Wave propagation and localisation in nonlocal and gradient-enhanced
damage models

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Abstract. Classical continuum descriptions of material degradation may cease to be mathematically meaningful in case of softening-induced localisation of deformation. Several enhancements of conventional models have been proposed to remove this deficiency. The properties of two of these so-called regularisation methods, the nonlocal and the gradient approaches, are examined and compared in a continuum damage context. It is shown that the enhanced models allow for the propagation of waves in the softening zone, in contrast to conventional damage models. For both types of enhancement wave propagation becomes dispersive. The behaviour under quasi-static loading conditions is studied numerically. Finite element simulations of a one-dimensional problem yield quite similar results for the nonlocal and a gradient-enhanced model. The gradient enhancement has been used to model concrete fracture, yielding results which are in good agreement with experimental data.

1. INTRODUCTION

The introduction of strain softening in classical plasticity or continuum damage theory generally leads to erroneous results. Numerical analyses show an extreme sensitivity to the fineness and orientation of the spatial discretisation. Upon refinement of the discretisation, convergence is observed to a solution in which deformation is localised in an infinitely narrow band. This response is unacceptable from a physical point of view because it does not allow for any energy dissipation in the fracture process. The underlying mathematical cause is that the partial differential equations which govern the deformation process locally lose hyperbolicity in dynamics and ellipticity in quasi-static analyses, rendering the initial-boundary value problem ill-posed. If the continuum context is to be preserved, the mathematical model must be enriched by additional terms to prevent this change of type. A number of approaches can be distinguished, which in one way or another introduce an internal length scale in the material description in order to regularise the localisation process [1–3].

Two of the regularisation techniques, the nonlocal and gradient approaches, are closely related. Both methods introduce a spatial interaction, either explicitly (nonlocal) or in the form of higher-order deformation gradients (gradient). Historically, gradient regularisation has been applied primarily in plasticity models, whereas nonlocal theory has found its application almost exclusively in damage mechanics [4–7]. Recently, however, gradient-enhanced damage models have become available [8–10], so that it is now possible to study the intrinsic similarities and differences of the two regularisation methods, without these being obscured by the differences between the damage and plasticity frameworks. To this end, the behaviour of a gradient-damage model for quasi-brittle fracture developed by the authors [10] is examined in a one-dimensional context in dynamics and statics and is compared with that of the underlying nonlocal formulation. Also, the effectiveness of the gradient approach is demonstrated by an application of the gradient model to concrete fracture.
2. CONSTITUTIVE RELATIONS

The classical constitutive equation for quasi-brittle damage reads
\[ \sigma = (1 - D)^4 \mathbf{C} : \varepsilon, \]
with \( \sigma \) and \( \varepsilon \) denoting the Cauchy stress tensor and the linear strain tensor; \( ^4\mathbf{C} \) stands for the fourth-order elasticity tensor. The damage variable \( D \) is zero for the initial, undamaged material; it can grow until \( D = 1 \), which characterises the complete loss of material cohesion, \textit{i.e.}, the situation where the material cannot transfer stresses anymore. The evolution of damage is governed by the deformation. A history variable \( \kappa \) is introduced for this purpose, which represents the most severe deformation the material has experienced and which acts as a threshold below which there is no damage development. The damage variable is given as an explicit function of this history variable. In the conventional, local damage model, \( \kappa \) is related to a scalar measure of the local deformation, the damage-equivalent strain \( \varepsilon_{\text{eq}}(\varepsilon) \). But in the nonlocal damage model \( \kappa \) is related to a weighted volume average of this (local) equivalent strain, the nonlocal equivalent strain \( \overline{\varepsilon}_{\text{eq}} \).

This quantity is defined in each material point \( \bar{x} \) by
\[ \overline{\varepsilon}_{\text{eq}}(\bar{x}) = \int_V g(\bar{\xi}) \varepsilon_{\text{eq}}(\bar{x} + \bar{\xi}) \, dV, \]
with \( \bar{\xi} \) a position vector relative to point \( \bar{x} \). The weight function \( g(\bar{\xi}) \) is often defined as
\[ g(\bar{\xi}) = \left( \frac{1}{\sqrt{2\pi l}} \right)^3 \exp\left( -\frac{||\bar{\xi}||^2}{4l^2} \right). \]
Notice that this weight function indeed introduces a length scale, \( l \), which has been shown to govern the width of the localisation band [5]. The history variable \( \kappa \) is now defined by the Kuhn-Tucker relations
\[ \kappa \geq 0, \quad \overline{\varepsilon}_{\text{eq}} - \kappa \leq 0, \quad \dot{\kappa}(\overline{\varepsilon}_{\text{eq}} - \kappa) = 0 \]
and the initial value \( \kappa_0 \).

