeed the sensitivity is made lower near the harmonics (i.e. \( \theta_1 = \pi/5 \)), at the cost of an amplification of the sensitivity at intermediate frequencies.

\[
\begin{align*}
W_{\text{opt}}(2, 1) &= (0.62, 0.38) \approx \left( \frac{2}{3}, \frac{1}{3} \right) \\
W_{\text{opt}}(3, 1) &= (0.49, 0.33, 0.17) \approx \left( \frac{3}{5}, \frac{2}{5}, \frac{1}{5} \right) \\
W_{\text{opt}}(4, 1) &= (0.37, 0.28, 0.22, 0.13) \approx \left( \frac{4}{10}, \frac{3}{10}, \frac{2}{10}, \frac{1}{10} \right) \\
W_{\text{opt}}(5, 1) &= (0.31, 0.25, 0.20, 0.15, 0.09) \approx \left( \frac{5}{17}, \frac{4}{17}, \frac{3}{17}, \frac{2}{17}, \frac{1}{17} \right).
\end{align*}
\]

The obtained weights are in correspondence with those obtained in [2]. The results are shown in Figure 6. Indeed, non-repeatable errors will be less amplified when compared with a standard repetitive controller.

The previously described situations are two extremes. A trade-off between robustness for changes in period time on one hand and sensitivity at intermediate frequencies on the other can be achieved by choosing an appropriate shaping function \( G(\theta) \).

**IV. APPLICATION TO A COMPACT DISC DRIVE**

In Figure 7 a schematic view of a Compact Disc mechanism is shown. The mechanism is composed of a turn-table DC-motor for the rotation of the Compact Disc, and a radial arm for track-following. An objective lens, suspended by two parallel leaf springs, can move in a vertical direction to give a focusing action.

In Figure 8 a block-diagram of the radial (tracking) control loop is shown. The difference between the radial track position and the spot position is detected by the optical pick-up; it generates a radial error signal [9]. The frequency response of the open loop system is depicted in Figure 9. In current systems, the servo controller is mostly a PID controller [10]. The tracking control loop has a cross-over frequency of 600 Hz.

The model of the radial actuator is a double integrator and resonances to account for parasitic dynamics as measured on the drive. The controller is a lead filter with integral action (PID controller [10]). The disc rotates at a frequency of 12.5 Hz.

**A. Design of the repetitive controller**

A straightforward choice for the learning filter \( L \) would be:

\[
L = k_r T^{-1}
\]
where $T$ is the complementary sensitivity function and $k_r$, the learning gain. However, in many applications, as is the case here, $T^{-1}$ is not proper and $T$ will be non-minimum phase. As a result, the computation of $T^{-1}$ will lead to a non proper or unstable $L$-filter. To overcome this problem, Tomizuka and others [13] developed the so-called Zero Phase Error Tracking Controller (ZPETC) algorithm, in which the non-minimum phase (or ‘unstable’) zeros in $T$ are approximated by stable poles in $L$. This has been applied and in Figure 10 the frequency response is shown of a proper and stable learning filter $L$. The present design is therefore not limited to systems in which the complementary sensitivity function is bi-proper or non-minimum phase.

As robustness filter $Q$ a symmetric and even order (type I) FIR filter is constructed. The corresponding Bode plot is shown in Figure 11. The cut-off frequency of $Q$ is specified at 200 Hz. This implies that up to 16 (200/12.5) harmonics will be suppressed. Raising the bandpass frequency results in disturbance rejection at higher harmonics, but then instability may occur, because $L$ does not match the inverse of $T$ well at high frequencies. Furthermore, the $Q$-filter order is set to 200. For lower orders the cut-off frequency did not correspond to the desired value. This is probably due to the high sample frequency of the filter ($f_s = 25$ kHz).

The magnitude of the resulting sensitivity function is plotted in Figure 12, for the system with and without repetitive control. From these figures it can be concluded that the repetitive controller provides a better disturbance rejection at the repetitive frequency and its harmonics at the cost of a degraded performance at intermediate frequencies, as expected.

As experiments

When the actual rotation period of the turntable motor is measured, it turns out that the number of revolutions per second is not a constant. The variation is in the range of $\pm 0.05$ Hz.
The measurements are performed for a high order repetitive controller (HORC) of order three \((p = 3)\), with the following set of weighting factors:

\[
\begin{align*}
W_1 &= (0, 0, 0) & \text{standard PID} \\
W_2 &= (1, 0, 0) & \text{single delay repetitive control} \\
W_3 &= (3, -3, 1) & \text{period-time-robust HORC} \\
W_4 &= \left(\frac{3}{6}, \frac{2}{6}, \frac{1}{6}\right) & \text{noise-robust HORC}
\end{align*}
\]

In Figure 13 the measured power spectrum of the error for the different configurations of the repetitive controller is depicted for a rotational frequency of 12.50 Hz. It can be seen that for the system without repetitive controller, \(W_1\), the spectrum of the error has large peaks at the rotational frequency of 12.50 Hz and its harmonics. The repetitive controller effectively reduces this peak. But looking at the different configurations, it can be seen that the \(W_3\) configuration almost perfectly reduces the disturbance at the periodic frequency, while for \(W_2\) and \(W_4\) there is still some power left. According to theory and simulations, the reduction at the repetitive frequency should be equal for all repetitive configurations. This difference can be related to the variation in the rotational frequency of the CD-player turntable \((\pm 0.05 \text{ Hz})\). Even small deviations of the disturbance period time from the delay time of repetitive controller result in a degradation of the sensitivity of the \(W_2\) and \(W_4\) configuration, while it does not affect the sensitivity of the \(W_3\) configuration.

Looking in Figure 13 at the frequency range between two successive harmonics, it can be seen that for the \(W_2\) and \(W_3\) configuration, the system sensitivity at intermediate frequencies is degraded due to the repetitive control action. Here the ‘noise proof’ repetitive controller, \(W_4\), shows its effectiveness. For this configuration the disturbance rejection between two harmonics is not noticeably deteriorated with respect to the system without repetitive control. These observations are in accordance with the magnitude plots of the modifying sensitivity function of Figure 5 and 6: the improved robustness for changes in the period time for \(W_3\) goes along with a reduced system sensitivity at intermediate frequencies. The \(W_4\) configuration is less robust for period time changes, but provides better disturbance reduction at intermediate frequencies. With respect to these properties, the single delay configuration lies between both high order configurations.

V. CONCLUSIONS

The use of memory loops is beneficial in systems with repetitive disturbances or tasks. In order to improve the capabilities of repetitive controllers for those cases where the periodicity is hard to measure and is subject to variation and/or where off-harmonics (or noise) occur, the possibility of high-order repetitive control is investigated. A new design algorithm has been developed, which uses simple linear programming techniques to design the repetitive controller. Both a noise-robust and a period-time-robust high order repetitive controller have been implemented successfully in a digital control setup of a Compact Disc player.

REFERENCES