Friction Induced Hunting Limit Cycles: 
An Event Mapping Approach

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Abstract

This paper studies the occurrence of limit cycles for a simple PID-controlled motion system subjected to different shapes of static frictional damping functions. The friction characteristics are compared with respect to the so-called hunting phenomenon. In particular, a classification with respect to the ability to predict both stable and unstable limit cycles is obtained. Furthermore, the local stability of equilibrium points is discussed and attractor basins are given. To perform this analysis, the closed-loop dynamics are represented with an equivalent one-dimensional map representing a so-called event map.

1 Introduction

In general, mechanical systems exhibit friction. A negative manifestation of friction is a degeneration of performance in terms of increasing tracking errors and friction induced limit cycling, i.e., periodic solutions of a nonlinear autonomous system, for instance for a servo system. Due to its oscillatory and persistent behavior, limit cycling is an undesirable phenomenon in controlled servo systems. Here, a friction induced phenomenon called hunting [1] is considered, which is mainly caused by the combination of a difference in the static friction level and the Coulomb friction and an integral action in the control loop for a simple regulator task.

The research on friction induced stick-slip oscillations has been extensive over the past decades [1, 2, 5, 6, 8, 10, 11, 12]. Friction induced limit cycles have been analyzed in literature with different techniques and various results, e.g., the Describing Function Analysis (DFA) [1, 2, 5] was widely used but resulted only in qualitative prediction and even these were often found to be incorrect [2, 13]. Other important analysis tools are (i) Phase Plane Analysis (PPA) [6, 12], (ii) exact algebraic analysis [2, 11] and (iii) analysis tools originating from nonlinear dynamics [7, 8, 10]. The latter seems to be the most powerful due to the ability to deal with nonlinearities in the friction model, however are mainly based on expensive numerical investigation of the closed-loop system. For the hunting phenomenon, [8] shows for two nonlinear friction models, i.e., the dynamic LuGre friction model [3] and the static Switch friction model [10], comparable results with respect to periodic solutions for the nonlinear closed-loop dynamics. A numerically constructed bifurcation diagram predicts for both models a disappearance of the hunting limit cycles for certain controller settings which is conform the practically observed behaviour, where industrial regulator problems are tuned by hand and indeed prevent stick-slip oscillations. In contrast to these results are the exact algebraic analysis by [2] for a PID-controlled simple mass system in the presence of static friction and a lower Coulomb friction level. For this friction model with a discontinuous jump in the friction level the authors state that the controlled system will lead to a frictional limit cycle for all stabilizing controllers. The major difference between this idealized friction model and the static Switch friction model used in [8] is the shape of the friction curve for velocities unequal to zero. Therefore the focus in this paper will be on the influence of the shape of the friction curve on the dynamic properties of the closed-loop system, i.e., stability of equilibrium points, types of periodic solutions and the possibility to predict the disappearance of the hunting phenomenon.

The various friction curves with a continuous drop in the friction level and a variable shape will be integrated in the Switch friction model [8, 10] and the resulting nonlinear closed-loop dynamics will be analyzed with an event mapping technique [6] which is a Phase Plane Analysis similar to the technique used in [12]. Here, the one-dimensional mapping technique is used to construct the so-called event map of the nonlinear closed-loop dynamics. Construction of the event map will be performed through numerical simulation of the closed-loop system and the map will give insight into the closed-loop dynamics together with stability properties and attractor basins of the periodic solutions and equilibrium set. Besides the numerical stability analysis of the equilibrium points, the exact algebraic results obtained by [2] will be used to perform an analytic analysis of the equilibrium points for different frictional damping characteristics.

The outline of this paper is as follows. A description of the closed-loop system will be given in Section 2 together with the various friction characteristics used. For the different frictional damping curves, a classification will be derived with respect to the local stability of the equilibrium set in Section 3. In Section 4, the construction of the equivalent one-dimensional map of the nonlinear discontinuous three-dimensional autonomous dynamic system will be discussed. The numerical results on the location of the limit cycles, both stable and unstable, together with their regimes of attraction will be given in Section 5 for the various friction curves. The paper will be concluded in Section 6 and future research topics will be addressed.