Two versions of gradient enhancement are considered here, which can both be derived from the nonlocal formulation [10]. Substitution of the weight function (3) and a Taylor expansion of the local equivalent strain into (2) gives
\[ \overline{\varepsilon}_{\text{eq}} = \varepsilon_{\text{eq}} + \frac{1}{2} l^2 \nabla^2 \varepsilon_{\text{eq}} + \frac{1}{8} l^4 \nabla^4 \varepsilon_{\text{eq}} + \ldots \]
Thus, neglecting terms of order four and higher, the integral definition (2) of the nonlocal strain can be approximated by the explicit differential expression
\[ \overline{\varepsilon}_{\text{eq}} = \varepsilon_{\text{eq}} + \frac{1}{2} l^2 \nabla^2 \varepsilon_{\text{eq}}. \]
The second, implicit form of gradient enhancement is obtained as follows. If (5) is differentiated twice, multiplied by \( \frac{1}{2} l^2 \) and the result subtracted from (5) itself, the following equivalent relation is obtained:
\[ \overline{\varepsilon}_{\text{eq}} - \frac{1}{2} l^2 \nabla^2 \overline{\varepsilon}_{\text{eq}} = \varepsilon_{\text{eq}} - \frac{1}{8} l^4 \nabla^4 \varepsilon_{\text{eq}} - \ldots \]
Comparing (5) and (7), it can be concluded that an approximation of the same order is introduced if
\[ \overline{\varepsilon}_{\text{eq}} - \frac{1}{2} l^2 \nabla^2 \overline{\varepsilon}_{\text{eq}} = \varepsilon_{\text{eq}} \]
is taken as the definition of \( \overline{\varepsilon}_{\text{eq}} \) instead of (6). Equation (8) must then be treated as an additional partial differential equation, complementary to the balance of momentum. Although both gradient methods preserve the internal length scale through the respective second-order terms and both methods introduce an approximation of the same order, it will be shown in the following section that the properties of the damage model based on (8) are much more similar to the properties of the nonlocal model than those of the model based on (6). This is believed to be caused by the fact that higher-order derivatives of \( \varepsilon_{\text{eq}} \) are still implicitly present in relation (8), while these terms have been explicitly neglected in (6).
3. WAVE PROPAGATION

Much insight in the properties of softening material models can be gained from an analysis of wave propagation in these models. In conventional continua, the change of type of the governing partial differential equations, caused by softening, prohibits the propagation of loading waves in the softening zone. The wave velocity locally becomes imaginary and deformation is trapped in an infinitely narrow band, in which the strain can grow unboundedly [11, 12]. Peerlings et al. [13] have shown that in a one-dimensional setting the hyperbolicity of the governing equations is preserved in the softening regime for both gradient models. As a result, waves can propagate in the softening zone and the deformation no longer localises in a band of zero width. Wave propagation is dispersive for these models, that is, the velocity of a harmonic wave depends on its wave number. In fact, dispersion is essential for the proper description of localisation under dynamic loading, since it enables the emergence of a standing wave of non-zero width, which can develop into a zone of localised deformation [3, 14].

The wave propagation behaviour of the nonlocal and gradient models is demonstrated here in a one-dimensional, unbounded domain. The linear comparison solid associated to the nonlinear models is considered, i.e., loading is assumed and the equations of motion are linearised around a homogeneous equilibrium state defined by the strain field $\varepsilon_0$ and the corresponding damage $D_0$. Furthermore, linear softening has been assumed and the equivalent strain has been set equal to the axial strain, which is taken positive (tension). A solution is sought in the form of a single harmonic displacement perturbation

$$\delta u = \hat{u} \exp\{i k (x - ct)\}$$

with wave number $k$ and corresponding phase velocity $c$. The non-trivial solution of the resulting equation gives the phase velocity as a function of the wave number. For the three models discussed above these relations read [13]:

nonlocal (Eq. (2)): $$c = c_e \sqrt{1 - D_0 - \varepsilon_0 \left( \frac{\partial D}{\partial \kappa} \right)_0 \exp \left\{ -\frac{1}{2} k^2 l^2 \right\}}$$

