Integrated control/structure design for planar tensegrity models

Bram de Jager
Department of Mechanical Engineering
Technische Universiteit Eindhoven
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
Email: A.G.de.Jager@wfw.wtb.tue.nl Fax: +31 40 2461418

Robert E. Skelton and Milenko Masic
Department of Mechanical and Aerospace Engineering
University of California at San Diego
La Jolla, California, 92093-0411
Email: bobskelton@ucsd.edu Fax: +1 619/534-7078

Abstract

A tensegrity structure is built using compressive members (bars) and tensile members (tendons).

We discuss how an optimal and integrated design of tendon length control and topology/geometry of the structure can improve the stiffness and stiffness-to-mass properties of tensegrity systems. To illustrate our approach we apply it on a tensegrity system build up from several elementary stages that form a planar beam structure.

The computations are done with a nonlinear programming approach and most design aspects (decentralized co-located control, static equilibrium, yield and buckling limits, force directionality, *etc.*, both for the unloaded and loaded cases) are incorporated.

Due to the way the control coefficients are constrained, this approach also delivers information for a proper choice of actuator or sensor locations: there is no need to control or sense the lengths of all tendons.

From this work it becomes clear that certain actuator/sensor locations and certain topologies are clearly advantageous. For the minimal compliance objective in a planar tensegrity beam structure, proper tendons for control are those that are perpendicular to the disturbance force direction, close to the support, and relatively long, while good topologies are the ones that combine different nodal configurations in a tensegrity topology that is akin to a framed beam, but, when control is used, can be quite different from a classical truss structure.

Keywords. Mechanical systems, tensegrity systems, structural optimization, actuator/sensor selection, nonlinear programming.

1 Introduction

Tensegrity systems are composed of tensile members (tendons, wires or strings) and compressive members (bars, sticks) [1]. This class of systems has been studied for a long time, see, *e.g.*, [2], whose terminology consisted of ties and struts instead of tendons and bars. The members in a tensegrity structure are connected in nodal points. In a class one node a bar end-point is connected to tendons only, while in a class two node two bars are connected together and to several tendons. A structure containing class one nodes derives its stability from pre-stressing the members.

Tendons in tensegrity structures have multiple roles, they:

- rigidize and stiffen the structure, also due to prestress,
- carry structural loads,
- provide opportunities for actuation/sensing [3].

Structural control can improve properties like damping and stiffness or stiffness-to-mass, and may be used in shape control strategies. For control, sensing and actuation are needed. Sensing provides information about the geometry of the structure, *i.e.*, the deformations or the actual lengths of tendons or bars. Actuation can be carried out by changing the length of tendons or bars. This can be done in several ways, by:

- shape memory alloys that enable the tendons to shorten and lengthen by changes in temperature,
- linear or rotary motors that can shorten a tendon by hauling it, *e.g.*, inside hollow bars,
- extensible bars.

When equipped with actuators and sensors, related to the tendon or bar length, tensegrity structures can change shape by controlling those lengths. This shape control ability sets them apart from most of the structures used in practice and may be used to adapt the shape according to requirements that change in time, *e.g.*, for deployment, or it may be used to keep the shape the same despite disturbances acting on the structure. Because elastic tendons connect bars via a class one node, and not a pinned joint, tensegrity structure of class one may not be sufficiently stiff when weight considerations play a role, because then stiffness cannot be increased by adding material to the structure. This reduction in stiffness may be off set by the shape control system if this is configured to counteract the effects of disturbing loads.

Here we consider only the tendons as elements that can sense their own length and can change that length. Also, we initially consider only tensegrity structures of class one, *i.e.*, with only class one nodes, although the optimization may deliver solutions with nodes that are very close together, and therefore could be combined to form a class two node.

A target area of application for tensegrity systems is where the shape of a structure needs to be changed dynamically, *e.g.*, in space technology with deployable structures or in medicine with expandable inserts.

Besides control we can also employ changes in topology and geometry to improve properties of structures. Optimization of topology/geometry of structures has been studied for a long time, see the short overview in [4]. Integrated structure/control design is rare, however, but can be quite profitable, so we aim to contribute to this integrated design process. Therefore, in this paper we want to simultaneously achieve

- optimal control of tendon length, with a limited number of actuators/sensors and a decentralized static controller,
- topology/geometry optimization, incorporating constraints for failure of the structure, like yield and buckling.