2 The Closed-loop System

For mechanical positioning systems, often modeled as a simple motor-driven mass system subjected to friction as shown in Figure 1, an important control problem is the regulator task, where due to a static friction level an integral action in the control loop is necessary to eliminate steady state errors. Hence, the simplest model of the closed-loop system reads as follows

\[ m\ddot{q} + P(q + Dq + I \int_0^t q(\tau) \, d\tau) = -F \]

where \( q \) is the position of the system with respect to the desired position, \( F \) the friction force, \( m \) the mass of the system to be controlled and \( P, I, D \) the controller parameters. For certain frictional
damping functions and PID-controller settings a limit cycle will exist around \( q = 0 \), which is called hunting [1]. Sufficient conditions to generate or simulate friction induced hunting limit cycles are:

- integral control,
- good stiction properties of the friction model, i.e., in the stick phase the sum of the external forces on the mass system are completely compensated with an equivalent friction force and
- a frictional damping function with a larger static friction level than the Coulomb friction level [2, 8, 12], i.e., either a continuous or discontinuous drop in the friction force for velocities unequal to zero.

An applicable friction model to simulate stick-slip oscillations, like the hunting phenomenon, is the discontinuous Switch friction model as demonstrated in [8, 10]. The Switch model [10] is a static friction model that can be considered as a modified version of the Karnopp model [9]. The idea is to model the friction force as a set-valued function at \( q = 0 \). The friction force is defined as

\[
F(q, u) = \begin{cases} 
\text{sgn}(\dot{q}) [g(\dot{q}) + b \dot{q}], & \dot{q} \neq 0 \text{ Slip} \\
\min\{u, F_s\} \text{sgn}(u), & \dot{q} = 0 \text{ Stick}
\end{cases}
\]

where \( u = -F(q + D\dot{q} + I \int_0^t q(\tau) d\tau) \) is the applied control effort, \( F_s \) is the maximum static friction at zero velocity, \( g(\dot{q}) \) is the frictional damping force for velocities unequal to zero and \( b \) is the viscous damping. The equation of motion, describing the closed-loop system, reads as

\[
\ddot{q} = \text{sgn}(\dot{q}) [g(\dot{q}) + b \dot{q}], \quad \text{for } \dot{q} \neq 0
\]

\[
\ddot{x} = J^{-1}(x_1, x_2) \dot{x}_2
\]

where \( J = [x_1, x_2, x_3]^T = [q \dot{q}, \int_0^t q(\tau) d\tau]^T \) and the control effort \( u = -F(x_1 + D \dot{x}_2 + I \dot{x}_3) \). The existence of solution of this Filippov-type of systems Eq. 1 is not trivial and therefore the uniqueness of solution is also questionable. In [4] it is shown that for this system a solution of the differential inclusion, corresponding to the time derivative of the second state equation, i.e.,

\[
m\ddot{x}_2 \in F_{\text{switch}}(\dot{x})
\]

exists if the set-valued function \( F_{\text{switch}}(\dot{x}) \) is upper semi-continuous, closed, convex and bounded for all \( \dot{x} \in \mathbb{R}^3 \). To analyze the existence and the uniqueness of the solution of the differential inclusion Eq. 2, a choice for the frictional damping function \( g(x_2) \) has to be made. Here, the frictional damping function is modeled with the widely used Striebeck curve, i.e.,

\[
g(x_2) = F_s + (F_e - F_s) \exp\left(-\frac{|x_2|}{v_s}\right),
\]

