Time dependent finite element analysis of the linear stability of viscoelastic flows with interfaces

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Abstract

In this paper we present a new time marching scheme for the time dependent simulation of viscoelastic flows governed by constitutive equations of differential form. Based on the ideas of Carvalho and Scriven [J. Comput. Phys. 151 (1999) 534], a domain perturbation technique is introduced that can be applied to viscoelastic flows with fluid/fluid interfaces or free surfaces. This work mainly focuses on the development and, consequently, benchmarking of finite element algorithms (FEM) that can efficiently handle the stability problems of complex viscoelastic flows. Since spurious or non-physical solutions are easily generated for this type of analysis using finite element techniques, both the new time stepping scheme and the domain perturbation technique are benchmarked in simple shear flows of upper convected Maxwell (UCM) fluids. Both single and two layer flows are considered for which the dominating mode and associated growth rate of a perturbation are solutions of the one-dimensional generalized eigenvalue problem (GEVP). We show that both the growth rate and the most dangerous eigenmode of the simple shear flows can be accurately captured by our transient algorithm.

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1. Introduction

Over the past decades, a number of numerical algorithms have been developed in order to be able to accurately and efficiently compute solutions of viscoelastic flows that exhibit both shear and elongation (i.e. complex flows). For instance, some of the most successful methods presently available employ the
streamline upwind Petrov–Galerkin (SUPG) method [1] or the discontinuous Galerkin (DG) method [2] to handle the hyperbolic constitutive equation, while lack of ellipticity of the momentum equations is resolved using the explicitly elliptic momentum equation (EEME) formulation of [3], the elastic viscous stress split (EVSS) formulation [4], or, more recently, discrete elastic viscous stress splitting (DEVSS) [5]. An extensive review on these mixed finite element methods was presented by Baaijens [6]. For the majority of these numerical techniques, emphasis is laid on the computation of steady state flow solutions. However, both from a material processing as well as a numerical point of view, simulation and prediction of transient viscoelastic flow phenomena probably represent the most challenging task of computational rheology today. For instance, the study of the stability of polymer solutions in complex flows, such as the flow around a cylinder [7,8] or the corrugated channel flow [9], has gained attention. Besides the study of complex flows of polymer solutions, there is an increasing interest in the numerical investigation of polymer melt flows that are more relevant to the polymer processing industry [10–13].

The development of numerical tools that are able to handle these dynamic flow phenomena (i.e. the transition from steady to transient flows as dictated by the stability problem) requires a number of important issues to be addressed properly. First of all, the constitutive equation that relates the polymer stress to the fluid deformation (and deformation history) needs to be defined. While most present research focused on the stability behavior of the upper convected Maxwell (UCM) model (e.g. [8,14,15]), there has been much less interest on how the choice of constitutive model affects the predicted stability of a given polymer melt flow [10,12,16–18]. This is a very important issue when the dynamics of polymer melt flows are predicted using numerical simulations. In two recent papers [17,18] we have shown that this is not a trivial task since generally accepted models for polymer melts can behave very different in terms of their stability characteristics.

In this paper we will address the second issue which involves the development of the computational techniques. We need to develop a numerical method (i.e. the discretized problem) that is able to predict the correct dynamics of the viscoelastic operator. In an earlier paper by Grillet et al. [13] we have set it our goal to study the stability behavior of complex processing flows with particular emphasis on the fountain flow problem as it occurs during injection molding. In this paper, the analysis of the stability behavior was limited to a fixed computational domain. This means that deformations of the domain which are the result of perturbations of the fountain flow surface were not included in the numerical analysis. To further complete the stability analysis of this type of complex viscoelastic flow, the domain perturbation should be taken into account. Hence, the goal of this paper is to develop an efficient numerical method that can to predict the correct linear dynamics of complex viscoelastic flows with interfaces.

There are some basic requirements the applied numerical scheme should meet. First, stable spatial finite element discretizations should be applied for the time-dependent stability problems. This is illustrated by the fact that numerical approximations that are not solutions of the continuous problem (i.e. spurious solutions) are easily generated for different spatial discretization schemes [19]. Second, in an effort to model real polymer melts, multimode models require the use of dedicated and efficient time marching schemes and finally, the governing set of equations should take the perturbation of the domain boundaries into account. In the sequel of this paper, we will develop an efficient numerical technique for time dependent problems that is suitable for complex flows with interfaces and benchmark this method for single- and multilayer shear flows for which the normal mode expansion provides an ‘exact’ solution of the linear stability problem.
2. Flows and governing equations

Here, we consider incompressible and isothermal creeping flows of viscoelastic fluids. These flows can be described by a set of equations which include the conservation of momentum:

$$\nabla \cdot \sigma = \vec{0},$$

(1)

and the conservation of mass:

$$\nabla \cdot \vec{u} = 0,$$

(2)

with $\nabla$ the gradient operator and $\vec{u}$ the velocity field. The Cauchy stress tensor $\sigma$ is defined as:

$$\sigma = -pI + \tau,$$

(3)

with isotropic pressure $p$ and extra stress tensor $\tau$. Within the scope of this work, a sufficiently general way to describe the evolution of the extra stress tensor is obtained by using a single-mode differential form constitutive equation:

$$\begin{align*}
\lambda \vec{\tau} + f(\tau) &= \eta(L + L^T),
\end{align*}$$

(4)

with relaxation time $\lambda$, viscosity $\eta$ and deformation gradient tensor $L = \nabla \vec{u}^T$. The upper convected time derivative of the extra stress tensor is defined as:

$$\begin{align*}
\tau = \frac{\partial \tau}{\partial t} + \vec{u} \cdot \nabla \tau - L \cdot \tau - \tau \cdot L^T.
\end{align*}$$

(5)

The upper convected Maxwell model is obtained for $f(\tau) = \tau$. Other models are obtained for different functions for $f(\tau)$ which include, for instance, the Giesekus model, the Phan–Thien–Tanner (PTT) models (without mixed upper and lower convected derivatives) and the recently proposed eXtended Pom–Pom (XPP) model of [20].

In order to assess the temporal stability characteristics of our operator splitting method as will be discussed in the next section, we investigate the viscometric shear flows as depicted in Fig. 1. The planar Couette and the planar Poiseuille flow are considered for both single fluid systems as well as multilayer systems consisting of two separate fluids. For these multilayer flows there is an additional equation that

![Diagram of planar shear flows](image)

**Fig. 1.** Planar shear flows. Couette flow (left) is driven by movement of one (or both) walls. Poiseuille flow (right) is driven by a stream wise pressure gradient. The distance between the parallel plates is defined as $H$ whereas $H_1$ defines the position of the interface relative to the bottom plate.
governs the deformation of the fluid/fluid interface. Here, we consider the displacement of the interface in normal direction:

\[
\vec{n} \cdot \frac{\partial \vec{x}}{\partial t} = \vec{n} \cdot \vec{u},
\]

(6)

where \(\vec{n}\) denotes the interface normal vector and \(\vec{x}\) the local position vector of the interface. Without loss of generality, we limit ourselves to application of the UCM model. It is important to note that there exist a fairly large amount of literature on the stability behavior of the UCM model in simple shear flows (see, e.g. Gorodtsov and Leonov [14], Ho and Denn [10], Renardy and Renardy [21], Lee and Finlayson [22], Wilson et al. [23]).

