A NUMERICAL ANALYSIS OF STEADY FLOW IN A THREE-DIMENSIONAL MODEL OF THE CAROTID ARTERY BIFURCATION

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Abstract—A finite element approximation of steady flow in a rigid three-dimensional model of the carotid artery bifurcation is presented. A Reynolds number of 640 and a flow division ratio of about 50:50, simulating systolic flow, was used. To limit the CPU- and I/O-times needed for solving the systems of equations, a mesh-generator was developed, which gives full control over the number of elements into which the bifurcation is divided. A mini-supercomputer, based on parallel and vector processing techniques, was used to solve the system of equations. The numerical results of axial and secondary flow compare favorably with those obtained from previously performed laser-Doppler velocity measurements. Also, the influence of the Reynolds number, the flow division ratio, and the bifurcation angle on axial and secondary flow in the carotid sinus were studied in the three-dimensional model. The influence of the interventions is limited to a relatively small variation in the region with reversed axial flow, more or less pronounced C-shaped axial velocity contours, and increasing or decreasing axial velocity maxima.

INTRODUCTION

Detailed information about the flow field in arterial systems is important for several reasons. Firstly, there is increasing evidence that there is a relation between atherosclerosis and hemodynamics. It is likely that atherosclerotic lesions coincide with low shear (Caro et al., 1971; Ku et al., 1985) rather than with high shear regions (Fry, 1968). Secondly, disturbances in the flow pattern are commonly used to diagnose atherosclerotic lesions at an early stage of the disease (van Merode et al., 1988). Therefore, it is necessary to distinguish flow disturbances, as induced by these lesions, from those normally occurring in non-stenosed arteries (Reneman et al., 1985). An arterial system of clinical interest is the carotid artery bifurcation. One of the daughter branches, the internal carotid artery, is characterized by a widening in its most proximal part, the carotid sinus or bulb, which is often affected by atherosclerotic plaque formation (Ku et al., 1985).

A study by Rindt et al. (1988a) and van de Vosse (1987) indicated that three-dimensional analysis of the flow field in the carotid artery bifurcation is necessary to better understand the in vivo flow situation. Especially in the region with reversed axial flow, large differences are observed between the axial velocities in the two-dimensional model and those in the plane of symmetry of the three-dimensional model. These differences are mainly caused by secondary flow, which is absent in the two-dimensional situation. To our knowledge, up to now only calculations of fluid flow in two-dimensional models of the carotid artery bifurcation (Fernandez et al., 1976; Rindt et al., 1987) or in a simplified three-dimensional model (Wille, 1984) have been presented, the latter dealing with an unphysiologically low Reynolds number of 10. Besides, Perktold et al. (1987) made a three-dimensional analysis of pulsatile blood flow in a carotid siphon model, using trilinear velocity-constant pressure elements. Only recently, Rindt et al. (1988b) and Perktold and Hilbert (1988) briefly described numerical results of blood flow through an arterial bifurcation model, both using the finite element method. The use of two-dimensional or simplified three-dimensional models of the more complicated in vivo situation can be explained by the rather complex geometry of a bifurcation, which is hard to divide into elements for the three-dimensional situation, and the relatively large computing times needed to solve the system of equations resulting from three-dimensional analysis. To overcome these problems a structured mesh-generator was developed and a mini-supercomputer, based on parallel and vector processing techniques, was used to solve the system of equations.

The present study deals with a finite element approximation of steady flow in a rigid three-dimensional model of the carotid artery bifurcation at physiological values of the Reynolds number (Ku et al., 1985), using triquadratic velocity-linear pressure elements. Although this study deals with steady flow calculations, it is believed that the steady flow characteristics show essential arterial flow phenomena. The geometry of the numerical model was similar to that of the model used in previous experiments (Rindt et al., 1988a), a geometry based upon the one described by Bharadwaj et al. (1982). First, the numerical model will...
be discussed and a description of axial and secondary flow is given. To validate the numerical model a comparison was made between the numerically predicted velocities and the laser-Doppler velocity measurements (Rindt et al., 1988), previously performed in an in vitro setup similar to the one described by Bovendeerd et al. (1987). Because the Reynolds number and the flow division ratio over the daughter branches vary considerably under pulsatile flow conditions, the influence of these features on both axial and secondary flow in the carotid sinus was studied in the numerical model. Besides, the influence of a smaller bifurcation angle is described because pulsed-Doppler measurements (Reneman et al., 1985) and a post-mortem cast study (unpublished results) indicated that this angle is essentially smaller than the one proposed by Bharadvaj et al. (1982).