(10)

explicit (Eq. (6)): $$c = c_e \sqrt{1 - D_0 - \varepsilon_0 \left( \frac{\partial D}{\partial \kappa} \right)_0 \left( 1 - \frac{1}{2} l^2 k^2 \right)}$$

(11)

implicit (Eq. (8)): $$c = c_e \sqrt{1 - D_0 - \frac{\varepsilon_0 \left( \frac{\partial D}{\partial \kappa} \right)_0}{1 + \frac{1}{2} l^2 k^2}}$$

(12)

where $c_e$ denotes the elastic wave speed $\sqrt{E/\rho}$. These wave velocities have been plotted in Figure 1(a) at the onset of damage growth (i.e., $D_0 = 0$). The parameters were chosen such that the elastic wave speed is $c_e = 1000$ m/s; the internal length was set to $l = \sqrt{2}$ mm. See reference [13] for the complete set of parameters. For small wave numbers, the three regularisation methods show almost the same behaviour. Below the critical wave number $k_c$, which at the peak stress is virtually equal for all three models, the wave speed is imaginary. In theory, the perturbation is unbounded for these values, but this effect is introduced by restricting the analysis to the linear comparison solid. In a ‘real’ softening solid, the large wave lengths associated to wave numbers smaller than $k_c$ cannot exist in the softening zone [13, 14]. For higher wave numbers, the approximation in deriving the gradient models from the nonlocal formulation becomes apparent. Higher-order terms play a more important role for these small wave lengths. The nonlocal model and the implicit formulation (8) have a horizontal asymptote equal to the elastic wave speed, but in the formulation with the explicit relation (6) $c$ is not bounded. Waves with an infinitely short wave length can propagate with an infinite speed in this model, which seems unrealistic from a physical point of view. The discrepancy between the two gradient models in this respect is probably caused by higher-order derivatives being indirectly present in the implicit formulation.
Figure 1: (a) Wave velocities and (b) critical wave lengths for the nonlocal and gradient models.

The critical wave number $k_c$ below which the phase velocity is imaginary depends on the strain level which is taken as a reference for the linear comparison solid. For the three localisation limiters $k_c$ can be derived as [13]:

nonlocal (Eq. (2)):  
$$k_c = \frac{1}{l} \sqrt{2 \ln \left( \frac{\varepsilon_0 \left( \frac{\partial D}{\partial \kappa} \right)_0}{1 - D_0} \right)},$$  
(13)

explicit (Eq. (6)):  
$$k_c = \frac{1}{l} \sqrt{2 \left( 1 - \frac{1 - D_0}{\varepsilon_0 \left( \frac{\partial D}{\partial \kappa} \right)_0} \right)},$$  
(14)

implicit (Eq. (8)):  
$$k_c = \frac{1}{l} \sqrt{2 \left( \frac{\varepsilon_0 \left( \frac{\partial D}{\partial \kappa} \right)_0}{1 - D_0 - 1} \right)},$$  
(15)

Figure 1(b) shows the critical wave length $\lambda_c = 2\pi/k_c$, with $k_c$ according to (13)–(15), as a function of the uniform strain $\varepsilon_0$. It is this critical wave length that sets the width of the localisation band [14]. For an increasing damage level $\lambda_c$ decreases, which is consistent with the narrowing of the damage evolution zone observed in static analyses [10, 13]. In the nonlocal and the implicit gradient damage models $\lambda_c$ becomes zero for the strain level which corresponds with complete fracture. Physically, this represents a line crack, and consequently a natural transition from a damaged zone into a line crack is obtained. In contrast, the explicit gradient formulation results in a finite width of the damage process zone also for complete loss of coherence, which precludes a gradual transition into a line crack.