In the remainder of this paper, the flows have been non-dimensionalized using a characteristic length scale (\(H/2\)) and the mean viscosity of the two fluid layers. The dimensionless flow strength of the different layers is then expressed by the viscoelastic Weissenberg number (\(We\)) which is defined as:

\[
We_i = \frac{\lambda_i V}{H},
\]

(7)

for Couette flow and:

\[
We_i = \frac{2\lambda_i Q}{H^2},
\]

(8)

for Poiseuille flow with \(Q\) the total imposed flow rate and \(i \in \{1, 2\}\). Furthermore, a dimensionless interface position is defined by \(\epsilon = H_1/H\).

3. Computational methods

To capture the temporal stability behavior of the viscoelastic flows, two intrinsically different computational methods are considered. A more or less standard one-dimensional spectral method is used to generate a generalized eigenvalue problem (GEVP) of the viscometric flows. This expansion into so called normal modes will serve as an ‘exact’ solution of the linear stability problems that are presented here. The method that will be developed and benchmarked against solutions of the eigenvalue problem is a finite element method (FEM) with temporal evolution based on an operator splitting method (\(\Theta\)-scheme) in order to efficiently integrate the governing equations in time. Special emphasis is paid to the flows of superposed fluids where the domains of the separate fluids should be able to deform as a consequence of disturbances of the fluid/fluid interface. Hence, a domain perturbation technique in relation with the linear stability characteristics of the bulk flow will also be discussed. For both the GEVP and the FEM the general approach for the linear stability analysis is as follows. The analysis requires an expansion of the governing equations on the computational domain in which only first order terms of the perturbation variables are retained. Hence, neglecting higher order terms, we may express the physical variables as the sum of the steady state and perturbed values. For instance, we can write for the polymeric stress:

\[
\tau(\vec{x}, t) = \bar{\tau}(\vec{x}) + \delta\tau(\vec{x}, t),
\]

(9)

where \(\bar{\tau}\) denotes the steady state value whereas \(\delta\tau\) denotes the perturbation value of the extra stress. In the sequel of this paper we will omit the \(\delta\) in front of the perturbation variables. For instance, \(\tau\) will denote the perturbed stress.
3.1. Spectral eigenvalue analysis

In order to obtain a discrete approximation of the eigenspectrum of the multilayer flows as depicted in Fig. 1, we discretize the governing variables into coefficients of a truncated Chebyshev expansion up to order $N$. To yield a complete set of equations we use a Chebyshev–Tau method as described in Gottlieb and Orszag [24] and which has previously been used in Bogaerds et al. [18] for single system shear flows. The stability behavior is examined by introducing small normal mode perturbations ($\hat{\delta}$) to the base flow:

$$\hat{\delta}(x, y, t) = \delta(y)e^{ikx + \sigma t},$$

(10)

where $\hat{\delta}$ are the perturbations of the polymer stress, pressure and the velocity. The perturbation of the interface position ($\hat{h}$, in the direction normal to the interface) is described by:

$$\hat{h}(x, t) = h e^{ikx + \sigma t},$$

(11)

with $k$ a real wavenumber and $\sigma$ the complex eigenvalue associated with the eigenmode $(\delta(y), h)$. Substitution of Eqs. (10) and (11) into the governing equations and retaining only first order terms of the perturbation variables yields a complete set of equations for the bulk flow:

$$\lambda [(\sigma + i\kappa \tilde{u}_x)\tau_{xx} - 2\tilde{u}'_x \tau_{xy} + i\kappa (\tilde{r}'_{xx}u_y - 2\tilde{r}_{xx}u'_y + \tau_{xx} - 2\eta i\kappa u_x = 0, \quad (12)$$

$$\lambda [(\sigma + i\kappa \tilde{u}_x)\tau_{yy} - 2i\kappa \tilde{r}_{xy}u_y] + \tau_{yy} - 2\eta u'_y = 0, \quad (13)$$

$$\lambda [(\sigma + i\kappa \tilde{u}_x)\tau_{xy} - \tilde{u}'_x \tau_{yy} + (\tilde{r}'_{xy} + i\kappa \tilde{r}_{xx})u_y] + \tau_{xy} - \eta (u_x + i\kappa u_y) = 0, \quad (14)$$

$$i\kappa(\tau_{xx} - p) + \tau_{xy}' = 0, \quad (15)$$

$$\tau_{yy}' - p' + i\kappa \tau_{xy} = 0, \quad (16)$$

$$i\kappa u_x + u'_y = 0, \quad (17)$$

and a description for the perturbation of the interface position:

$$(\sigma + i\kappa \tilde{u}_x)h - u_y = 0. \quad (18)$$

The base flow solution is denoted by the tildes and gradients in the wall normal direction are denoted by the primes. Both on the confining walls and on the fluid/fluid interface, boundary conditions need to be imposed. These boundary conditions are the no-slip conditions ($u_x = u_y = 0$) on the walls and continuity of interfacial velocity, normal stress and shear stress. The correct linearizations of these boundary conditions were first defined by Chen [25] and are continuity of $x$-velocity:

$$[[u_x]] + [[\tilde{u}'_x]]h = 0, \quad (19)$$

continuity of $y$-velocity:

$$[[u_y]] = 0, \quad (20)$$

continuity of normal stress:

$$[[\tau_{xy} + i\kappa(\tau_{xx} - \tau_{yy})]] = 0. \quad (21)$$
and continuity of shear stress on the interface:

\[ \text{i} \kappa ([\tilde{\tau}_{xx} - \tilde{\tau}_{yy}])h - [\tau_{xy}] = 0, \]  

(22)

with \([\{\alpha\}]\) the jump operator of \(a\) over the interface. If the bulk flow perturbation variables are defined as \(\delta(y) = (u_x, u_y, p, \tau_{xx}, \tau_{yy}, \tau_{xy})(y)\) and the necessary boundary conditions are imposed, a generalized eigenvalue problem is obtained:

\[ A \left[ \begin{array}{c} \delta \\ h \end{array} \right] = \sigma M \left[ \begin{array}{c} \delta \\ h \end{array} \right], \]  

(23)

which is solved using a QZ-algorithm. The Chebyshev–Tau method as applied here, has shown to produce the correct eigenspectra for several benchmark flows [17,18]. The computed eigenspectrum for similar flow conditions as those used by Wilson et al. [23] (Fig. 9) is shown in Fig. 2. It can be seen that there is excellent agreement with the spectrum computed in [23].

### 3.2. Finite element analysis

One of the major concerns when the linear stability behavior of viscoelastic flows is investigated by means of temporal integration of the governing equations using a finite element method is the appearance of ‘spurious’ solutions that result from the spatial discretization of the differential operator. For the 1D-GEVP, these artifacts of the numerical solution methods are fairly well documented [18,21,26]. However, on complex flows, for which our FEM has its practical relevance, the amount of literature is far less extensive [9,19,27,28].