**APPLIED METHODOLOGY**

This study focusses on the analysis of steady flow of an incompressible and Newtonian fluid. The flow of such a fluid in rigid walled three-dimensional geometries is described by the Navier-Stokes and continuity equations together with the boundary conditions. In dimensionless form these equations read:

\[ \begin{align*}
\mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + f + 1/Re \nabla^2 \mathbf{u} \\
\nabla \cdot \mathbf{u} &= 0
\end{align*} \]  

(1a) \hspace{1cm} (1b)

with \( \mathbf{u} \) the velocity vector, \( p \) the pressure, \( f \) the volume forces, \( V \) the gradient operator and \( \nabla^2 \) the Laplacian operator. \( Re \) denotes the Reynolds number and is defined as \( Re = VD/v \) with \( V \) the mean axial velocity in, and \( D \) the diameter of, the common carotid artery, and \( v \) the kinematic viscosity. Using Galerkin's finite element approximation (Cuvelier et al., 1986), these equations are transformed into the following set of non-linear equations:

\[ \begin{align*}
[S + N(U)] U + L^T P &= F + B \\
LU &= 0.
\end{align*} \]  

(2a) \hspace{1cm} (2b)

The first equation proceeds from the discretization of the Navier-Stokes equation and the second equation from the discretization of the continuity equation. The vector \( U \) contains the velocity unknowns and \( P \) the pressure unknowns in discrete points. The term \( SU \) represents the viscous forces, \( N(U)U \) the convective acceleration forces and \( L^T P \) the pressure gradient forces. The right-hand side of the first equation consists of the volume forces \( F \) and the boundary forces \( B \). The equation \( LU = 0 \) represents the discretized continuity equation.

In the solving process of the above set of equations a Newton-Raphson iteration method is used to linearize the convective term (Cuvelier et al., 1986). This term is replaced by:

\[ N(U^{t+1})U^{t+1} = J(U^t)U^{t+1} - N(U^t)U^t \]  

(3)

with \( t \) the iteration number and \( J \) the Jacobian matrix of \( N \). As stop criterion for the iteration process, the maximal difference between two successive solutions in the same discrete point is used. Because no pressure unknowns appear in the discretized continuity equation (2b), partial pivoting is necessary to solve the velocity and pressure unknowns from the equations (2a) and (2b). To overcome the problem of partial pivoting, which is very time consuming and has a negative effect on the band structure of the matrix, a penalty function method is used (Cuvelier et al., 1986), enabling the elimination of the pressure unknowns from the discretized Navier-Stokes equation (2a). In this case the discretized continuity equation reads:

\[ LU^{t+1} = \varepsilon M_p P^{t+1} \]  

(4)

with \( M_p \) the pressure mass matrix and \( \varepsilon \) a very small parameter \([O(10^{-5})]\) so that the right-hand side of the above equation becomes very small.

In addition to the Navier-Stokes and continuity equations, boundary conditions must be considered. In the experimental set-up, the length of the inlet section ensures the flow to be fully developed before it reached the bifurcation model. Therefore, flow at the entrance of the main branch is equal to Poiseuille flow. At the wall, all velocity components are made equal to zero (no-slip condition). Because the geometry on either side of the bifurcation plane is the same for the model used, it is sufficient to consider only half of the bifurcation. In the plane of symmetry the velocity component normal to this plane and the tangential stresses are supposed to be zero (symmetry condition). At the end of the daughter branches, the normal stress and both tangential stresses are made equal to zero (stress-free condition). For a more detailed description of the prescribed boundary conditions, one is referred to van de Vosse et al. (1988).

Elimination of the pressure unknowns, using equation (4), and linearization of the convective term, using equation (3), results in a set of linear equations with only velocity unknowns. These equations are solved by a standard direct profile method. A disadvantage of the direct method is the necessity of a computer with a large central memory. However, the penalty function method applied does not allow the use of an iterative method. Alternative approaches which can be used in combination with a conjugate gradient method are presently under study. The construction of the system of equations as well as the post-processing of the velocity data are carried out with the finite element package Sepran (Segal, 1982).