where \( F_e \) is the static friction level, \( F_s \) the Coulomb friction level, and \( v_s \) the so-called Striebeck velocity and \( \beta \) determines the shape of the friction curve for velocities unequal to zero. Examples of these friction models with different values for the shaping parameter \( \beta \) are given in the left plot of Figure 2. For the different damping curves the solution of the differential inclusion Eq. 2 exists since \( F_{\text{switch}}(\dot{x}) \) is upper semi-continuous, non-empty, closed, and convex. The set \([-F_e, F_e] \) at \( x_e = 0 \) is the smallest closed convex set that contains the left and right limits of \( F_{\text{switch}}(\dot{x}) \) to \( x_e = 0 \), as depicted in the right plot of Figure 2. As shown in the figure, an attraction sliding mode at \( x_e = 0 \) exists, if \( -F_e < u < F_e \), then \( x_e < 0 \) for \( x_e > 0 \) and \( x_e > 0 \) for \( x_e < 0 \). Moreover, at \( x_e = 0 \) a transversal intersection exists if \( |u| > F_e \). Therefore, uniqueness of the solution of the differential inclusion Eq. 2 and the equation of motion Eq. 1 is ensured.

As for the Karnopp friction model, a narrow band \( \eta \ll 1 \) around zero velocity is introduced to numerically integrate the system of equations of the Switch model. Due to this narrow stick band the model distinguishes three situations: (i) the slip phase, (ii) the stick phase and (iii) the transition phase, which all will be described by ordinary differential equations (ODEs). The transition phase describes the transition from stick to slip or velocity reversals without stiction. In the stick phase, the acceleration of the mass is set to \(-\alpha x_2\) and forces the velocity \( x_2 \) to zero for \( \alpha > 0 \). This acceleration term ensures that the ODE belonging to the stick phase does not suffer from numerical instabilities as in the Karnopp friction model.

3 Local stability of equilibrium points

The equilibrium points of the autonomous system, as described in Eq. 1, satisfy the condition \( \frac{\dot{x}}{\dot{x}} = F_{\text{switch}}(\dot{x}) = 0 \). Hence, \( x_1 \) and \( x_2 \) should be zero, which results for the closed-loop system to be in the stick phase and therefore \( |x_e| \leq \frac{F_e}{p_T} \). Since it involves a regulator task, it is important to realize that the equilibrium points of interest are indeed those for which the position error \( x_1 \) and velocity error \( x_2 \) are zero. The integral term \( x_3 \) should not necessarily be zero. Hence, the set of equilibrium points are located on the line

\[
x^* = \gamma [0 \ 0 \ 1]^T, \quad \forall |\gamma| \leq \frac{F_e}{p_T}
\]

For the equilibrium points, the applied control effort \( u = -P I x_e^* \) is compensated by an equivalent portion of the static friction \( F_s \). The local stability of such a set of equilibrium points on a line is not trivial. Here, the results presented by [2], which are obtained with an exact algebraic analysis of the same closed-loop system, are used to analyze the equilibrium set. With respect to the ability to promote hunting limit cycling they report:

By analyzing the time-domain response of a servo with

![Figure 1: Simple mass system subjected to friction.](image1)

![Figure 2: Different frictional damping characteristics and the attraction sliding mode.](image2)
PID position control and Coulomb or Coulomb + static friction, it has been shown that no system with Coulomb friction alone will show a limit cycle in the neighborhood of the origin, and that every system with minimal Coulomb friction and nonzero static friction has such a limit cycle as a possible motion.... when the Coulomb + static friction model is valid and the relative magnitudes of Coulomb and static friction lie in their normal range, there is no combination of PID control parameters that will eliminate stick-slip.

where the terms "nonzero static friction" and "normal range" express the range of the parameters $F_c > F_s > 0$.