We show that this can be done by integrated control/topology/geometry optimization.

The paper is structured as follows. First, we outline a model for static equilibria for pre-stressed systems. Then, we explain our optimization problem formulation. This is followed by an application for a planar beam structure built up from elementary tensegrity stages. A set of conclusions finishes the paper.

2 Static equilibria of pre-stressed systems

The equilibrium conditions for a frame structure with nodal point coordinates p under a load f acting on those nodes and causing a displacement u of these nodes can be posed as

$$C^T \operatorname{kron}(\Lambda, I_{\dim}) C(p+u) = f, \tag{1}$$

which is just the balance of element forces under load at each of the nodes. Boundary conditions, *e.g.*, for a support, are handled by removing the balance equations for the relevant nodes from (1). Equation (1) is a classical result in the analysis of equilibria for mechanical structures.

Here, $p \in \mathbb{R}^{\dim \cdot n_n}$ is a column containing the nodal coordinates (n_n is the number of nodes and dim is the dimensionality of the problem, either 2 or 3, note that f and u are elements of the same sized space as p). The matrix $C \in \mathbb{N}^{\dim(n_m \times n_n)}$ represents the connectivity of the frame, with n_m the number of elements or members. Matrix C is a sparse block matrix whose i, j-th block is $I_{\dim}$ or $-I_{\dim}$ if the element i ends at or emanates from node j, otherwise it is $0_{\dim}$. It is a member-node incidence matrix. By using this formulation, it is assumed that a maximum set of allowed element connections of a tensegrity structure and its associated oriented graph have been adopted.

The diagonal matrix $\Lambda \in \mathbb{R}^{n_m \times n_m}$ contains the force coefficients (note that the member force itself is the force coefficient (a scaling factor) times the element vector g_i). The sign convention is that Λ is positive for tensile and negative for compressive forces. Λ is a function of the displacement u, so the equations are nonlinear, and it also depends on the control action, member volume, and material properties. Pre-stress is incorporated because Λ is not necessarily zero in the unloaded case with $f = 0$, so without load the member forces do not vanish in general, and the structure is stabilized. By using the Kronecker product we expand Λ by a factor dim, so it matches in the equation.

The vector $g \in \mathbb{R}^{\dim \cdot n_m}$, representing the orientation of the elements, here in the loaded equilibrium, is computed as

$$g = C(p+u)$$

while the length of member i is derived from

$$l_i = \|g_i\|_2.$$

Depending on the material model chosen, the relationship between force coefficients, Λ, and physical parameters of the structure may be different. For this analysis the linear elastic material model is used. Then the relation for Λ is smooth, algebraic, and monotonous. It depends on material properties like Young's modulus and on the control coefficients, because these together determine the changes in length of the tendons due to changes in tendon force. Effectively, proportional control based on changes in length has the same effect as a change in Young's modulus, so the control can also be regarded as a way to change the material properties, even to a range that cannot be achieved by material science. This is called parametric control. Expressed in terms of initial length, pre-stress and elongation of member i, it holds that

$$\Lambda_i l_i - \Lambda_{0_i} l_{0_i} = (E_i + e_i) v_i \frac{l_i - l_{0_i}}{l_i l_{0_i}} \tag{2}$$

with E_i the modulus of elasticity of member i, e_i its parametric control coefficient, and v_i its volume. The subscript 0 indicates the values of force coefficients and lengths for the unloaded ($f = 0$, $u = 0$) case.

3 Formulation of optimization problem

The objective of this analysis is to design a controlled tensegrity structure, *i.e.*, a structure and a proportional diagonal feedback controller, that, for a given mass of the material available and for a given sum of feedback coefficients, has an optimal stiffness. Assuming that all the elements are made of the same material, fixing the mass available is equivalent to specifying total volume S_v of the material used.

The optimization algorithm to a tensegrity structure whose number of nodes and number of members available are n_n, n_m respectively, assigns structural parameters collected in vectors of the nodal positions p, pre-stress of the elements Λ_0, volumes of the elements $v \in \mathbb{R}^{n_m}$, and control coefficients $e \in \mathbb{R}^{n_m}$. For a given vector of applied external nodal forces f, this set of parameters defines a structure, whose static response, defined in the vector of nodal displacements u, yields a compliance energy $\frac{1}{2} f^T u$, that is guaranteed to be improved from the value corresponding to an initial design. Note that compliance is used as a measure of the stiffness of the structure.