The discrete points, in which velocities or pressures are calculated, are determined by partition of the three-dimensional geometry into elements. The three-dimensional element used is known as the \( P_2-P_1 \) element, consisting of 27 nodes for the velocity (tri-quadratic, 81 velocity unknowns) and 1 node for the pressure (linear, 4 pressure unknowns) (Cuvelier et al., 1986). The main reasons for using this element are that it can be employed for the calculation of steady flow in
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a 90° curved tube (van de Vosse et al., 1988) and that the accuracy of the velocity approximation was found to be $O(h^3)$ for a simple test example (Segal, 1986), with $h$ a characteristic element size. For a proper investigation of the numerical errors, it would be necessary to perform calculations with various element sizes. This, however, was not possible, due to the limited capacity of the computer system used.

From the three-dimensional calculations of steady flow in a 90° curved tube, it appeared that these calculations were very time consuming. One iteration on an Appolo-dsp90 took about 24 h of CPU-time and 48 h of I/O-time. For a Reynolds number of 500, about 15 iterations were needed. Because it was expected that the computations of steady flow in a three-dimensional model of the carotid artery bifurcation should take much more computing time, these calculations were carried out on a mini-supercomputer, a Convex-cl-xp. Nevertheless it was necessary to minimize the number of elements into which the bifurcation was divided to limit the I/O-time needed for solving the system of equations by a standard direct profile method. Therefore, a structured mesh-generator was developed, which gave full control over the number of elements and the axial and radial coarseness of the element division. In essence, the plane of symmetry was divided into rectangular two-dimensional elements. Then, this two-dimensional element division was expanded in the third direction by placing three-dimensional elements on it. Finally, the surface of this three-dimensional element division was deformed in order to describe the exact geometry of the carotid artery bifurcation. Due to the complex geometry of the bifurcation near the flow divider some elements in this region were ill-shaped which means that the angle between two edges of an element is essentially larger than 90°, causing numerical errors in the solution of the flow field near the flow divider. The element sizes were taken to be small in the direction of large velocity gradients to reduce the numerical errors resulting from the finite accuracy of the element used.

Figure 1 shows a hidden line plot after division of the carotid artery bifurcation into elements. This three-dimensional mesh consists of 1474 elements and 14,019 nodes. Substitution of the prescribed degrees of freedom resulted in 32,975 velocity unknowns. To equalize the flow division ratio for the numerical case to that for the experimental case (52% through the internal carotid artery), the lengths of the internal and external carotid arteries were chosen to be ten and two times the diameter of the main branch, respectively, thus increasing the flow resistance of the former branch. Besides, the downstream pressures of both branches were taken as equal (zero normal stresses). For the mesh, as shown in Fig. 1, one iteration on a Convex-cl-xp took about 1 h of CPU-time. For convergence of the solution at a Reynolds number of 640, 18 iterations were needed. The total computing time could be reduced by using interpolation of the solution from a coarse to a fine mesh. First, the problem was solved for a coarse mesh for which one iteration took about one quarter of an hour of CPU-time. After convergence the solution was interpolated to the fine mesh, as shown in Fig. 1, and several more iterations were performed until convergence was reached. By using this method, the total computing time was reduced by a factor of 2 to about 10 h of CPU-time.

The numerical results are presented by axial velocity contours and secondary velocity vectors. To compare the numerical with the experimental results, axial velocity profiles in the plane of symmetry as well as axial velocity contours and secondary velocity profiles are presented in one figure for both situations. To enable a quantitative comparison between the computations and the measurements, some characteristic flow quantities are also calculated.

RESULTS

Description of the flow field

For the presentation of the flow field the cross-sections are considered as given in Fig. 2. In Fig. 3a and b, axial flow is presented by means of velocity contours and secondary flow is visualized by means of velocity vectors, at a Reynolds number of 640. The axial flow direction at each level is defined as the direction perpendicular to the cross-sectional plane, whereas the secondary velocities are parallel to this plane. Contour level 0 corresponds with zero axial velocity, and level 10 with maximal axial velocity at the entrance of the common carotid artery. The secondary velocities in the main branch are scaled up 10 times with respect to the other levels.

In the common carotid artery just before the bifurcation (Cl.5), the axial flow field hardly differs from a parabolic flow field. Secondary flow at this site is completely directed from the internal side, the side of the internal carotid artery, towards the external side, pointing at upstream influences due to flow branching. These secondary velocities result in a larger volume
flow through the external carotid artery than expected on the basis of the geometry alone.