The basic idea to analyze the local stability of the equilibrium points is to linearize the different frictional damping characteristics from above and from below to $x_c = 0$, i.e., linearize the state equations for the slip phase of Eq. 1. If the resulting closed-loop system, with the linearized slip phase embedded in Eq. 1, can locally be represented as a PID-controlled system with Coulomb friction alone the equilibrium points are locally stable. The exact algebraic results show for such a closed-loop system that no hunting limit cycle exist on an open interval near the origin in $x_1$ direction, i.e., solutions which start in the stick phase. Starting on this open interval the equilibrium set will be reached either with asymptotic convergence or with a contracting sequence of stick and slip transitions. A sufficient condition for these algebraic results is that the linear part of the closed-loop dynamics, i.e., the homogeneous part of Eq. 1, is asymptotically stable. To perform the analysis on the set of the equilibrium points for the above discussed frictional damping functions, i.e., $g(x_c) = F_s + (F_c - F_s) \exp\left[-\frac{|x_c|}{v_s}\right]$, the following Taylor expansion at $x_c = 0$ for $\beta > 1$ can be constructed:

$$g(x_c) = g(0) + \frac{\partial g(y)}{\partial y} |_{y=0} x_c + R(x_c) \approx F_s,$$

where $R(x_c)$ is the rest term of this expansion with $R(0) = 0$ and $\frac{\partial g(y)}{\partial y} |_{y=0} = 0$ and $v_s$ is supposed to be positive.

For $\beta = 1$ this damping function $g(x_c)$ is not differentiable at $x_c = 0$, but let us analyse this function for positive velocity $x_c$ going to zero:

$$g(x_c) = \lim_{y \to 0} g(y) + \frac{\partial g(y)}{\partial y} |_{y=0} x_c + R(x_c)$$

$$\approx F_s + \frac{(F_c - F_s)}{v_s} x_c,$$

where $R(x_c)$ is again the rest term with $\lim_{y \to 0} R(y) = 0$ and $\frac{\partial g(y)}{\partial y} |_{y=0} = 0$. For negative velocity $x_c$ going to zero the following approximation holds $g(x_c) \approx F_s - \frac{(F_c - F_s)}{v_s} x_c$. Combining both results for $\beta = 1$ the linearization can be written as:

$$g(x_c) \approx F_s + \frac{(F_c - F_s)}{v_s} |x_c|.$$

For the range $0 < \beta < 1$ of the shaping parameter, the function $g(x_c)$ is also not differentiable at $x_c = 0$ and moreover its first derivative with respect to $x_c$ is undefined for both positive and negative velocity $x_c$ approaching zero. Hence, the nonlinear frictional damping function in the slip phase can not be linearized for $0 < \beta < 1$.

Using the different linearizations of the nonlinear frictional damping characteristics in the slip of the friction force, the following linearized friction forces are found about $x_c = 0$:

$$F(x_c) \approx \begin{cases} F_s \sgn(x_c) + bx_c, & \forall \beta > 1 \\ F_s \sgn(x_c) + \frac{(F_c - F_s)}{v_s} x_c + bx_c, & \text{if } \beta = 1 \\ \text{undefined}, & \forall 0 < \beta < 1 \end{cases}$$

Substitution of either the first or second linearization in the slip phase of the state space representation of Eq. 1, the closed-loop system is locally reduced to a PID controlled simple mass system subjected to Coulomb friction only, where the Coulomb friction level is equal to the maximum static friction level of the original system. Moreover for $\beta = 1$, the linear dynamics in the slip phase of the closed-loop system is modified by the linearization. Hence, a sufficient condition for the local stability of the equilibrium set can be described as follows: if the linear part of the dynamics in the slip phase are asymptotically stable the equilibrium set is locally stable. For the first and second region of $\beta$, the characteristic polynomial of the homogeneous part of the linearized closed-loop dynamics about $x_c = 0$ are:

$$s^3 + \frac{PD+b}{m} s^2 + \frac{P+PI}{m} \quad \forall \beta > 1$$

$$s^3 + \left(\frac{PD+b}{m} + \frac{F_c - F_s}{mv_s}\right) s^2 + \frac{P+PI}{m} \quad \forall \beta = 1.$$