We will first describe our FEM on stationary domains before moving on to domains that do not need to remain stationary due to the presence of free surfaces or fluid/fluid interfaces. Hence, if the governing set of linearized equations are considered:

\[ \lambda \left( \frac{\partial \tau}{\partial t} + \bar{u} \cdot \nabla \tau + \bar{u} \cdot \nabla \tilde{\tau} - \tilde{G} \cdot \tau - \tau \cdot \tilde{G}^T - G \cdot \tilde{\tau} - \tilde{\tau} \cdot G^T \right) + \tau - \eta(G + G^T) = 0, \]  

(24)

\[ -\nabla \cdot (\tau + \alpha(\nabla \bar{u} - G^T)) + \nabla p = \bar{p}, \]  

(25)

\[ \nabla \cdot \bar{u} = 0, \]  

(26)

\[ G^T - \nabla \bar{u} = 0, \]  

(27)
with $G$ the auxiliary velocity gradient tensor and $\alpha$ the stabilizing parameter as defined for the DEVSS method [5]. Again, the tildes denote the base flow variables. In order to obtain an efficient time marching scheme for Eqs. (24)–(27) we consider an operator splitting method to perform the temporal integration. The major advantage of operator splitting methods is the decoupling of the viscoelastic operator into parts which are ‘simpler’ and can be solved more easily than the full problem. Due to the stiffness of the viscoelastic stability problem as defined above, it is worthwhile to consider a $\Theta$-scheme to perform the temporal integration [29,30]. Hence, if we write Eqs. (24)–(27) as:

$$\frac{dx}{dt} = A(x) = A_1(x) + A_2(x),$$

the $\Theta$-scheme is defined following [30]:

$$\frac{x^{n+\theta} - x^n}{\Theta \Delta t} = A_1(x^{n+\theta}) + A_2(x^n),$$

$$\frac{x^{n+1-\theta} - x^{n+\theta}}{(1 - 2\Theta) \Delta t} = A_1(x^{n+\theta}) + A_2(x^{n+1-\theta}),$$

$$\frac{x^{n+1} - x^{n+1-\theta}}{\Theta \Delta t} = A_1(x^{n+1}) + A_2(x^{n+1-\theta}),$$

with time step $\Delta t$ and $\Theta = 1 - 1/\sqrt{2}$ in order to retain second order accuracy. Formally, only the constitutive equation and the perturbation equation for the interface contain the temporal derivatives which implies that the first term of Eq. (28) should be multiplied by a diagonal operator with the only non-zero entrees being the ones corresponding to these equations. The remaining problem is to define the separate operators $A_1$ and $A_2$. In essence, we like to choose $A_1$ and $A_2$ in such a way that solving Eqs. (29)–(31) requires far less computational effort as compared to solving the implicit problem while the stability envelope of the time integrator remains sufficiently large. If the simplified problem $A_1 = \beta A$ and $A_2 = (1 - \beta)A$ is considered, the stability envelopes are plotted in Fig. 3 for different values of $\beta$. The arrows point towards the region of the complex plane for stable time integration. Obviously, setting $\beta = 0$ yields only a small portion of the complex plane whereas $0.5 \leq \beta \leq 1.0$ results in a scheme that is unconditionally stable. Based on the argument that we split the viscoelastic operator into a kinematic problem and a transport problem for the advection of polymer stress, we can define $A_1$ and $A_2$ from the approximate location of the governing eigenvalues. For instance, the viscous (Stokes) problem has eigenvalues which are essentially real and negative. The absolute value has the tendency to grow very fast with mesh refinement (for a one-dimensional diffusion problem using a low order FEM, $\max(\mu) = O(N^2)$ with $N$ the number of grid points) and it is convenient to define $A_1$ as the kinematic problem for given polymer stress. On the other hand, the eigenspectrum of the remaining advection operator in the constitutive equation is located close to the imaginary axis (however not on the imaginary axis due to the introduction of Petrov–Galerkin weighting functions later on) and we define $A_2$ as the transport of extra stress. We can then define our $\Theta$-scheme ($A_1$ and $A_2$) for the temporal evolution of polymer stress.
Fig. 3. Stability envelope of the \( \Theta \)-scheme for \( A_1 = \beta A \) and \( A_2 = (1 - \beta)A \) with \( \mu \) the spectrum of \( A \). Regions for stable time integration are indicated by the arrows for different values of \( \beta \) and it can be seen that for \( 0.5 \leq \beta \leq 1.0 \) this \( \Theta \)-scheme is unconditionally stable for this special choice of \( A_1 \) and \( A_2 \).

perturbations as:

\[
A_1 = -\begin{bmatrix}
\lambda (\ddot{\vec{u}} \cdot \nabla \vec{\tau} - \vec{G} \cdot \vec{\tau} - \vec{\tau} \cdot \vec{G}^T) + \tau - \eta(G + G^T) \\
-\nabla \cdot (\tau + \alpha(\nabla \ddot{\vec{u}} - \vec{G}^T)) + \nabla p \\
\nabla \cdot \dddot{\vec{u}} \\
G^T \cdot \ddot{\vec{u}}
\end{bmatrix},
\]

(32)

and:

\[
A_2 = -\begin{bmatrix}
\lambda (\ddot{\vec{u}} \cdot \nabla \dot{\vec{\tau}} - \vec{G} \cdot \dot{\vec{\tau}} - \dot{\vec{\tau}} \cdot \vec{G}^T) \\
0 \\
0 \\
0
\end{bmatrix},
\]

(33)

which represent the linearized stability equations as defined by Eqs. (24)–(27). Based on the above definitions for \( A_1 \) and \( A_2 \), the kinematic (elliptic saddle point) problem for \( \dddot{\vec{u}}, p, \vec{G} \) is updated implicitly in the first (Eq. (29)) and last (Eq. (31)) step of the \( \Theta \)-scheme whereas the transport of polymeric stress is updated explicitly. Conceptually, this operator splitting is very similar to the \( \Theta \)-scheme that was developed by [31] and later applied to study the linear dynamics of complex viscoelastic flows by [7,8]. The difference being the fact that the term:

\[
\lambda (\ddot{\vec{u}} \cdot \nabla \vec{\tau} - \vec{G} \cdot \vec{\tau} - \vec{\tau} \cdot \vec{G}^T) - \eta(G + G^T),
\]

(34)

is now contained within \( A_1 \) rather than \( A_2 \). The redefinition of \( A_1 \) and \( A_2 \) is motivated by the fact that the earlier \( \Theta \)-schemes can only be applied to assess the linear stability dynamics of viscoelastic flows with a sufficiently large purely Newtonian contribution to the extra stress. For real polymer melts this generally is an unacceptable modification of the material description.
Notice that, based on Eq. (32) there is hardly any gain in computational efficiency since the constitutive relation cannot be decoupled from the remaining equations in a weighted residual formulation. However, the updated polymeric stress at $t = t^{n+\Theta}$ can be written as:

$$\tau(t^{n+\Theta}) = P^{-1} \left[ \frac{\lambda}{\Theta \Delta t} \tau(t^n) - F_1(\tilde{u}(t^{n+\Theta}), G(t^{n+\Theta})) - F_2(\tau(t^n)) \right],$$  