At the entrance of the internal artery (10), high axial velocities are found near the divider wall which is primarily caused by division of the flow field at the site of the divider. A region with negative axial velocities with a diameter of about 30% of the local diameter of the bulb is seen opposite to the flow divider. Secondary flow at this site is almost entirely directed towards the divider wall. In a small region near the side wall, secondary flow is directed towards the non-divider wall. When axial flow is defined as the velocity component parallel to the axis of the common carotid artery and secondary flow as the velocity components perpendicular to this axis, it is found that secondary flow at the entrance of the bulb is almost zero over the whole region except for a small region along the side wall. In this region secondary flow is directed towards the non-divider wall. These findings indicate that the main flow direction at the entrance of the internal carotid artery is still parallel to the axis of the main branch.

Halfway along the bulb (11) the geometry of the region with negative axial velocities has changed and in the plane of symmetry increased to a diameter of about 60% of the local bulb diameter. The maximum of axial velocity is shifted towards the divider wall and the low-numbered axial velocity contours are C-shaped. All these effects are strongly related to secondary flow at this site. The secondary vector plot shows some resemblance with a Dean vortex; near the plane of symmetry the secondary velocities are directed towards the divider wall and near the side wall they point circumferentially back towards the non-divider wall.

At the end of the bulb (12) no reversed axial flow region is found. High axial velocities are observed near the divider wall and a region with almost equal axial velocities is found near the non-divider wall. The bending of the axial velocity contours has shifted to the more high-numbered ones. Secondary flow at this site has grown in strength with regard to secondary flow halfway along the bulb. Near the non-divider wall, secondary flow still shows great resemblance with a Dean vortex, but near the divider wall all secondary velocities are directed towards the opposite wall. The latter effect originates from the tapering of the bulb near its end which causes high secondary velocities directed towards the center of the branch in regions with high axial velocities.

In the external carotid artery no reversed axial flow is found. The highest axial velocities are found near the divider wall. Downstream in the branch the high-numbered axial velocity contours become C-shaped. At both sites in the external carotid artery the secondary velocities are directed towards the divider wall near the plane of symmetry and circumferentially back near the side wall. Near the flow divider the secondary velocities are directed towards the opposite wall probably due to boundary layer development.

Descriptive comparison with experiments

In Fig. 4 the axial velocity profiles in the plane of symmetry are presented for both the measurements and the calculations. The agreement between the experimental and numerical data is good. The shape of the region with reversed axial flow in the plane of symmetry as well as the axial velocity plateau is estimated well by the numerical model. Downstream in the external carotid artery the numerical velocities are consequently somewhat higher than the experimental ones. These differences are probably caused by small errors in the adjustment of the volume flow in the experiments.

In Fig. 5a and b for both the measurements and the calculations some axial velocity contours are shown together with some velocity profiles of the x- and y-component of secondary flow. The x- and y-directions of secondary flow are defined as depicted in Fig. 6. Again, contour level 0 corresponds to zero velocity, and level 10 to the maximal axial velocity at the entrance of the main branch. The secondary velocities in the main branch are scaled up 10 times with respect to the other levels. There is a good agreement between the numerically predicted and experimentally measured axial velocities. The largest differences occur at the entrance of the internal carotid artery with regard to the region with reversed axial flow. Within this region the axial velocities are of the order of 0.05 times $U_{max}$ near the plane of symmetry and 0.001 times $U_{max}$ near the side wall, $U_{max}$ being the maximal axial velocity at this site. Therefore, small errors in the axial velocity measurements may cause relatively large differences in the determination of the region with reversed axial flow, while the absolute values of the experimental and the numerical velocities are quite the same. Close to the flow divider numerical
errors due to ill-shaped elements in this region could partly explain the small deviations between experimental and numerical values found there. Downstream in the external carotid artery, the experimental values of axial flow are consequently somewhat lower than the numerical ones. This points to smaller volume flow at this site for the experiments than for the computations. Table 1 presents the relative volume flows at all levels for both the measurements and the calculations together with the 95%-confidence inter-
vals for the experiments. The 95%-confidence intervals for the experiments consist of stochastic errors which are caused by measuring errors of the axial velocity component and the diameter of the level analysed. Systematic errors in the integration procedure for the determination of volume flow, as occurring in both the numerical and the experimental case, are neglected.

From this table, it is concluded that at almost all sides the experimental and numerical volume flows are equal. Only the experimental volume flow downstream in the external carotid artery is essentially smaller than the numerical one. This may be caused by a systematic error in the adjustment of the volume flow through the main branch or by an alteration of the volume flow ratio through the daughter branches during the experiments.