Our approach is based on nonlinear programming (NLP), in which we can embed decentralized control, pre-stress, failure conditions and changes in geometry due to displacements.

For the optimization of control/topology/geometry we consider the following set of design or optimization variables, as sketched above,

$$x^T = \begin{bmatrix} p & \Lambda_0 & v & e & u \end{bmatrix}.$$

All columns stacked together give the design vector $x \in \mathbb{R}^n$ with $n = 2 \cdot \dim \cdot n_n + 3n_m$. Appropriate modifications are made when some of the variables are not supposed to change, *e.g.*, for the position of the nodes where the force is applied, or no deformations occur, *e.g.*, for the position and displacement of the nodes that are connected to the support, or for members whose length is not controlled, *e.g.*, the bars, so the size of the design vector is slightly less than indicated above.

It is clear that with this design vector the geometry can be influenced. Also the topology can be determined, when we allow a starting grid of nodal points and members that is more detailed than required, *e.g.*, for accuracy of shape control. Members are allowed to vanish, when their volume approaches zero, making a change in topology. Actuators/sensors are not needed for members who's feedback coefficients are zero.

To make the structure stiff, our objective is to minimize compliance

$$\min_x f^T u,$$

the inner product between the load f and the displacement u of the nodal points under load, in the presence of a set of (nonlinear) constraints. The composition of the load vector f is given. The nonlinear equality and inequality constraints are:

1. equilibrium constraint

   $$C^T \operatorname{kron}(\Lambda(x), I_{\dim}) C(p + u) - f = 0$$

   employing the equations for a static equilibrium,

2. sign restricted force coefficients constraint

   $$-z_i \Lambda_i \leq 0$$

   which enforces tensile forces in the tendons (for which $z_i = 1$) and compressive forces in the bars (for which $z_i = -1$),

3. yield constraint

   $$|\Lambda_i| l_i^2 - v_i \sigma_i \leq 0$$

   so the stress in member i is always bounded by the yield-stress σ_i,

4. buckling constraint for the bars

   $$|\Lambda_i| l_i^5 - \frac{\pi}{4} E_i v_i^2 \leq 0$$

   which considers Euler buckling of a round cross section bar,

5. minimum length constraint for the unloaded case

   $$-l_{0_i} + a_i \leq 0, \quad a_i > a_{\min}$$

   imposed because the member length cannot be too short, otherwise there is no place for the joint construction or for the device that is needed to actively change the tendon lengths, or, when using memory type alloys, a short tendon allows only a limited range of length changes.

There are also a number of linear constraints and bounds. The sign requirement on Λ_0 is implemented as a bound.

A fixed volume, or mass, for the system is obtained with the linear equality constraint

$$\sum_i v_i = S_v, \quad v_i \geq 0$$

where we sum over all members of the structure.

Likewise it holds for the control coefficients:

$$\sum_i e_i \leq S_e, \quad e_i \geq 0$$

where we sum over all tendons.

With this type of constraint on the control coefficients we expect to be able to allocate actuators and sensors, because a relatively small number of control coefficients is expected to be nonzero, see [5, Section 3.8].

When only one load case is considered the constraints 1–4 have to hold for both unloaded ($f = 0$, $u = 0$) and loaded case. Several load cases can be handled simultaneously by extending the vector of design variables x with the nodal displacements u for the other loads, by using a linear combination of compliances as criterion, and by requiring constraints 1–4 to hold for the additional load cases also.

All nonlinear constraints are incorporated in a constraint vector $c \in \mathbb{R}^{(n_l+1)(\dim\cdot n_n+2n_m+n_n/2)}$, where $n_n/2$ is the number of bars and n_l the number of loaded cases. The part of c related to inequality constraints is required to be nonpositive, and the part related to equality constraints is required to be zero. The actual number of constraints contained in c is slightly less than indicated above, due to the boundary conditions. The linear constraints are specified directly by their coefficients

Using actuation, sensing, and control we expect to improve performance. A physical motivation for this statement is that we are now able to decouple the requirements for compliance and failure conditions. Improved compliance is achieved by control and failure conditions are met by redistribution of material. There is less need to redistribute material to improve compliance.