(35)

with:

$$P = \frac{\lambda + \Theta \Delta t}{\Theta \Delta t} I,$$

(36)

for the UCM model. The tensor functionals $F_1$ and $F_2$ depend only the kinematics of the flow at $t^{n+\Theta}$ and the polymeric stress at $t^n$:

$$F_1(\tilde{u}(t^{n+\Theta}), G(t^{n+\Theta})) = \lambda (\tilde{u} \cdot \nabla \tilde{\tau} - \tilde{G} \cdot \tilde{\tau} - \tilde{\tau} \cdot G^T) - \eta (G + G^T),$$  

(37)

and:

$$F_2(\tau(t^n)) = \lambda (\tilde{u} \cdot \nabla \tau - \tilde{G} \cdot \tau - \tau \cdot G^T).$$  

(38)

Substitution of Eqs. (35) into (32) yields a modified momentum equation in which the only degrees of freedom are the kinematics of the flow at $t^{n+\Theta}$. The kinematics of the flow and the constitutive relation can then be solved separately. If the finite element approximation spaces for $(\tilde{u}, \tau, p, G)$ are defined by $(U^h, T^h, P^h, G^h)$, the complete weak formulation of our $\Theta$-scheme can be written as follows:

**Problem $\Theta$-FEM step 1a.** Given the base flow $(\tilde{u}, \tilde{\tau}, \tilde{G})$ and $\tilde{\tau} = \tau(t^n)$, find $\tilde{u} \in U^h$, $p \in P^h$ and $G \in G^h$ at $t = t^{n+\Theta}$ such that for all admissible test functions $\Phi_u \in U^h$, $\Phi_p \in P^h$ and $\Phi_G \in G^h$,

$$\left( \nabla \tilde{u}, P^{-1} \left[ \frac{\lambda}{\Theta \Delta t} \tilde{\tau} - F_1(\tilde{u}, G) - F_2(\tau(t^n)) \right] + \alpha (\nabla \tilde{u} - G^T) \right) - (\nabla \cdot \Phi_u, p) = 0,$$

(39)

$$(\Phi_p, \nabla \cdot \tilde{u}) = 0,$$

(40)

$$(\Phi_G, G^T - \nabla \tilde{u}) = 0,$$

(41)

with $(, , )$ the usual $L_2$-inner product on the (stationary) domain $\Omega$. The kinematics of the flows are solved separately from the constitutive relation and an update for the polymeric stress is now readily obtained from:

**Problem $\Theta$-FEM step 1b.** Given the base flow $(\tilde{u}, \tilde{\tau}, \tilde{G})$ and $\tilde{\tau} = \tau(t^n)$, $\tilde{u} = \tilde{u}(t^{n+\Theta})$ and $G = G(t^{n+\Theta})$ find $\tau \in T^h$ at $t = t^{n+\Theta}$ such that for all admissible test functions $\Phi_\tau \in T^h$,

$$\left( \Phi_\tau + \frac{h^\tau}{|\tilde{u}|} \tilde{u} \cdot \nabla \Phi_\tau, \lambda \left( \frac{\tau - \tilde{\tau}}{\Theta \Delta t} + \tilde{u} \cdot \nabla \tilde{\tau} + \tilde{u} \cdot \nabla \tilde{\tau} - \tilde{G} \cdot \tilde{\tau} - \tilde{\tau} \cdot G^T - G \cdot \tilde{\tau} - \tilde{\tau} \cdot G^T \right) \right)$$

$$+ \tau - \eta (G + G^T) = 0.$$  

(42)

The second step involves the solution of the transport problem:
**Problem Θ-FEM step 2.** Given the base flow \( (\tilde{u}, \tilde{\tau}, \tilde{G}) \) and \( \tilde{\tau} = \tau(t^n + \Theta), \tilde{u} = \bar{u}(t^n + \Theta) \) and \( \tilde{G} = G(t^n + \Theta) \)

find \( \tau \in \mathcal{T}_h \) at \( t = t^n + 1 - 2\Theta / \Delta t \) such that for all admissible test functions \( \Phi_\tau \in \mathcal{T}_h \),

\[
\left( \Phi_\tau + \frac{h^c}{|\tilde{u}|} \bar{u} \cdot \nabla \Phi_\tau, \lambda \left( \frac{\tau - \tilde{\tau}}{(1 - 2\Theta)\Delta t} + \tilde{\tau} \cdot \nabla \tilde{\tau} + \tilde{\tau} \cdot \nabla \tau - \tilde{G} \cdot \tau - \tau \cdot \tilde{G}^T - \tilde{G} \cdot \tilde{\tau} - \tilde{\tau} \cdot \tilde{G}^T \right) \right) + \eta (\tilde{G} + \tilde{G}^T) = 0. \tag{43}
\]

The third fractional step (Eq. (3)) corresponds to symmetrization of the Θ-scheme and is similar to step 1. In the above weak formulation, additional stabilization of the hyperbolic constitutive equation is obtained by inclusion of SUPG weighting functions [32] using some characteristic grid size \( h^c \).

The above decoupling of the constitutive relation from the remaining equations provides a very efficient time integration technique which is second order accurate for linear stability problems. The efficiency becomes even more evident when real viscoelastic fluids are modeled for which the spectrum of relaxation times is approximated by a discrete number of viscoelastic modes. For simplicity, the procedure is described for the UCM model. However, if nonlinear models like the PTT, Giesekus or the XPP model are considered as is common for polymer melts, a generalization of the θ-scheme is readily obtained since this requires only a redefinition of \( P \) in Eq. (36) and \( P^{-1} \) can be evaluated either analytically or numerically.

A choice remains to be made for the approximation spaces \( (U^h, T^h, P^h, G^h) \). As is known from solving Stokes flow problems, velocity and pressure interpolations cannot be chosen independently and need to satisfy the LBB condition. Likewise, interpolation of velocity and extra stress has to satisfy a similar compatibility condition. We report calculations using low order finite elements using similar spatial discretizations as were defined by [19, 27] (continuous bi-linear interpolation for viscoelastic stress, pressure and \( G \) and continuous bi-quadratic interpolation for velocity).

Evidently, the set of equations should be supplemented with suitable boundary conditions on parts of the boundary \( (\Gamma) \) of the flow domain. On both walls Dirichlet conditions are enforced on the velocity to account for the no-slip conditions on the fluid/solid interface. Also, the in- and outflow boundaries of the domain are considered to be periodic. The boundary conditions on the fluid/fluid interface and the deformation of both fluid domains will be described in the next section.

