Softening, singularity and mesh sensitivity in quasi-brittle and fatigue damage

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Abstract

Continuous descriptions of material damage are prone to nonphysical responses beyond a certain level of damage. Numerical simulations then become extremely sensitive to the spatial discretisation. These phenomena are studied in this contribution for continuum damage models of quasi-brittle and fatigue fracture. A linear stability analysis shows that the issue is less relevant for fatigue crack initiation than for quasi-brittle fracture. However, when the standard fatigue model is used to describe crack propagation, it is found that the crack growth rate strongly depends on the fineness of the finite element discretisation and goes to infinity when the element size approaches zero. A gradient-enhanced damage model gives a finite growth rate and does not exhibit this pathological mesh sensitivity.

1 Introduction

The renewed interest in nonlocal and higher-order solids which we have seen since the end of the past decade largely stems from efforts to model material degeneration and failure. Continuum models that have been developed for this purpose invariably exhibit strain softening. However, classical continuum theories cannot correctly describe the strongly localised deformations which accompany softening. This difficulty becomes apparent in finite element analyses: the predicted fracture behaviour is highly sensitive to the spatial discretisation of the problem. The introduction of nonlocality – either directly in an integral format or in the more practical form of higher-order gradients – provides an effective remedy for this deficiency.

Perhaps the simplest class of nonlinear continuum models which suffer from mesh sensitivity is that of elasticity-based damage models. Two domains of application can be distinguished for these models: quasi-brittle fracture (e.g., concrete cracking) and high-cycle fatigue. Localisation of deformation and mesh dependence of brittle damage models have been studied extensively and are believed to be understood reasonably well. In statics the local equilibrium equations lose ellipticity near the onset of softening. The deformation and damage growth then localise in a line or, in finite element analyses, in the narrowest band that can be captured by the spatial discretisation. Upon refinement of the discretisation the predicted fracture energy approaches zero. Moreover, the direction of damage growth tends to follow the finite element grid. In dynamic analyses, the reverse transition causes similar complications, as well as an inability to propagate loading waves [1–3].

The possibility of localisation and mesh sensitivity in fatigue damage seems to have received less attention, although fatigue damage – particularly high-cycle fatigue damage – is often more localised than quasi-brittle damage. In this contribution the relevance of localisation phenomena and mesh dependence in fatigue modelling is examined using a fatigue damage formulation which is similar to quasi-brittle damage descriptions [4]. Both the original, local model and a gradient-enhanced formulation are considered. The damage process before crack initiation and the pathological localisation which may result from it are examined analytically and the behaviour of the fatigue model is compared with that of brittle damage. The influence of mesh fineness on the computed crack growth is studied by finite element analyses of fatigue crack propagation using the local and the gradient model.

2 Constitutive modelling

The basic premise of Continuum Damage Mechanics is that microstructural material damage (micro-cracks, microvoids, etc.) can be represented in a continuous fashion by introducing a damage variable...
as an additional field variable. In its most general form this damage variable is a tensorial quantity, but it is defined as a scalar variable $D$ here for simplicity, where $D = 0$ represents the initial, undamaged material and $D = 1$ a state of complete loss of material integrity. Taking into account the effect of damage, Hooke’s law becomes (see for instance [5])

$$\sigma = (1 - D) C : \varepsilon,$$

where $\sigma$ is the Cauchy stress tensor, $C$ the fourth-order elastic stiffness tensor and $\varepsilon$ the linear strain tensor. This constitutive relation can be applied to processes in which the growth of material damage is the predominant dissipation mechanism and which do not result in permanent deformations. Applications are therefore limited to quasi-brittle fracture and high-cycle fatigue.

For both application domains, the evolution of damage can be related to the deformation. A loading surface $f = 0$ is defined, similar to the yield surface in plasticity models, but formulated in terms of strains:

$$f(\varepsilon, \kappa) = \tilde{\varepsilon}(\varepsilon) - \kappa,$$

with $\tilde{\varepsilon}$ an equivalent strain measure and $\kappa$ a threshold variable. For strain states within the loading surface ($f < 0$) the response is fully elastic. The role of the loading surface in determining the growth of damage differs slightly for the quasi-brittle and fatigue damage models. In quasi-brittle damage the appropriate Kuhn-Tucker conditions for the history variable ensure that the loading surface grows such that strain states always remain on or within the loading surface ($f \leq 0$). Damage growth is only possible for increasing $\kappa$ (cf. elastoplasticity). In the fatigue model, on the other hand, the loading surface is kept fixed and the strain can also traverse the loading surface, in which case the damage variable can grow (cf. overstress viscoplasticity models); it is further assumed that damage can be progressive only for increasing deformation, i.e., for $\dot{f} > 0$ [4].