Regarding secondary flow, there is a fair agreement between the numerical and experimental data. The largest differences are found near the flow divider in both daughter branches. This again may be caused by numerical errors due to the ill-shaped elements near the flow divider, but it cannot be excluded that measuring problems near the flow divider also contribute to this discrepancy.

Quantitative comparison with experiments

A quantitative comparison of the numerical and experimental results of axial flow is possible by its first moment $\langle X/R \rangle$, as defined by Olson and Snyder (1985):

$$\langle X/R \rangle = \frac{\int_A u_{ax} x dA}{\int_A u_{ax} dA}$$

![Figure 4](image-url) Comparison between the numerically predicted axial flow field and the experimentally measured one in the plane of symmetry. Solid line: numerical data, circles: experimental data.

![Figure 5](image-url) (a) Comparison of total axial and secondary flow in the main branch and the external carotid artery. Solid line: numerical data, dashed line, circles: experimental data.
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Fig. 5. (b) Comparison of total axial and secondary flow in the carotid sinus. Solid line: numerical data, dashed line, circles: experimental data.

Fig. 6. Definition of circumference $S$ at a cross-section with radius $R$.

Table 1. Relative volume flows at the six levels analysed for the experimental and numerical case. Experimental confidence intervals $\pm 2\%$

<table>
<thead>
<tr>
<th>Level</th>
<th>Cl.5</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>E0</th>
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<td>51%</td>
<td>51%</td>
<td>48%</td>
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<td>52%</td>
<td>52%</td>
<td>48%</td>
<td>48%</td>
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</table>

with $u_\alpha$, the axial velocity component, $A$ the surface of the cross-sectional plane, $x$ the distance along the $x$-axis towards the center and $R$ the radius of the level considered (Fig. 6). A positive value in the daughter branches means a shift of axial flow towards the divider wall and in the main branch towards the side of the external carotid artery. Table 2 gives the experimental and numerical values of $\langle X/R \rangle$ for the six levels analysed, together with the $95\%$-confidence intervals. Again, for both cases the integration errors are neglected. The agreement between the experi-

Table 2. First moments of axial flow at the six levels analysed for both the measurements and the calculations. Experimental confidence intervals $\pm 0.02$

<table>
<thead>
<tr>
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<th>12</th>
<th>E0</th>
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<td>0.48</td>
<td>0.17</td>
<td>0.18</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Table 3. Some integrals at the six levels analysed for the experimental and numerical case. Experimental confidence intervals $\pm 0.02$

<table>
<thead>
<tr>
<th>Level</th>
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<td>0.15</td>
</tr>
<tr>
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<tr>
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mental and numerical values is satisfactory. At all positions in the daughter branches, the axial velocity profile is shifted towards the divider wall. At the entrance of the internal carotid artery and halfway along the bulb the first moment of axial flow is large, as compared with its value at the end of the bulb and in the external carotid artery. This is due to the existence of a region with reversed axial flow opposite to the flow divider.

Some features of secondary flow can be characterized by the following integrals:

\[ \text{Int}_1 = \frac{1}{U_{\text{com}} A} \int_A \| u_x \| \, dA \]
\[ \text{Int}_2 = \frac{1}{U_{\text{com}} A} \int_A u_{xy} \, dA \]
\[ \text{Int}_3 = \frac{1}{U_{\text{com}} A} \int_A u_{yy} \, dA \]