### 3.3. Domain perturbation technique

The Θ-method described in Section 3.2 is discussed for flows defined on stationary domains. As a consequence the important class of free surface flows and flows with internal fluid/fluid interfaces cannot be simulated yet. Due to the deformation of the interfaces as a result of the perturbation of the flow, the computational domain should also be allowed to deform according to the interface perturbation. Hence, next to the perturbation variables defined previously \( (\tilde{u}, \tau, p, \tilde{G}) \) also the perturbation of the computational domain \( \tilde{x} \) should be considered. However, the nature of linear stability analysis implies that the disturbances are infinitely small and the domain deformation can be localized to the interface. Hence, in order to account for these disturbances it is not necessary to consider the full computational domain but rather the part of the domain near the interface. In the sequel, we will use a domain perturbation technique that was derived by [33] for the prediction of the (Newtonian) ribbing instability observed in deformable roll coating.
If the base flow domain \((\Omega_0, \Gamma_0)\) is considered, as schematically depicted in Fig. 4, and allow the domain to be perturbed into \((\Omega, \Gamma)\), the governing equations are solved on the perturbed domain. For instance:

\[
\int_{\Omega} (\tilde{a} + a) \, d\Omega = \int_{\xi, \eta} (\tilde{a} + a) J \, d\xi \, d\eta, \tag{44}
\]

in which the tilde denotes the base flow and \(J\) the Jacobian of the mapping:

\[
J = \frac{\partial (\tilde{x} + x)}{\partial \xi} \frac{\partial (\tilde{y} + y)}{\partial \eta} - \frac{\partial (\tilde{x} + x)}{\partial \eta} \frac{\partial (\tilde{y} + y)}{\partial \xi}, \tag{45}
\]

which equals approximately:

\[
J \approx J_0 + \left( \frac{\partial x}{\partial \xi} \frac{\partial \tilde{y}}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial \tilde{y}}{\partial \xi} \right) + \left( \frac{\partial \tilde{x}}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial \tilde{x}}{\partial \eta} \frac{\partial y}{\partial \xi} \right), \tag{46}
\]

when only linear terms are retained. Hence, linearization of Eq. (44) yields:

\[
\int_{\Omega_0} \tilde{a} \, d\Omega + \int_{\Omega_0} a \, d\Omega + \int_{\xi, \eta} \tilde{a} \left[ \left( \frac{\partial x}{\partial \xi} \frac{\partial \tilde{y}}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial \tilde{y}}{\partial \xi} \right) + \left( \frac{\partial \tilde{x}}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial \tilde{x}}{\partial \eta} \frac{\partial y}{\partial \xi} \right) \right] \, d\xi \, d\eta. \tag{47}
\]

The first term of Eq. (47) corresponds to the base state equations which is zero whereas the second term is part of the conventional stability equations (Eqs. (39)–(43)). The perturbation of the domain is restricted to the interface by the introduction of the function \(H^\delta\) which equals 1 on the interface and vanishes
smoothly inside the domain (Fig. 5). With the definition:
\[
\tilde{x} = hH^0\xi_0 = hH^0\left(\frac{\partial \tilde{y}}{\partial \Gamma_0}, -\frac{\partial \tilde{x}}{\partial \Gamma_0}\right)
\]
with \(H^0 = \lim_{\delta \to 0} H^\delta\),

the domain perturbation is governed by the disturbance of the interface in normal direction multiplied by \(H^0\). Eq. (47) can now be simplified significantly since the definition of \(H^0\) implies:

\[
\int_\xi H^0 d\xi = 0 \quad \text{and} \quad \int_\xi \frac{\partial H^0}{\partial \xi} d\xi = \pm 1.
\]

Effectively, this means that linearization of Eq. (44) now reads:

\[
\int_{\Omega} \partial(\tilde{a} + a) \partial(\tilde{x} + x) d\Omega \approx \int_{\Omega_0} a d\Omega + \int_{\Gamma_0} \tilde{a} h d\Gamma;
\]

and the last integral has been reduced to a boundary integral on the interface with the addition of the scalar function \(h\) (i.e. perturbation in normal direction).

Similar to Eq. (44), we can write:

\[
\int_{\Omega} \frac{\partial(\tilde{a} + a)}{\partial(\tilde{x} + x)} d\Omega = \int_{\xi,\eta} \left[\frac{\partial(\tilde{a} + a)}{\partial \xi} \frac{\partial(\tilde{y} + y)}{\partial \eta} - \frac{\partial(\tilde{a} + a)}{\partial \eta} \frac{\partial(\tilde{y} + y)}{\partial \xi}\right] d\xi d\eta,
\]

using the definition of spatial derivatives for isoparametric mappings:

\[
\frac{\partial a}{\partial x} = \frac{1}{J} \left[\frac{\partial a}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial a}{\partial \eta} \frac{\partial y}{\partial \xi}\right].
\]

Retaining only the first order terms, Eq. (51) now reduces to:

\[
\int_{\Omega} \frac{\partial(\tilde{a} + a)}{\partial(\tilde{x} + x)} d\Omega \approx \int_{\Omega_0} \frac{\partial \tilde{a}}{\partial \tilde{x}} d\Omega + \int_{\Omega_0} \frac{\partial a}{\partial \tilde{x}} d\Omega + \int_{\Gamma_0} h \frac{\partial \tilde{x}}{\partial \Gamma} \frac{\partial \tilde{a}}{\partial \Gamma} d\Gamma,
\]

where the first term of the right-hand side corresponds to the base flow solution and therefore vanishes from the stability equations. The extra contribution due to the perturbation of the computational domain is again reduced to a boundary integral over the stationary interface.

In order to arrive at a complete set of stability equations, the governing equations are expanded using the technique described above. The extra approximation space \((\mathcal{H}^l)\) required for the unknown function
Problem $\Theta$-FEM step 1*. Given the base flow $\tilde{u}, \tilde{r}, \tilde{G}$ on $\Omega_0$, $\tilde{r} = \tau(t^\theta)$ and $\tilde{h} = h(t^\theta)$, find $\tilde{u} \in U^h$, $p \in P^h$, $G \in G^h$ and $h \in H^l$ at $t = t^{n+\theta}$ such that for all admissible test functions $\tilde{\Phi}_u \in U^h$, $\tilde{\Phi}_p \in P^h$, $\tilde{\Phi}_G \in G^h$ and $\tilde{\Phi}_h \in H^l$, the transport problem as defined by Eq. (43) is obtained from:

$$\{\text{LHS Eq. (39)}\} - \int_{\Gamma_0} \nabla \tilde{\Phi}_u^T : P^{-1} \cdot \left[ \lambda \left( \tilde{u}_x \frac{\partial \tilde{\tau}}{\partial \Gamma} + \tilde{u}_y \frac{\partial \tilde{\tau}}{\partial \Gamma} \right) \frac{\partial \tilde{\tau}}{\partial \Gamma} + \tilde{\mathcal{L}} \right] h \, d\Gamma$$

$$+ \int_{\Gamma_0} \left( \frac{\partial \tilde{x}}{\partial \Gamma} \frac{\partial \tilde{\Phi}_u}{\partial \Gamma} \tilde{e}_x + \frac{\partial \tilde{y}}{\partial \Gamma} \frac{\partial \tilde{\Phi}_u}{\partial \Gamma} \tilde{e}_y \right) + \tilde{\mathcal{L}} h \, d\Gamma$$