This nonlinear programming problem is solved with SNOPT 6.1-1(2) [6]. This program employs a sequential quadratic programming (SQP) approach with active set strategy to solve the problem.

This software is extended with MAD [7], a library of functions in Matlab, using a class library and operator overloading, for automatic differentiation, to compute the Jacobian J of the constraint vector c. This is advantageous from a numerical point of view, because finite differencing is not needed anymore.

4 Application

The basic design problem used to illustrate our approach is stiffness optimization for the tensegrity beam in Fig. 1.

This beam structure

- is built up from 3 planar tensegrity crosses
- with an aspect ratio of 7
- while the support at the left side removes the 3 degrees-of-freedom of a rigid body when $\dim = 2$
- and is loaded by a unit vertical load at the top/right node.

For this example $n_m = 26$ and $n_n = 12$. Using the NLP formulation, we obtain the results in Figs. 2–11. Figures 2 and 7 give the results for an optimization without using control, so $S_e = 0$, see also [4]. The other figures show the additional benefits of active control, were $S_e = 300$ and $S_e = 2000$ have been chosen to show the influence of the allowable sum of the control coefficients. In the shallow figures we do not allow the positions of the nodal points p to move outside the horizontal lines at $y = 0$ and $y = 1$, so the y-components of p are removed from x, while in the other figures the nodal points p can move freely in the plane, except for the nodes at the support and for the node with the load. The size of J is then 182×120 and 182×129, respectively. Figures 4, 6, 9, and 11 present the tendons that are controlled, so for which $e_i > 0$, and the width of the lines indicate the magnitude of the control coefficients per unit length. The width of the lines in the other figures is proportional to the diameter of the bars and tendons, so gives an indication of the volume that is assigned to them by the optimization process.

The color coding in the figures is as follows

- unstressed member: light gray (green),
- pre-stressed bars: dark gray (red),
- pre-stressed tendons: black (blue).

Table 1 gives an overview of the performance objective that could be achieved in the cases presented.

Table 1: Overview of compliances

	Restricted p	Free p
No optimization	0.46588	0.46588
$S_e = 0$	0.20716	0.14287
$S_e = 300$	0.14216	0.10509
$S_e = 2000$	0.05523	0.05009

We note the following

- optimizing topology/geometry improves the objective by more than 50%, and leaving the nodal points free to move is quite advantageous,
- integrated control/topology/geometry design improves performance, as expected, in this case by 30% for the case that $S_e = 300$ and with 60-70% for the case that $S_e = 2000$, with respect to the situation without control, so the advantage is mainly determined by the choice of design parameter S_e,
- for larger values of S_e the number of tendons used tends to become smaller and more of the tendons are controlled,
- for larger values of S_e the difference in objective between the restricted and the free nodal point case diminishes because the stiffness of the structure is mainly determined by the controlled tendons and less influenced by geometry,
- for the optimized structures the sum of force coefficients is increased considerably, due to more short members, but all forces are within the failure constraints,
- only few tendons are controlled, those close to the support, with long lengths, and horizontal orientation (perpendicular to the load), this agrees with results in [8], where a more involved method based on H_∞-criteria was used,
- the tendons that are controlled tend to be quite long, because this is clearly advantageous, so expressing control coefficients per unit length may make sense,
- the number of members in the optimal structure is smaller with control, reducing the complexity of the structure,
- the optima tends to include class two nodes because some nodes move close to each other, this also causes members to be close to each other, so some members are hidden from view in the figures.

5 Conclusions

The conclusions are as follows

- the integrated control/topology/geometry optimization, with a set of diverse constraints and cast in the form of a nonlinear program, is effectively solvable, providing an appropriate design tool,
- shape control improves performance, reduces the complexity of the structure, and can be implemented by using a small number of actuators/sensors and a decentralized control scheme,
- information about optimal actuator/sensor location is available from the optimal design,
- topologies including class two nodes are preferred,
- although tensegrity beams do not always excel at stiffness, in the application considered here the compliance is excellent, and better than for an uncontrolled optimal structure without tensegrity constraints, see [4], a so-called Michell beam [9].