with \( U_{\text{com}} \) the mean axial velocity in the common carotid artery, \( u_{xx} \) and \( u_{yy} \) the \( x \) - and \( y \) -component of secondary flow respectively and \( \| u_x \| \) the length of the secondary velocity vector. \( \text{Int}_1 \) gives information about the mean value of the absolute secondary velocity with respect to the mean axial velocity in the common carotid artery. \( \text{Int}_2 \) and \( \text{Int}_3 \) supplies information about the appearance of secondary flow. For example, \( \text{Int}_2 \) and \( \text{Int}_3 \) will be small for a Dean vortex, whereas \( \text{Int}_3 \) will be negative and large, as compared with \( \text{Int}_2 \), for a secondary flow field directed towards the center of the branch. A positive value of \( \text{Int}_2 \) in the daughter branches means that the \( x \) -component of secondary flow is directed towards the divider wall, whereas a positive value of \( \text{Int}_2 \) in the main branch means that this component is directed towards the side of the external carotid artery. A negative value of \( \text{Int}_3 \) points to a net secondary flow direction towards the plane of symmetry. Table 3 gives the experimental and numerical values of the integrals at the six levels together with the 95%-confidence intervals for the experiments. From the data presented in this table it can be concluded that there is a good agreement between the experimental and numerical values. The largest differences occur in \( \text{Int}_1 \) and \( \text{Int}_2 \) at the entrance of both daughter branches. These differences are probably caused by measuring problems near the flow divider, through which the secondary flow field found by the calculations is much more detailed than the secondary flow field found in the experiments. Also numerical errors due to the ill-posedness of the problem may be excluded. From Table 3 it is furthermore concluded that the secondary flow patterns halfway along the bulb and in the external carotid artery resemble a Dean vortex. The highest secondary velocities are found at the tapering end of the bulb. Secondary flow at the entrance of the bulb is almost completely directed towards the flow divider.

Finally, secondary flow may be quantified by its mean axial vorticity \( \xi \) (Olson and Snyder, 1985):

\[ \xi = \frac{R_{\text{com}}}{U_{\text{com}}} \int_s u_a \, ds \]

with \( s \) the boundary of a region with surface \( A \), \( u_a \) the tangential secondary velocity at \( s \) and \( R_{\text{com}} \) and \( U_{\text{com}} \) the radius of, and mean axial velocity in the common carotid artery, respectively. For the six levels considered, \( s \) is defined as depicted in Fig. 6. The radius of the region surrounded by \( s \) was chosen so that for most of the levels, \( \xi \) reached its maximal value. A negative value of \( \xi \) means that the vorticity occurs clockwise. Table 4 gives the values of \( \xi \) for the six levels analysed, together with the 95%-confidence intervals. From the data presented in this table it can be concluded that the resemblance between the experimental and numerical results is quite satisfactory. The relative large confidence intervals for the experiments are mainly caused by positioning errors of the measuring volume. As a consequence of the large velocity gradients near the wall, small positioning errors may cause relatively large measuring errors of the secondary flow at boundary \( s \). The mean axial vorticity at the end of the bulb is large compared to the other values, especially at the entrance of and halfway along the bulb.

### Table 4. Mean axial vorticities at the six levels analysed for both the measurements and the calculations. Experimental confidence intervals ± 0.05

<table>
<thead>
<tr>
<th></th>
<th>Cl.5</th>
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<th>I1</th>
<th>I2</th>
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<td>0.50</td>
<td>1.13</td>
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<td>-0.66</td>
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</table>

**Influence of various parameters**

Because the Reynolds number and flow division ratio vary essentially during a heart cycle (Ku et al., 1985), computations were also performed at a Reynolds number of 300 and a flow division ratio of 63/37. In an *in vivo* study performed by Reneman et al. (1985), it was found that the angle between the internal and common carotid artery is 10.8° ± 5.2° for a group of 11 adults between the ages of 20 and 30 yr and 12.6° ± 3.5° for a group of 9 adults between the ages of 50 and 60 yr. Determination of the geometry of the carotid artery bifurcation by measuring several dimensions from post-mortem casts of this bifurcation, indicated that the angle between the internal and common carotid artery was 10.5° ± 7°, essentially smaller than the angle proposed by Bharadva et al. (1982). Therefore, a computation was also carried out in a model of the bifurcation with an angle between the internal and common carotid artery of 10°, instead of 25°. When varying one parameter, the other two were kept unchanged. Table 5 gives an overview of the calculations performed. The flow division ratio for
the Reynolds case (Re = 300, flow division ratio 48/52, angle 25°) differed somewhat from that for the reference case (Re = 640, flow division ratio 52/48, angle 25°) due to the influence of the Reynolds number on the flow division ratio.