The relation between deformation and damage growth is given by the evolution law

$$\dot{D} = g(D, \tilde{\varepsilon}) \dot{\tilde{\varepsilon}}$$

if the appropriate growth conditions are satisfied; otherwise we have $\dot{D} = 0$. For quasi-brittle damage this evolution law is usually integrated, which, using the consistency relation $\dot{f} = 0$, results in an explicit expression for the damage variable in terms of the history parameter $\kappa$. However, the differential form (3) is retained here in order to keep the formulation consistent with the fatigue damage model.

Nonlocal and gradient formulations of the damage models follow from the local formulations by replacing the (local) equivalent strain measure $\tilde{\varepsilon}$ in (2) and (3) by the nonlocal equivalent strain $\bar{\varepsilon}$. In the nonlocal formulation $\bar{\varepsilon}$ is defined as a weighted volume average of $\tilde{\varepsilon}$ [6]. We will limit ourselves to the

![Figure 1: Typical damage evolution for the quasi-brittle and fatigue damage models.](image)
corresponding gradient model, in which $\bar{\varepsilon}$ is the solution of the partial differential equation [3, 7]

$$\bar{\varepsilon} - c \nabla^2 \bar{\varepsilon} = \tilde{\varepsilon}. \quad (4)$$

The parameter $c$, which has dimension length squared, introduces the scale of the microstructure of the material in the macroscopic constitutive model. This parameter determines the width of the band in which deformation and damage localise.

Typical curves of damage growth versus time for quasi-brittle and fatigue damage have been plotted in Figure 1. A homogeneous deformation has been assumed, so that the gradient enhancement is not active and the respective local and gradient models coincide. Furthermore, a linear increase of the deformation in time has been assumed for the brittle model, whereas the frequency and strain amplitude were kept constant for the fatigue model.

3 Localisation study

Localisation of deformation before crack initiation can be studied effectively in a one-dimensional setting. The possibility of the emergence of localised solutions is investigated by considering the stability of small perturbations of an otherwise homogeneous solution of the one-dimensional equilibrium equations. Note that this stability analysis can be regarded as the static equivalent of a linearised wave propagation analysis or dispersion analysis [3,8,9]. In fact, the wave numbers which follow from the stability analysis are equal to the critical wave numbers resulting from a fully dynamic analysis.

The equations which govern the deformation of a homogeneous bar loaded in tension can be written as a quasi-linear system

$$P(u) \frac{\partial u}{\partial t} + Q(u) \frac{\partial u}{\partial x} = r(u), \quad (5)$$

where, for the local damage model, the column matrices $u$ and $r$ are given by $u^T = [\varepsilon \ D]$ and $r = 0$ and the matrices $P$ and $Q$ by

$$P = \begin{bmatrix} 0 & 0 \\ -g(D, \varepsilon) & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} (1 - D)E & -E \varepsilon \\ 0 & 0 \end{bmatrix}. \quad (6)$$

For the gradient model Equation (4) can be included in the system by splitting it in two first-order equations (see [3]).

We now assume a solution of the form

$$u = u_0 + \hat{u} e^{ikx + yt}, \quad (7)$$

with $u_0$ a stationary, homogeneous solution and $\hat{u} e^{ikx + yt}$ a small harmonic perturbation with wave number $k$ and corresponding growth factor $\gamma$. Substitution of (7) into (5) and linearisation result in:

$$\left( \gamma P(u_0) + i k Q(u_0) - \left( \frac{\partial r}{\partial u} \right)_0 \right) \hat{u} e^{ikx + yt} = 0. \quad (8)$$

A nontrivial solution requires that

$$\text{det} \left( \gamma P(u_0) + i k Q(u_0) - \left( \frac{\partial r}{\partial u} \right)_0 \right) = 0, \quad (9)$$

which, for the local damage model, gives the dispersion relation

$$ik \gamma E(1 - D_0 - \varepsilon_0 g(D_0, \varepsilon_0)) = 0. \quad (10)$$

It can easily be verified that $E(1 - D_0 - \varepsilon_0 g(D_0, \varepsilon_0))$ is exactly the tangential stiffness for $\varepsilon = \varepsilon_0, \ D = D_0$. For the situation where this tangential stiffness is positive only a homogeneous perturbation
(\(k = 0\)) can grow and the solution thus remains uniform along the bar. However, when the tangential modulus becomes zero, all wave numbers are admissible and the corresponding growth factors are undetermined. The deformation and damage growth can then localise in a line or, in a numerical setting, in the smallest possible band that can be captured by the spatial discretisation. The steep initial growth of damage in the quasi-brittle model (Figure 1) results in softening – and thus in localisation – at the onset of damage. The situation is quite different in the fatigue model, which shows a slow initial growth and fast growth near the end of the fatigue life. Consequently, softening can only occur in the very last stages of fatigue damage growth. Indeed, if we set \(\epsilon_0\) to the strain amplitude \(\epsilon_a\), the evolution law used in Figure 1 gives a critical damage \(D_c \approx 0.9\) as a lower bound for localisation. Since most of the damage development has then already taken place, it is likely that the effects of localisation on crack initiation are less pronounced in fatigue damage than in the brittle model and that the need to introduce nonlocality in the constitutive model is less compelling. Indeed, fatigue crack initiation analyses of smooth specimens show no undesirable influence of the spatial discretisation.