4. STATIC LOADING

Complications quite similar to those encountered in dynamic localisation problems may also occur in quasi-static problems. It is therefore useful to study the behaviour of the gradient-enhanced and nonlocal models also for this situation. For this purpose, finite element analyses of a simple, one-dimensional bar problem have been performed with both the nonlocal and the implicit gradient damage model. The explicit gradient model is less suited for a finite element implementation [10] and is therefore left out of consideration here. The geometry of the bar is given in Figure 2; the cross-section has been reduced by 10% in the centre in order to trigger localisation of damage in this area. The bar is subjected to a uniaxial, tensile stress which is applied by an indirect displacement control procedure [15]. Reference is made to [10] for the complete problem statement and a discussion of the numerical solution procedure.
Upon refinement of the finite element mesh both models show convergence to load-displacement curves with a finite energy dissipation [13]. Figure 3 gives the converged load-displacement curves of both models. The predicted tensile strengths are practically equal and the initial softening paths also agree well. The gradient-dependent formulation exhibits a somewhat more brittle post-peak behaviour than the nonlocal model. Using the gradient model, the softening path could be followed throughout the snap-back phase until \( \sigma = 0 \) was reached. For the nonlocal model convergence of the solution procedure was lost at the onset of snap-back. Figure 4 shows the strain (Figure 4(a)) and damage profiles (Figure 4(b)) for \( \Delta L = 0.01, 0.02, 0.03 \) and 0.04 mm. For relatively small displacements, these profiles almost coincide. For larger elongations the deformation is slightly more localised in the gradient model than in the nonlocal model, which explains the more brittle softening branch in Figure 3. Both the gradual narrowing of the process zone and the increasing deviation between the two models for increasing deformations are consistent with the evolution of the critical wave lengths as presented in Section 3.
5. APPLICATION TO CONCRETE FRACTURE

The (implicit) gradient damage model has been used to simulate the fracture of a single-edge notched beam subjected to an anti-symmetric four-point loading. The geometry and loading conditions (Figure 5) have been taken from experiments of Schlangen [16]. The beam has been modelled using 1362 elements with a quadratic interpolation of the displacements and a separate, bilinear interpolation of the nonlocal equivalent strain. The two loading platens near the centre of the beam have been modelled as rigid bodies by introducing appropriate linear dependencies between the nodal displacements. The loading platens at the ends of the beam have been modelled as a nodal support and a nodal force, respectively. Plane stress has been assumed and Young’s modulus and Poisson’s ratio were set to $E = 35,000$ MPa and $v = 0.2$; an internal length of $l = \sqrt{2}$ mm was used. The damage growth law was defined such that exponential softening results in uniaxial tension. For the equivalent strain the so-called modified von Mises definition was used. This definition is based on the standard von Mises flow criterion, but has been extended with a dependence on the first stress invariant in order to model the different responses of concrete in tension and compression; reference is made to Peerlings et al. [17] for the exact definition of the equivalent strain as well as the complete problem definition.

![Figure 5: Single-edge notched beam configuration.](image)

The evolution of the damage variable has been plotted in Figure 6. In the first stage of the fracture process, damage is initiated at the right corner of the notch and opposite to the central loading platens. The damage growth at the latter locations is arrested at a certain stage; in the experiments, some cracking was indeed observed in these areas [18]. The damage zone at the notch continues to grow, following a curved path which ends to the right of the lower right loading platen. The final damage distribution is given in Figure 7, along with the experimentally determined crack paths in three different specimens as

![Figure 6: Damage evolution in the single-edge notched specimen.](image)
given by Schlangen [16]. The agreement of the experimental and numerical paths is satisfactory, with the experimental cracks lying within the damage band obtained in the numerical analysis. In Figure 8 the computed crack mouth opening and sliding displacements (CMOD and CMSD) are compared with the measured data [16]. Again the numerical curves lie within or close to the band formed by the experimental curves.

6. CONCLUDING REMARKS

It has been shown in the preceding sections that gradient-enhanced and nonlocal damage models can show similar behaviour, while the former is more attractive from a computational point of view [10]. Physically realistic results are obtained, which are insensitive to the spatial discretisation if the discretisation is sufficiently fine. The width of the damage band is independent of the element size and the application in Section 5 shows that the direction of damage growth is also independent of the mesh orientation. Nevertheless, the wave propagation analysis has revealed that the precise response of the gradient model critically depends on the format in which the gradient term is included in the constitutive model. This shows the importance of carefully examining the behaviour of material descriptions which result from the introduction of gradient dependence – or in fact any regularisation method.
REFERENCES