In Fig. 7 the influence of the Reynolds number on the velocity profiles in the plane of symmetry and on axial and secondary flow in the carotid sinus is shown. In the plane of symmetry the region with reversed axial flow is somewhat smaller in axial and radial extent for a Reynolds number of 300 than for a Reynolds number of 640. For the Reynolds case it has a maximal diameter of about 50% of the local bulb diameter. The shift of the maximum of axial velocity towards the divider wall and the occurrence of an axial velocity plateau is less pronounced than for the reference case. The influence of the Reynolds number on the axial flow field is small at the entrance of the bulb and restricted to a smaller zone with reversed axial flow. Halfway along and at the end of the carotid sinus the maximum of axial velocity is somewhat smaller for the Reynolds case than for the reference case, also due to the larger volume flow for the latter case. The region with reversed axial flow halfway along the bulb is essentially smaller in the plane of symmetry and larger in the direction perpendicular to this plane. The low-numbered axial velocity contours halfway along and at the end of the bulb are less C-shaped than for the reference case. The influence of the Reynolds number

Table 5. Parameters for the calculations performed

<table>
<thead>
<tr>
<th>Case</th>
<th>Reynolds number</th>
<th>Flow division ratio</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference</td>
<td>640</td>
<td>52/48</td>
<td>25°</td>
</tr>
<tr>
<td>Reynolds</td>
<td>300</td>
<td>48/52</td>
<td>25°</td>
</tr>
<tr>
<td>Flow</td>
<td>640</td>
<td>63/37</td>
<td>25°</td>
</tr>
<tr>
<td>Angle</td>
<td>640</td>
<td>52/48</td>
<td>10°</td>
</tr>
</tbody>
</table>

Fig. 7. Influence of the Reynolds number on both axial and secondary flow in the carotid sinus. Solid line: reference case, dashed line, circles: Reynolds case (for definition see Table 5).
The influence of the volume flow ratio is pointed out in Fig. 8. The effect of an increasing volume flow ratio on the region with reversed axial flow in the plane of symmetry is almost the same as the influence of a decreasing Reynolds number: smaller in axial and radial extent. The shift of the maximal axial velocity towards the divider wall is about the same as for the reference case. For the flow case (Re = 640, flow division ratio 63/37, angle 25°), higher axial velocities are found due to a larger volume flow through the internal carotid artery. An increase of volume flow through the internal carotid artery with about 20% has little effect on secondary flow in this branch.

A smaller bifurcation angle causes the region with reversed axial flow to grow at the entrance of the bulb, smaller in the plane of symmetry but larger in the direction perpendicular to this plane. For the angle case (Re = 640, flow division ratio 52/48, angle 10°), the regions with high axial velocities are somewhat wider and the axial velocity contours are less C-shaped than for the reference case. The secondary velocities at the entrance of the bulb decrease with decreasing bifurcation angle. Halfway along and at the end of the bulb, the influence on secondary flow is small.

Discussion

The findings in the present study indicate that the numerical model, as described, can be used to predict axial as well as secondary velocities for steady flow at physiological Reynolds numbers in rigid-walled three-dimensional models of the carotid artery bifurcation. A comparison of the data obtained with laser-Doppler
et al., 1988a) reveals good agreement between the numerical and experimental data. The total CPU-time needed could be restricted to reasonable values by using a structured mesh-generator, which gives full control over the number of elements into which the bifurcation is divided, and a mini-supercomputer to solve the system of equations.

A comparison of axial and secondary flow with the laser-Doppler velocity measurements revealed that the flow phenomena occurring in the carotid artery bifurcation were well predicted by the numerical model. Small discrepancies between the axial and secondary flow fields were observed near the flow divider because of the ill-shaped elements in this region and the measuring problems near the flow divider. The discrepancy found regarding the region with reversed axial flow is mainly caused by the relatively large measuring errors due to the low axial velocities in this region. A quantitative comparison between the numerical data and those obtained with the laser-Doppler measuring technique was performed regarding the first moment of axial flow, the axial vorticity and some characteristic parameters of secondary flow. Taking into account the 95%-confidence intervals for the experiments, a good agreement was observed between the numerical and experimental values of these quantities.

From a study by van de Vosse et al. (1988), dealing with steady flow in a three-dimensional model of a 90° curved tube, it was concluded that (mini-)supercomputers were needed for the calculation of steady flow in a three-dimensional model of the carotid artery bifurcation. Indeed, these computers reduced the CPU-times needed with a factor ranging from 10 to 100, but still it was necessary to limit the number of elements into which the bifurcation was divided. This was necessary because large numbers of elements could give rise to long I/O-times, due to the relative small central memories of (mini-)supercomputers and the standard direct profile method used to solve the
system of equations. For a too-small central memory, the I/O-time needed could be 10 times the CPU-time needed for a calculation. To control the number of elements into which the bifurcation was divided, a structured mesh-generator was developed. With this mesh-generator, it was possible to divide an arbitrary three-dimensional model of a bifurcation into 27-noded brick elements. For post-processing purposes, programs were developed to enable the presentation of the axial and secondary flow field at arbitrary cross-sections of the carotid artery bifurcation.