$$- \int_{\Gamma_0} \left( \frac{\partial \tilde{x}}{\partial \Gamma} \frac{\partial \tilde{\Phi}_u}{\partial \Gamma} + \frac{\partial \tilde{y}}{\partial \Gamma} \frac{\partial \tilde{\Phi}_u}{\partial \Gamma} \right) \tilde{\mathcal{L}} h \, d\Gamma = 0,$$  

$$\{\text{LHS Eq. (40)}\} + \int_{\Gamma_0} \tilde{\Phi}_p \left( \frac{\partial \tilde{x}}{\partial \Gamma} \frac{\partial \tilde{u}_x}{\partial \Gamma} + \frac{\partial \tilde{y}}{\partial \Gamma} \frac{\partial \tilde{u}_y}{\partial \Gamma} \right) h \, d\Gamma = 0,$$  

$$\int_{\Gamma_0} \tilde{\Phi}_h \left[ \frac{h - \tilde{h}}{\Theta \Delta t} + \left( \tilde{u}_x \frac{\partial \tilde{h}}{\partial \Gamma} + \tilde{u}_y \frac{\partial \tilde{h}}{\partial \Gamma} \right) \frac{\partial \tilde{h}}{\partial \Gamma} - \vec{n}_0 \cdot \tilde{u} \right] \, d\Gamma = 0,$$  

with LHS the left-hand sides of the original $\Theta$-equations evaluated on the stationary domain ($\Omega_0$). The Lagrangian residual of the stationary constitutive relation is denoted by $\tilde{\mathcal{L}}$ and defined as:

$$\tilde{\mathcal{L}} = -\lambda (\tilde{G} \cdot \tilde{r} + \tilde{r} \cdot \tilde{G}^T) + \tilde{r} - \eta (\tilde{G} + \tilde{G}^T).$$  

Eqs. (54)–(56) are supplemented by Eq. (41) which has no additional terms under the conditions discussed in the next paragraph. Eq. (56) is the linearized kinematic condition on the interface. The weak form of the constitutive equation in step 1$^h$ now reads:

Problem $\Theta$-FEM step 1$^h$. Given the base flow $\tilde{u}, \tilde{r}, \tilde{G}$ on $\Omega_0$, $\tilde{r} = \tau(t^\theta)$ and $\tilde{h} = h(t^\theta)$ find $\tau \in \mathcal{T}^h$ at $t = t^{n+\theta}$ such that for all admissible test functions $\Phi_\tau \in \mathcal{T}^h$,

$$\{\text{LHS Eq. (42)}\} + \int_{\Gamma_0} \left( \Phi_\tau + \frac{h^e}{|\tilde{u}|} \tilde{u} \cdot \nabla \Phi_\tau \right) \left[ \lambda \left( \tilde{u}_x \frac{\partial \tilde{\tau}}{\partial \Gamma} + \tilde{u}_y \frac{\partial \tilde{\tau}}{\partial \Gamma} \right) \frac{\partial \tilde{\tau}}{\partial \Gamma} + \tilde{\mathcal{L}} \right] h \, d\Gamma = 0.$$  

Notice that in the above formulation, all kinematics including the perturbation of the interface, are governed by the operator $\mathcal{A}_3$.

A solution of the transport problem as defined by Eq. (43) is obtained from:

Problem $\Theta$-FEM step 2. Given the base flow $\tilde{u}, \tilde{r}, \tilde{G}$ on $\Omega_0$, $\tilde{r} = \tau(t^{n+\theta})$ and $\tilde{h} = h(t^{n+\theta})$, find $\tau \in \mathcal{T}^h$ and $h \in H^l$ at $t = t^{n+1-\theta}$ such that for all admissible test functions $\Phi_\tau \in \mathcal{T}^h$.

$$\{\text{LHS Eq. (42)}\} + \int_{\Gamma_0} \left( \Phi_\tau + \frac{h^e}{|\tilde{u}|} \tilde{u} \cdot \nabla \Phi_\tau \right) \left[ \lambda \left( \tilde{u}_x \frac{\partial \tilde{\tau}}{\partial \Gamma} + \tilde{u}_y \frac{\partial \tilde{\tau}}{\partial \Gamma} \right) \frac{\partial \tilde{\tau}}{\partial \Gamma} + \tilde{\mathcal{L}} \right] h \, d\Gamma = 0.$$  

Notice that in the above formulation, all kinematics including the perturbation of the interface, are governed by the operator $\mathcal{A}_3$. 
and $\Phi_h \in \mathcal{H}^1$,

$$\{\text{LHS Eq. (43)}\} + \int_{\Gamma_0} \left( \Phi_\tau + \frac{h^e}{|\tilde{u}|} \tilde{u} \cdot \nabla \Phi_\tau \right) : \left[ \lambda \left( \tilde{u}_x \frac{\partial \tilde{x}}{\partial \Gamma} + \tilde{u}_y \frac{\partial \tilde{y}}{\partial \Gamma} \right) \frac{\partial \tilde{x}}{\partial \Gamma} + \tilde{Z} \right] \tilde{h} \, d\Gamma = 0, \quad (59)$$

$$\int_{\Gamma_0} \Phi_h \left[ \frac{h - \tilde{h}}{(1 - 2\Theta)\Delta t} + \left( \tilde{u}_x \frac{\partial \tilde{x}}{\partial \Gamma} + \tilde{u}_y \frac{\partial \tilde{y}}{\partial \Gamma} \right) \frac{\partial \tilde{h}}{\partial \Gamma} - \tilde{n}_0 \cdot \tilde{z} \right] d\Gamma = 0. \quad (60)$$

In the above equations, the line integrals related to the natural boundary conditions have been omitted. For instance, for free surface flows without capillary forces, the perturbation of the normal and shear stress vanishes. On the other hand, when superposed fluids are considered, the force balance on the interface requires continuity of the perturbation of the normal and shear stress which is obtained in a weak sense by omitting these integrals. The linearized boundary condition that imposes continuity of $x$-velocity (Eq. (19)) implies that the velocity perturbations are discontinuous across the interface. In order to simplify the computations, we restrict ourselves to multilayer flows of fluids with equal viscosity. In this way the velocity perturbations are continuous across the interface ([[$\tilde{u}_x$]] = 0) and computations can be performed using continuous velocity fields whereas the remaining degrees of freedom are discontinuous across the fluid/fluid interface.

Also notice that the line integrals related to the DEVSS parts of the equations have been omitted from the governing equations. This can be seen if the weak form of the Laplace operator is considered:

$$-(\tilde{\Phi}_\mu, \nabla \cdot \alpha \nabla \tilde{u}) = (\nabla \tilde{\Phi}_\mu^T, \alpha \nabla \tilde{u}) - \int_{\Gamma} \tilde{\Phi}_\mu \cdot (\alpha \tilde{n} \cdot \nabla \tilde{u}) \, d\Gamma; \quad (61)$$

and linearization of the boundary integral introduces gradients of the domain perturbation normal to the boundary (i.e. $\partial h/\partial n \neq 0$). However, we cannot compute normal gradients of $h$ which is defined only on the interface and it is necessary to define $\alpha$ in such a way that the linearized boundary integral resulting from (61) vanishes. Without loss of generality we can set $\alpha = 0$ on the boundary and $\alpha = 1$ inside the domain (with $\alpha$ increasing continuously from 0 to 1). It is easily observed that linearization of Eq. (61) with $\alpha = 1 - H^0$ yields a set of equations that only need to be evaluated on the steady state domain.