References

[1] R. Adhikari, R. Skelton, and W. Helton, “Mechanics of tensegrity structures,” Feb. 1998. Report 1998-01 UCSD Structural Systems and Control Lab.

[2] J. C. Maxwell, *The Scientific Papers of James Clerk Maxwell*. New York: Dover, Forew. 1890.

[3] R. E. Skelton and R. Adhikari, “An introduction to smart tensegrity structures,” in *Proc. 12th ASCE Engineering Mechanics Conf.*, (La Jolla, CA), pp. 24–27, May 1998.

[4] B. de Jager, M. Masic, and R. E. Skelton, “Optimal topology and geometry for controllable tensegrity systems,” in *Preprints 15th IFAC World Congress*, (Barcelona, Spain), IFAC, July 2002. To appear.

[5] M. van de Wal and B. de Jager, “A review of methods for input/output selection,” *Automatica—J. IFAC*, vol. 37, pp. 487–510, Apr. 2001.

[6] P. E. Gill, W. Murray, and M. A. Saunders, “User’s guide for SNOPT 5.3: A FORTRAN package for large-scale nonlinear programming,” Feb. 1999.

[7] S. Forth, “User guide for MAD – a MATLAB automatic differentiation toolbox, version 1, beta release, the forward mode,” June 2001. AMOR Report 2001/5.

[8] B. de Jager and R. Skelton, “Input/output selection for planar tensegrity models,” in *Proc. 40th IEEE Conf. on Decision and Control*, (Orlando, FL), pp. 4280–4285, IEEE, Piscataway, NJ, Dec. 2001.

[9] A. G. M. Michell, “The limits of economy in frame structures,” *Phil. Mag.*, vol. 8, pp. 589–597, 1904.

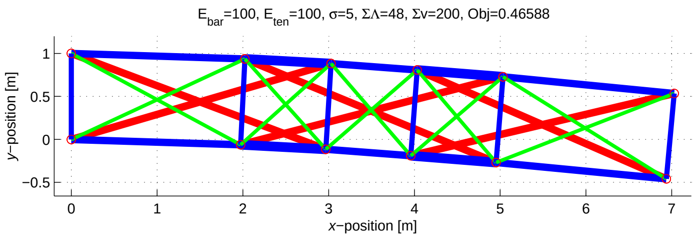

Figure 1: Basic tensegrity beam system (not optimized)