For a Reynolds number of 640 and a flow division ratio of 52/48, the axial velocity profiles in both daughter branches of the common carotid arteries are skewed towards the divider wall, whereas a region with reversed axial flow is observed in the carotid sinus opposite to the flow divider. The three-dimensional shape of this region is largely determined by secondary flow, caused by centrifugal forces. Secondary flow also causes C-shaped axial velocity contours and axial velocity plateaux downstream in the daughter branches. At the entrance of the bulb opposite the flow divider, both axial and secondary velocities are low, whereas at the end of the bulb, secondary flow is highly influenced by the tapering of the branch at this site. The influence of the Reynolds number, the flow division ratio, and the bifurcation angle are restricted to a relatively small variation in the region with reversed axial flow, more or less pronounced C-shaped axial velocity contours and increasing or decreasing axial velocity maxima. The influence of these variations on secondary flow is mainly restricted to the end of the bulb for a smaller Reynolds number and to the entrance of the bulb for a smaller bifurcation angle. The influence of these physiological and geometrical parameters on both the axial and secondary flow fields may be important for methods used to diagnose atherosclerotic lesions at an early stage of the disease, especially, because in vivo measurements with pulsed-Doppler systems provide velocity information which combines axial and secondary velocity components.

It is likely that the flow patterns as observed are realistic because they are in good agreement with those observed by Bharadvaj et al. (1982). They performed laser-Doppler measurements of axial and secondary flow at several sites in the bifurcation for different physiological Reynolds numbers and flow division ratios. In the carotid sinus they found a region with reversed axial flow opposite the flow divider. For a Reynolds number of 800 and a flow division ratio of 70/30 (70% through the carotid sinus) the maximal diameter of this region was about 45% of the local bulb diameter. In the present study the maximal diameter at a Reynolds number of 640 was about 50% of the local bulb diameter at a flow division ratio of 63/37 and about 60% at a flow division ratio of 52/48. They also observed high axial velocities near the divider wall of the carotid sinus and an axial velocity plateau opposite to this wall at the end of the bulb. At the entrance of the bulb they found that secondary flow was almost completely directed towards the flow divider. Halfway along the bulb they found the high secondary velocities directed towards the non-divider wall near the side wall and low secondary velocities directed towards the divider wall near the plane of symmetry, resulting in helical patterns of the fluid flow. At the end of the bulb high secondary velocities were found directed towards the non-divider wall near the side wall and directed towards the divider wall near the plane of symmetry. These features were also observed in the present study.

The study of Bharadvaj et al. (1982) also revealed that at a flow division ratio of 70/30, a decreasing Reynolds number resulted in a reduction of the radial as well as the axial extent of the region with reversed axial flow in the plane of symmetry. The present study reveals that this is also valid for a flow division ratio of about 50/50. One should realise, however, that this is not necessarily valid for the direction perpendicular to the plane of symmetry. At a Reynolds number of 400 they found that in the carotid sinus the region with reversed axial flow decreased as the flow rate through the internal carotid artery increased. In the present study the same phenomena are observed at a Reynolds number of 640.

Wille (1984) performed a finite element calculation of steady flow in a symmetrical bifurcation at a Reynolds number of 10. Instead of a direct solver, he used a conjugate gradient iteration method to solve the set of linear equations. A division of one quadrant of the symmetrical bifurcation into 300 brick elements was performed, which resulted in 5508 unknowns. Computation of the flow field at a Reynolds number of 10 took about two months CPU-time on an unknown computer system. In spite of the low Reynolds number, the results were believed to be representative of the overall flow patterns in human branching systems. He found skewed velocity profiles in the daughter branches. The shear forces were high near the flow divider and relatively low just upstream of the bifurcation.

With the numerical model as presented in this paper, it is possible to perform detailed analysis of axial and secondary flow in a rigid-walled three-dimensional model of the carotid artery bifurcation for various Reynolds numbers, flow division ratios and geometries of the bifurcation. In the near future shear stresses also will be calculated. Then, extension of the numerical model towards unsteady flow will give the opportunity to analyse the shape of the region with reversed axial flow and the shear stresses as functions of time.

REFERENCES

A numerical analysis of steady flow


Segal, A. (1986) Test problem for the \((Q_2^{271})-P_1\) element. Personal communication.