For $0 < \beta < 1$, the linearization is undefined, which disables us to construct a characteristic polynomial for this region. Using the Routh Hurwitz stability criterion for the linear part of the linearized closed-loop dynamics, the following classification on the stability of the set of equilibria can be obtained (under the assumption that the linear part of dynamics of the original closed-loop system, i.e., represented by the characteristic polynomial corresponding to $\beta > 1$, are asymptotically stable):

□ Stable for damping functions with continuous stick-slip transitions and $\beta > 1$, since the linear part of the dynamics of the linearized closed-loop system are the same as for the original system.

□ Stable for damping functions with continuous stick-slip transition with $\beta = 1$ if $\left(\frac{PD+b}{m} - I\right) > \frac{(F_c - F_s)}{mv_s}$, which is a sufficient condition.

□ For damping functions with continuous stick-slip transitions where $0 < \beta < 1$, no sufficient condition on the local stability of the set of equilibrium points can be obtained, since the linearization is undefined.

4 Event Mapping: a Phase Plane Analysis

To study the global three-dimensional non-linear dynamics of the closed-loop system for the various damping functions, a technique similar to a Phase Plane Analysis [1] is used. With respect to the friction induced hunting limit cycles the system might be described by a one-dimensional iterated mapping as shown by [12]. Such a reduction of dimension can only take place for a system with sufficiently strong dissipation in the phase space [6]. For the hunting stick-slip oscillations, an infinite dissipation occurs when the closed-loop system enters the stick phase, i.e., a finite volume in the slip phase is reduced to zero volume in finite time in the stick phase. To illustrate the event mapping technique, as presented by [6], the closed-loop system of Eq. 1 is considered for a damping function with $\beta = 2$. The system is initialized in the stick phase with $x(0) = [0.1 0 0]^T$ and the closed-loop parameters used in this
simulation study are given in Table 1. In the upper plot of Figure 3, the time response of the system is shown for all three states. The equilibrium points define an continuous line \( E \) with an upper and a lower bound for \( x_3 \) as depicted in the phase portrait in the lower plot of Figure 3.

**Table 1: Closed-loop system parameters.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>( m = 1 ) [kg]</td>
</tr>
<tr>
<td>Static friction level ( F_s )</td>
<td>( 4 ) [N]</td>
</tr>
<tr>
<td>Coulomb friction level ( F_c )</td>
<td>( 2 ) [N]</td>
</tr>
<tr>
<td>Strubeck velocity ( v_s )</td>
<td>( 0.1 ) [m/s]</td>
</tr>
<tr>
<td>Shape parameter ( \beta )</td>
<td>( 2 ) [-]</td>
</tr>
<tr>
<td>Viscous damping ( b )</td>
<td>( 1 ) [N.s/m]</td>
</tr>
<tr>
<td>Controller gain ( P )</td>
<td>( 10 ) [N/m]</td>
</tr>
<tr>
<td>Derivative action ( D )</td>
<td>( 1 ) [s]</td>
</tr>
<tr>
<td>Integral action ( I )</td>
<td>( 10 ) [1/s]</td>
</tr>
<tr>
<td>Narrow stick width ( \eta )</td>
<td>( 1 \times 10^{-8} ) [m/s]</td>
</tr>
<tr>
<td>Acceleration coefficient ( \alpha )</td>
<td>( 10 ) [1/s]</td>
</tr>
</tbody>
</table>

These three points, depicted with big dots, are part of the event map and describe only a very small part of the global dynamics of the system.