4. Results

In this section results are presented for the linear stability problem of shear flows of UCM fluids. We will first discuss the behavior of the $\Theta$-scheme in single layer flows (i.e. planar Couette and Poiseuille flows). Next, two-layer flows are considered. The corresponding eigenspectra that determine the stability of these multilayer flows consist of the spectra of the bulk flows of the two fluids (similar to the single layer flows) and a number of interfacial modes $[23,34]$. For the flows considered here, there is always an interfacial mode that is the dominating one which is usually well separated from the remaining parts of the eigenspectra as opposed to single fluid flows. For single layer flows the most dangerous eigenmodes are often very close to the so-called continuous spectrum of singular eigenfunctions which is a notorious source for the generation of spurious solutions of the stability problem $[35]$. 

4.1. Single fluid flows

Some typical eigenspectra of shear flows of UCM fluids can be seen in Fig. 6. For the Couette flow, it was shown by Gorodtsov and Leonov that the eigenspectrum consists of two discrete wall modes and a so-called continuous spectrum which is associated with a regular singular point of the governing ordinary differential equation. The leading modes are the discrete Gorodstov–Leonov modes with maximum growth rate \( \sigma_r = -1/(2\text{We}) \) for \( \kappa \to \infty \) and \( \text{We} \ll 1 \). The remaining modes are part of the continuous spectrum which is defined by the line segment \( \sigma = -1/\text{We} \) to \( \sigma = -1/(\text{We} - i\kappa V) \).

Fig. 7 shows the evolution of the \( L_2 \)-norm of an initially random perturbation of Couette flows for different lengths of the computational domain (\( L = H \) and \( L = 5H \)). Based on different ideas but leading to a similar criterion, both [8,35] recognized the importance of refinement of the grid aspect ratio next to absolute refinement of the computational domain. Using a discontinuous Galerkin spatial discretization scheme for the extra stress variables, Smith et al., discussed the wavenumber dependence of the structure of the cross streamwise velocity perturbation of the Gorodstov–Leonov modes in the wall normal direction. This
results in steep boundary layers with decreasing layer depth as the wavenumber increases. For instance, the boundary layer is shown in Fig. 8 (right) by means of the $\tau_{xx}$ component of the GL wall mode for the stationary wall at $We = 10$ and $\kappa = H/2\pi$. Since finer streamwise discretization allows for higher wavenumbers to occur, directional refinement can therefore not be performed independently. In other words, if the steep boundary layers (which depend on the wavelength that is allowed by the grid) are not resolved accurately, we cannot expect mesh converged estimates of the computed growth rate. This is clear from Fig. 8 (left) where the estimated growth rates are plotted for both the grids used in Fig. 7 which are equidistant and grids that were refined near the walls in order to capture the wall modes. It should be noted that although the results in Fig. 7 do not yet represent the true dynamics of the UCM model since the mesh still needs to be refined in order to capture the boundary layer, the estimation of the growth rate of the Gorodstov–Leonov modes is underpredicted.

This is in contrast with the findings of [8,28] using discontinuous Galerkin interpolation for the extra stress variables where underresolved solutions seem to overpredict the growth of a perturbation of the flow. Stability of the EVSS-G/SUPG spatial discretization was first observed by Brown et al. [19] and later by Bogaerds et al. [28] for the DEVSS-G/SUPG method using fully implicit temporal integration. Smith et al. [8] claimed that the instability found using the DG scheme could be attributed to the fact that DG is less diffusive in a model problem as compared to Petrov–Galerkin methods. Consequently, there may be less damping of short wave disturbances which can result in boundary layers that cannot be resolved on the actual spatial grid.

An alternative explanation was given by [35] who derived a wavelength criterion based on the failure of the numerical approximation to capture the remaining spectrum (continuous spectrum). It can be seen from Fig. 6 that, for the spectral method, the approximation of the continuous spectrum converges only linearly due to the singular nature of the eigenfunctions. In fact, the error contained in the approximation of the continuous spectrum is almost constant when the product $\kappa N$ is kept constant [35]. Hence, insufficient cross-streamwise discretization as compared to streamwise discretization can easily allow the leading mode to be part of the continuous spectrum. For the finite element approximation of the stability of the
flow, streamwise refinement now results in higher wavenumbers that can be accommodated by the mesh which can poses severe problems on the ability of any numerical technique to capture the true dynamics of the flow [36]. From this point of view, the ability of the chosen spatial discretization (continuous or discontinuous) could also be a determining factor for the numerical scheme to capture the true flow dynamics. Whatever the problems are that may arise in a complex stability problems such as these, we show in Fig. 8 (left) that our method is capable of predicting the dynamics of the Couette flow up to very high Weissenberg numbers.

Application of the Θ-scheme to Poiseuille flow of the UCM model also results in accurate results with respect to the long time temporal growth of a perturbation of the flow. Fig. 9 shows mesh converged results for different values of the Weissenberg number. The spectrum of Poiseuille flow was already shown in Fig. 6 (right). There is again a continuous spectrum of singular eigenmodes with real part $-1/\text{We}$. In addition to this spectrum, there are now six discrete eigenmodes [23]. In the short wave limit, the leading modes behave as the Gorodstov–Leonov wall modes with real parts at $-1/(2\text{We})$. Fig. 9 (right) shows estimates of the growth rates obtained with the Θ-scheme which can be observed to follow the $-1/(2\text{We})$ line.

4.2. Two-layer flows

For the multilayer flows, we first study the Couette flow of two UCM fluids with $\text{We}_1 = 0.1$ and $\text{We}_1 = 1.0$. Figs. 10 and 11 show the eigenspectra for flows with layer thickness ratio $\epsilon = 0.60$ and 0.25. Next to the continuous spectra of the different fluids and the discrete wall modes (although barely visible there exist one wall mode for each fluid layer), there are five discrete modes present that can be attributed to the presence of the fluid/fluid interface. The trajectories of the leading interface modes as a function of the imposed wavenumber (or wavelength $\Delta$) are also shown in the blowups of the spectra. If the real parts of the leading mode are plotted against the wavelength (Figs. 10 and 11, right) it is apparent that both flows exhibit a shortwave instability with similar growth rates. This is consistent with the shortwave analysis of [34] where this instability was shown to be independent of the interface position $\epsilon$. The main difference between the two flows is the fact that for $\epsilon = 0.60$ the growth rate exhibits a maximum at $\Delta \approx 4.5$ whereas the long time dynamics for $\epsilon = 0.25$ are fully determined by the shortwave instability.