The dispersion relation for the gradient model can be derived along the same lines as for the local model, which results in:

\[
 ikyE \left(1 - D_0 - \epsilon_0 g(D_0, \bar{\epsilon}_0) + k^2 \epsilon (1 - D_0) \right) = 0. \tag{11}
\]

As long as \(1 - D_0 - \epsilon_0 g(D_0, \bar{\epsilon}_0) > 0\), this equation can again only be satisfied by the trivial solution \(k = 0\). At the onset of softening, a nontrivial solution becomes possible with a wave number

\[
 k = \sqrt{\frac{1}{c} \left( \frac{\epsilon_0 g(D_0, \bar{\epsilon}_0)}{1 - D_0} - 1 \right)} \tag{12}
\]

The wave length \(\lambda = 2\pi/k\) associated to this wave number has been plotted as a function of the homogeneous damage level \(D_0\) in Figure 2 for the brittle and fatigue models. Since the wave length remains positive for \(D < 1\), the growth of deformation does not localise in a line. Instead, a band of damage growth is formed, the width of which is proportional to the square root of the gradient parameter \(c\). For \(D \to 1\) the width of the damage band goes to zero \((k \to \infty)\), which is necessary to have a gradual transition into a line crack.

![Figure 2: Unstable wave lengths for the gradient-enhanced damage models.](image)

### 4 Crack growth

Continuum damage models and related descriptions of material degeneration have been used to model not only damage growth before the initiation of a macrocrack, but also the growth of this crack. The latter aspect is sometimes called the ‘local approach to fracture’ [10]. In order to study the influence of the spatial discretisation on crack propagation, the fatigue damage formulation of Section 2 has been...
applied to a problem in which the propagation stage is dominant: a plate with a sharp notch in plane stress (Figure 3). The material parameters were set to those of a mild steel. The computation was carried out on the upper half of the specimen, with the appropriate symmetry conditions being imposed at the axis of symmetry. A constant amplitude displacement cycle was applied to the top clamp. Three finite element meshes of quadratic elements, with an increasingly finer discretisation in the relevant area have been used (Figure 4).

Figure 5(a) shows the crack length $a$ versus the number of cycles $N$ as computed with the local fatigue model, where the crack is defined as the set of Gauss points in which $D = 1$. The constant displacement amplitude boundary condition results in a constant growth rate for almost the entire propagation phase.
The magnitude of this growth rate, however, is influenced dramatically by the discretisation: a finer finite element grid results in a much faster growth of the crack. This trend, which has been observed before in creep damage problems [11, 12], can be understood by realising that the actual solution of the crack problem contains a strain singularity at the crack tip. Theoretically, this singularity results in an infinite damage growth rate at the crack tip and thus in an infinite crack propagation rate. When the finite element grid is refined, the numerical solution converges to this nonphysical solution. Similarly, the singularity at the notch tip results in the immediate initiation of a crack, which is observed in the numerical analyses as a vanishing initiation life [12].

In the gradient-enhanced model, the rate of damage growth is governed by the nonlocal strain instead of the local equivalent strain. Since this nonlocal equivalent strain is not singular at the crack tip, the damage and crack growth rates remain finite. The growth curves obtained with the gradient-enhanced formulation (Figure 5(b)) show that the crack growth rate indeed converges to a finite value. The initiation life is still rather sensitive to the element size, but this is due to the fact that the strain singularity at the notch tip is of a geometrical nature rather than the consequence of damage. A relatively coarse discretisation may result in an inaccurate description of the equivalent strain field near the crack tip, but the fact that the total deformation in the vicinity of the tip remains more or less the same means that the nonlocal equivalent strain field will nevertheless be quite accurate. At the notch tip, where the singularity is imposed by boundary constraints, this compensation does not exist and a finer mesh is thus needed in order to represent the nonlocal equivalent strain with sufficient accuracy.

5 Concluding remarks

The example of fatigue damage shows that numerical crack growth analyses may exhibit mesh dependence — or in fact convergence to a non-physical weak solution — even when localisation does not seem to be a critical issue. The introduction of spatial interaction terms (nonlocality or gradient enhancement) provides an effective remedy for this difficulty. It should be remarked that the influence of the spatial discretisation on predicted crack growth rates is not necessarily related to time-dependent models, but is also encountered in for instance quasi-brittle damage. In fact, it plays an important role in the vanishing fracture energy in any model where crack growth contributes significantly to the total energy dissipation.

References