To investigate the global dynamics of the closed-loop system, the function \( P(d^r) \) has to be computed, which will be constructed numerically since it is impossible to derive an analytic expression for the event map of the closed-loop systems with non-linear damping functions. To construct the event map, the system is initialized in the stick phase for different initial positions \( x_3 = d^s \) and integrated numerically until the system enters the stick phase again. Hence, on a grid \( d^1 = [0, \Delta d, \ldots, d - \Delta d, d] \) the next iterated points \( d^2 \) of the map are computed. These points are elements of the single valued function \( P(d^r) \) defining the global dynamics of the closed-loop system. Due to the fact that the closed-loop system is odd with respect to the origin the function \( P(d^r) \) will also be odd, i.e., \( P(d^r) = -P(-d^r) \) and therefore the numerical integration is limited to positive initial positions. The elements of the function \( P(\{d^r, d^r + \Delta d, \ldots, d - \Delta d, d\}) \) with \( \Delta d = 0.001 \) and \( d = 0.5 \), as depicted with small dots in Figure 4, give an impression of the global dynamics. To locate a limit cycle, i.e., a closed trajectory, the second iterated event map is of interest, since the first iterated map \( P(d^r) \), as shown in the figure, maps an initial state \( d^r \to a state d^{r+1} \) which has an opposite sign after one event and will not close the trajectory immediately, as can be seen Figure 3. However, the map describing an initial state \( d^r \to a state d^{r+2} \) which has possibly the same sign after two events might close the trajectory at once when \( d^r = d^{r+2} \). This second iterated map is defined by

\[
d^{r+2} = P(d^{r+1}) = P(P(d^r)) = P^2(d^r)
\]

and the fixed points of the map \( P^2 \) locate both (i) the equilibrium points of the system and (ii) the limit cycles. The fixed points of the map are located at

\[
d^r = \{d^r \mid d^{r+2} = P^2(d^r) = d^r\}
\]

and the local stability of the fixed points of the map can be determined by the contraction theorem [14]. A fixed point of the map are locally stable if the response is locally contracting to this point, i.e., the slope of the map \( P^2 \) in the fixed point of interest has an absolute value smaller than 1. The fixed point are locally unstable if the response is locally diverging from this point, i.e., the slope of the map \( P^2 \) in the fixed point of interest has an absolute value larger than 1.

In Figure 5, the second iterative event map \( P^2(d^r) \) is shown together with its fixed points \( d^r \in \{d_1, d_2, d_3\} \). The exact location of the limit cycles are computed by approximating the single
value function \( P^2(d^*) \) in their vicinity with a third order polynomial through four neighbouring data points. Through the origin, defining the equilibrium set, the function \( P^2(d^*) \) is approximated with a line through three data points including the origin. These approximation are used to locate the fixed points of the map and their local stability can be determined. In Table 2, the occurring types of solution together with their location and local stability are given. The equilibria of the closed-loop system, for a damping function with shaping factor \( \beta = 2 \), are indeed locally stable as predicted analytically in the previous section. Moreover, there exist for this system two limit cycles, where one is stable and one is unstable. Another interesting feature of the one-dimensional event map is the ability to describe the attractor basins for the different solutions, i.e., \( d^* \in (d_1^*, d_2^*) \) will eventually end up in the origin and therefore in the equilibrium set and starting with a line through three data points including the origin. These approximations are used to locate the fixed points of the map and their local stability can be determined. In Table 2, the occurring types of solution together with their location and local stability are given.

**Table 2:** Fixed points of the second iterated event map \( P^2(d^*) \).

<table>
<thead>
<tr>
<th>Fixed point</th>
<th>Type of solution</th>
<th>Local stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_1^* = 0 )</td>
<td>Equilibrium set</td>
<td>Stable</td>
</tr>
<tr>
<td>( d_2^* = 0.0034 )</td>
<td>Limit cycle</td>
<td>Unstable</td>
</tr>
<tr>
<td>( d_3^* = 0.2523 )</td>
<td>Limit cycle</td>
<td>Stable</td>
</tr>
</tbody>
</table>

Figure 5: Second iterated event map locating the equilibrium set and limit cycles.