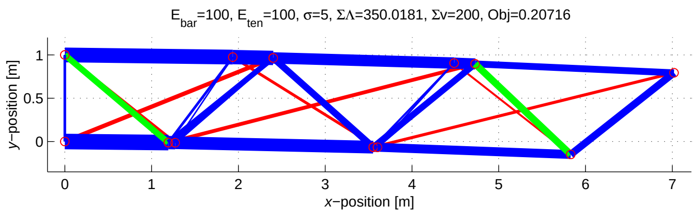

Figure 2: Optimal topology/geometry, displacement under load

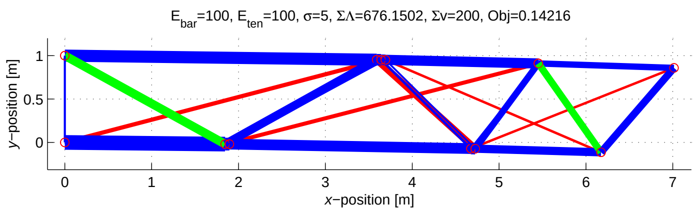

Figure 3: Optimal control/top/geo, displacement under load

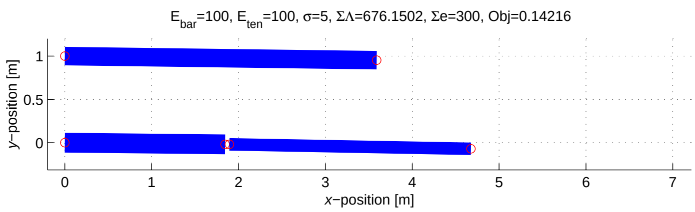

Figure 4: Optimal control/top/geo, controlled tendons

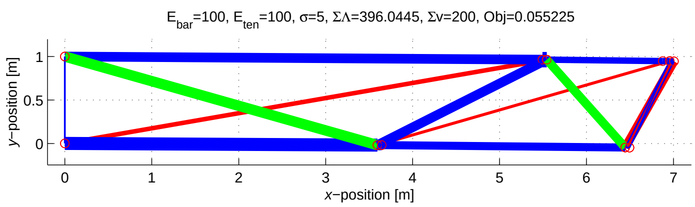

Figure 5: Optimal control/top/geo, displacement under load

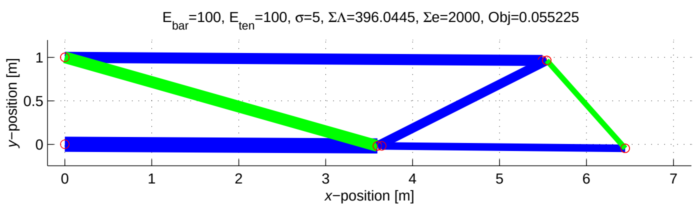

Figure 6: Optimal control/top/geo, controlled tendons

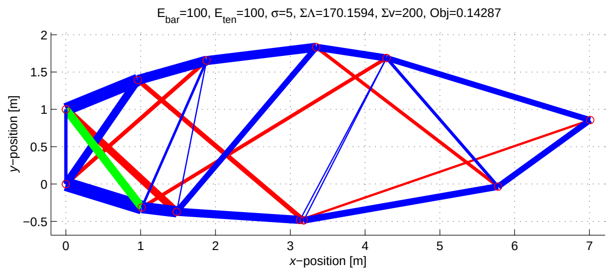

Figure 7: Optimal topology/geometry, displacement under load

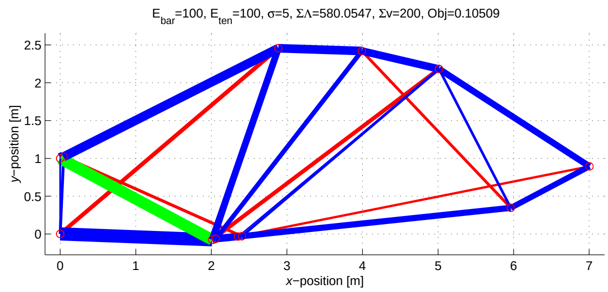

Figure 8: Optimal control/top/geo, displacement under load

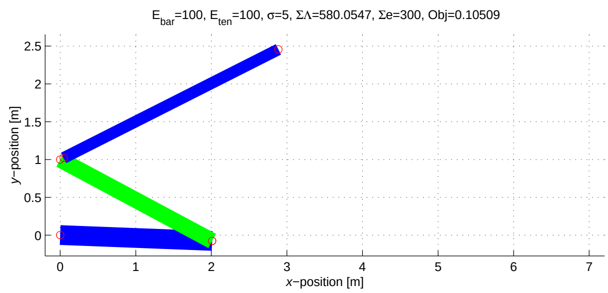

Figure 9: Optimal control/top/geo, controlled tendons

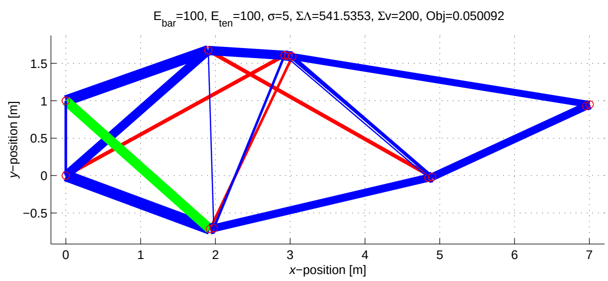

Figure 10: Optimal control/top/geo, displacement under load

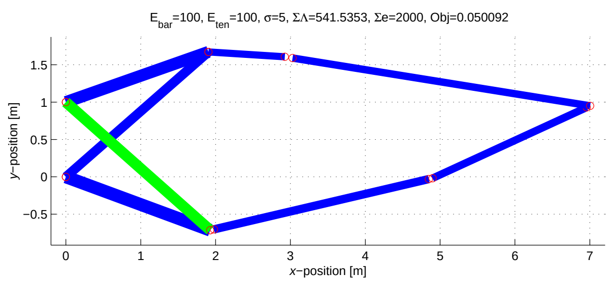

Figure 11: Optimal control/top/geo, controlled tendons