![Fig. 9. Evolution of the $L_2$-norm of an initially random perturbation of steady Poiseuille flow with channel length $L = 2H$ (left). Exponential growth for different Weissenberg numbers (right). Also shown are the limit lines for the growth of the continuous spectrum (dashed) and the growth of the Gorodstov–Leonov modes (solid).](image-url)
In the previous section we have shown that our $\Theta$-method is able to predict the correct dynamics of shear flows of single layer UCM fluids. The question which we shall address here is whether we are also able to predict these dynamics for multilayer flows using the technique described in Section 3.3. Table 1 summarizes results obtained from the GEVP (Figs. 10 and 11) and finite element simulations. Depending on the wavelengths associated with the maximum growth rate computed using the GEVP, a number of different streamwise lengths ($L$) have been selected for the FEM computations. It can be seen that the growth rate is accurately predicted for both values of the relative layer depth.

Fig. 12 shows computational results obtained from the one-dimensional eigenvalue problem for a superposed Poiseuille flow with $(\epsilon, W_1, W_2) = (0.60, 0.1, 1.0)$. From the locus of the leading eigenmode in the complex plane, it can be seen that there exist a wavelength $\Delta_m^{\text{GEVP}} = 1.9$ for which the real part
Table 1

Results for Couette flow of superposed UCM fluids using the GEVP and FEM

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\text{We}_1$</th>
<th>$\text{We}_2$</th>
<th>$\Delta_{m}^{\text{GEVP}}$</th>
<th>$\text{max}(\sigma_i)$</th>
<th>$\Delta_{m}^{\text{FEM}}$</th>
<th>$\tilde{\sigma}$</th>
<th>$L$</th>
<th>$\Delta x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.60</td>
<td>0.1</td>
<td>1.0</td>
<td>4.5</td>
<td>0.025</td>
<td>3.0</td>
<td>0.018</td>
<td>3.0</td>
<td>0.15</td>
</tr>
<tr>
<td>0.60</td>
<td>0.1</td>
<td>1.0</td>
<td>4.5</td>
<td>0.025</td>
<td>4.5</td>
<td>0.023</td>
<td>4.5</td>
<td>0.15</td>
</tr>
<tr>
<td>0.60</td>
<td>0.1</td>
<td>1.0</td>
<td>&lt;0.6</td>
<td>0.015</td>
<td>6.0</td>
<td>0.015</td>
<td>2.0</td>
<td>0.08</td>
</tr>
<tr>
<td>0.25</td>
<td>0.1</td>
<td>1.0</td>
<td>&lt;0.6</td>
<td>0.015</td>
<td>8.0</td>
<td>0.021</td>
<td>9.0</td>
<td>0.30</td>
</tr>
</tbody>
</table>

The maximum growth rate of a normal mode ($\text{max}(\sigma_i)$) is obtained from Figs. 10 and 11 together with the dominant wavelength $\Delta_{m}^{\text{GEVP}}$. Corresponding finite element calculations have been performed on a computational domain with length $L$ and spatial resolution in streamwise direction $\Delta x$. Estimates of the computed growth rate using the FEM are expressed by $\tilde{\sigma}$ whereas $\Delta_{m}^{\text{FEM}}$ denotes the approximate dominant wavelength after exponential growth at $t = 100$.

Fig. 12. Eigenspectrum of two-layer Poiseuille flow of UCM fluids (left) for $(\epsilon, \text{We}_1, \text{We}_2) = (0.60, 0.1, 1.0)$ and $N = 75$. The blowup shows the locus of the leading mode as the wavelength $\Delta$ is varied. The (right) graph shows the maximum real part of this mode as a function of the imposed wavelength.

Fig. 13. Evolution of the $L_2$-norm of a perturbation of a two-layer Poiseuille flow with $(\epsilon, \text{We}_1, \text{We}_2) = (0.60, 0.1, 1.0)$ together with the estimate of exponential growth (left). Also shown are the stress components of the leading eigenmode at $\Delta_{m}^{\text{GEVP}} = 1.9$ using GEVP (middle) as well as the structure of the perturbation after exponential growth at $t = 100$ (right). The length of the computational domain is $L = 2\Delta_{m}^{\text{GEVP}}$. 
Fig. 14. Eigenspectrum of two-layer Poiseuille flow of ucm fluids (left) for \((\epsilon, \text{We}_1, \text{We}_2) = (0.60, 1.0, 2.0)\) and \(N = 75\). The blowup shows the locus of the leading mode as the wavelength \(\Delta\) is varied. The (right) graph shows the maximum real part of this mode as a function of the imposed wavelength.

Fig. 15. Leading eigenmode for Poiseuille flow using the GEVP for \((\epsilon, \text{We}_1, \text{We}_2) = (0.60, 1.0, 2.0)\) and \(\Delta_{\text{m}}^{\text{GEVP}} = 0.7\) (upper graph). Structure of the solution of the perturbation after exponential growth \((\hat{\sigma} = 0.0154)\) using the \(\Theta\)-scheme with channel length equal to \(\Delta_{\text{m}}^{\text{GEVP}}\) (lower graph).

of this mode exhibits a maximum. Fig. 13 shows finite element calculations of the same problem on a computational domain \(L = 2\Delta_{\text{m}}^{\text{GEVP}}\). With our method, we can accurately predict both the growth rate of a disturbance as well as the structure of the eigenfunction of the dominant mode.

A similar Poiseuille flow \((\epsilon, \text{We}_1, \text{We}_2) = (0.60, 1.0, 2.0)\) has also been investigated. Fig. 14 shows the governing spectrum for this flow together with the locus of the leading mode for varying wavelengths. The wavelength for which the growth rate exhibits a maximum has shifted to \(\Delta_{\text{m}}^{\text{GEVP}} = 0.7\). Results of the dominating mode of the GEVP and the long time solution of of the FEM are shown in Fig. 15. The computed growth rate after exponential growth for this flow \(\hat{\sigma} = 0.0154\) for a computational domain equal to \(\Delta_{\text{m}}^{\text{GEVP}}\). The upper and lower graphs show all components of the perturbation vectors which are clearly consistent for both computational methods.

5. Conclusions

The stability of viscoelastic planar shear flows of single and two-layer superposed ucm fluids has been investigated. Based on the work of Carvalho and Scriven [33], we have developed a numerical technique
that is able to efficiently handle stability problems of viscoelastic flows on complex domains with fluid/gas (free surface) or fluid/fluid interfaces. Using a generalized eigenvalue analysis of the shear flows, we have benchmarked our new time marching technique and the handling of the perturbations of the computational domain. It was confirmed that our method is capable of accurately predicting exponential growth of both single and multilayer flows up to considerable values of the Weissenberg number. In addition, the GEVP does not only provide for the dominant growth rate of the normal modes, it also provides for the structure of the eigenfunctions associated with this leading mode. Making use of the fact that, for multilayer flows, the dominant mode is usually well separated from the remaining part of the eigenspectrum, we were also able to predict these eigenfunctions from the solution of the perturbed flow after exponential temporal growth.

References