In this section, the location of the equilibrium points and limit cycles together with their local stability, are determined for two other damping functions, i.e., with shaping parameter \( \beta = 1 \) and \( \beta = 0.5 \). Again the corresponding event map \( P^2(d^*) \) is used to analyse the various systems where the closed-loop system parameters are chosen equal to the parameters in Table 1 except for the shaping parameter \( \beta \). In Figure 6 the various second iterated event maps are given and for the various damping functions the closed-loop system shows only one limit cycle, which are depicted with \( d_2^* \), which is clearly different with the two limit cycles found for the damping function with \( \beta = 2 \) in Section 4. For \( \beta = 1 \) the equilibrium set \( d_1^* \) is unstable as shown in the magnified part of Figure 6 a), since the slope through the origin is larger than 1. Using the sufficient condition for the stability of the equilibrium set as derived in Section 3, this result can be tested analytically, i.e., \((\frac{2d_1^*}{m} - 1) = 1 \) [1/s] should be larger than \( (\frac{d_1^*}{m} - 1) \) = 20 [1/s], which is not satisfied. The numerical results are in line with the analytic result, however this condition cannot be used in this sense since the condition is only a sufficient one. The details about the fixed points for the event map for \( \beta = 1 \) are given in Table 3. For the damping function where \( \beta = 0.5 \) an interesting phenomenon appears for the equilibrium set, i.e., in the neighbourhood of the origin. The event maps, as depicted in Figure 6 b), seem to be discontinuous around the origin predicting unstable sets of equilibrium points. This numerical result is also in line with the analytic results on the stability of the equilibrium set for damping functions where \( 0 < \beta < 1 \), since the homogeneous part for this linearized closed-loop system are always unstable. Moreover, the attractor basin for the stable limit cycle is the entire domain excluded the origin. The details on these two damping functions are also given in Table 3.

An interesting question is what will happen when for instance the controller gain \( P \) is altered. Moreover, it is of interest to know how the system characteristics change for the different damping functions. Hence, the fixed points of the second iterated event maps together with their local stability for the various damping functions, on a grid \( P = \{0, \ldots, 0.31\} \) [N/m] with step-size 1, are computed and shown in Figure 7. The so-called bifurcation diagram of Figure 7 a), predicts for the frictional damping function with \( \beta = 2 \), a disappearance of the hunting limit cycles for controller gains larger than approximately \( P = 21 \) [N/m]. The equilibrium set is for all computed controller gains locally stable as expected by the analytic results presented in Section 3. Figure 7 b) presents the changing dynamics for the system with \( \beta = 1 \), where the unstable equilibrium set becomes stable and intersects with the stable limit cycles for controller gain \( P \approx 27 \) [N/m]. Larger controller gains predict only stable equilibrium points and no limit cycles. The sufficient condition derived for this closed-loop system predicts locally stable

**Figure 6:** Second iterated event map locating the equilibrium set and limit cycles for various damping functions.

**Table 3:** Fixed points of \( P^2(d^*) \) for the various damping functions.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Fixed point</th>
<th>Type of solution</th>
<th>Local stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta = 1 )</td>
<td>( d_1^* = 0 )</td>
<td>Equilibrium set</td>
<td>Unstable</td>
</tr>
<tr>
<td>( \beta = 1 )</td>
<td>( d_2^* = 0.2314 )</td>
<td>Limit cycle</td>
<td>Stable</td>
</tr>
<tr>
<td>( \beta = 0.5 )</td>
<td>( d_1^* = 0 )</td>
<td>Equilibrium set</td>
<td>Unstable</td>
</tr>
<tr>
<td>( \beta = 0.5 )</td>
<td>( d_2^* = 0.1786 )</td>
<td>Limit cycle</td>
<td>Stable</td>
</tr>
</tbody>
</table>
where this table can be concluded that practically identified friction curves with a continuous transition of the friction force from the stick phase to the slip phase is given in Table 4 with respect to the control of machines with friction”, *Automatica*, **30**, 1083-1138, (1994).


