Effective properties of a viscoplastic constitutive model obtained by homogenisation

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Abstract

Heterogeneous materials are used more and more frequent due to their enhanced mechanical properties. If the relation between the microscopic deformation and the macroscopic mechanical behaviour can be obtained, it can be used to design new materials with desired properties such as high strength, high stiffness or high toughness. A method for obtaining this relation is called homogenisation, by which the heterogeneous material is replaced by an equivalent homogeneous continuum. In this paper, a homogenisation method is proposed which offers the possibility to determine effective material properties for the homogeneous equivalent continuum, modelled by Perzyna’s viscoplastic constitutive law. To this end, finite element calculations are performed on a representative volume element, the geometry of which is defined by the microstructure of the considered material. The mechanical behaviour of this RVE will also be described by a viscoplastic model, clearly with a given parameter set. The proposed homogenisation strategy provides a way to acquire the constitutive parameters for the equivalent medium. To validate the results of the homogenisation, finite element calculations of the deformation behaviour of a perforated plate that is subjected to different loading histories are performed. The global mechanical behaviour of the homogenised simulations and direct calculations, where the heterogeneous structure is completely discretised, will be compared. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In the past few decades, tremendous effort has been put into the development of new materials with enhanced mechanical properties, such as high stiffness, high strength and toughness. These materials all have in common an heterogeneous nature of the microstructure. Extensive experimental research has been reported on this subject, clearly proving the considerable influence of the microstructure on the macroscopic mechanical behaviour. Nevertheless, it seems apparent that more fundamental studies are needed for a better understanding of the deformation behaviour of these advanced materials. Relations between the microstructural phenomena and the macroscopic deformation behaviour are indispensable when predicting macroscopic properties from the microstructure of a material. In this respect, the works of Hill (1965), Gurson (1977) and Hashin (1983) were pioneering.
Generally speaking, we can distinguish three different routes to predict the macroscopic response of a heterogeneous material subject to a mechanical load: (1) choice of a macroscopic constitutive model, of which the material parameters are fitted onto experimental data, (e.g. Geers, 1997; Meuwissen, 1998); (2) the application of a multilevel finite element method, where on the different levels, simultaneous numerical simulations are performed, (e.g. Ghosh et al., 1995; Ghosh and Moorthy, 1995; Ghosh et al., 1996; Smit et al., 1998; Kruch et al., 1998); (3) homogenisation of the heterogeneous material towards a closed form macroscopic constitutive equation, either by analytical (Ponte Castañeda, 1996; Garájeu and Suquet, 1997), or by numerical approach (Vosbeek, 1994; Van der Sluis et al., 1999).

The first possibility seems to be the most obvious one and is most widely used. A macroscopic constitutive equation is chosen and consequently, by collecting sufficient experimental data, values of the used parameters in the constitutive equation can be determined. Although no direct microstructural information is available in the thus determined constitutive model, valuable insight can be gained from it.

The most important benefit of the multilevel method appears to be the readily available information of the evolution of the microstructure during loading of the macroscopic structure and therefore, the results of this method are quite accurate (Ghosh et al., 1996; Smit et al., 1998; Kruch et al., 1998). Nevertheless, this approach poses severe limitations on practical applications with complex microstructures as well as complex macrostructures, since in that case, such simultaneous simulations require an enormous amount of computer memory and computational time.

Another possibility is to predict the macroscopic response from microscopic analyses followed by an homogenisation technique that results in a so-called Homogeneous Equivalent Continuum (HEC). According to Maugin (1992) and Vosbeek (1994), this homogenisation process aims at replacing the macroscopic heterogeneous structure with a continuum model that ‘best’ represents the structural model (Fig. 5). The fundamental assumption in these homogenisation procedures is the statistical homogeneity of the heterogeneous material (Hashin, 1983), that is, all statistical properties of the state variables are the same at any point in the material. In that case, it is possible to identify an element whose mechanical behaviour is representative for the heterogeneous medium as a whole. Such an element is called a representative volume element, or short, RVE. Several analytical approaches have been proposed for elastic, viscoelastic and elasto-plastic models (Hashin, 1983; Mura, 1987; Nemat-Nasser and Hori, 1993; Boutin, 1996; Ju and Tseng, 1996).

More recently, also for viscoplastic models, effective potentials have been studied (Ponte Castañeda, 1996; Li and Weng, 1997a,b; Garájeu and Suquet, 1997). It is evident that simplifying assumptions have to be made (e.g. concerning the stiffness of the microscopic inclusions or the constitutive behaviour of the individual materials), to be able to derive expressions for the overall behaviour.

The numerical homogenisation method proposed in this paper, does not require any simplifying assumptions on account of the microstructure of the material. Instead, finite element calculations are performed on the microstructural level, hereby circumventing the difficulties encountered in the analytical treatment. This procedure can be summarised as follows (Vosbeek, 1994; Van der Sluis et al., 1999). First, relations between the microscopic and macroscopic state variables (e.g. stresses and strains) have to be defined. Next, appropriate boundary conditions are developed which follow from these micro–macro relations. These boundary conditions are prescribed on the RVE to generate the volume averaged state variables, necessary to determine the constitutive equations of the equivalent homogeneous continuum.

When considering strain softening material behaviour, attention should be focussed on the constitutive modelling. When trying to capture this phenomenon in finite element analyses by using conventional constitutive models, the obtained results will reveal a strong sensitivity with respect to the employed spatial discretisation. In this case, the obtained results will not converge to a unique solution upon mesh refinement (see for instance
Needleman, 1988; De Borst and Mühlhaus, 1991; De Borst et al., 1993). The initial boundary value problem of the underlying mechanical process becomes ill-posed upon entering the strain softening regime. Locally, the character of the partial differential equations changes from elliptic to hyperbolic in quasi-static problems, and from hyperbolic to elliptic in dynamic analyses. In literature, several methods have been proposed to preserve well-posedness of the mechanical problem and to obtain mesh-independent results in numerical analyses. Examples of the mentioned methods include non-local models (Bazant, 1991; Brekelmans, 1993; Tvergaard and Needleman, 1995) and gradient models (Triantafyllidis and Aifantis, 1986; Peerlings et al., 1996), Cosserat continua (Toupin, 1962; De Borst, 1990), and rate-dependent models (Wu and Freund, 1984; Needleman, 1988; Sluys and De Borst, 1992; De Borst et al., 1993; Wang et al., 1996). For an overview, see De Borst et al. (1993). In effect, these so-called regularisation models introduce an internal length scale parameter which regularises the localisation process.

In order to be able to deal with softening materials, a rate-dependent model is adopted. More specifically, Perzyna’s viscoplastic model (Perzyna, 1966, 1971; Owen and Hinton, 1980) has been selected. The corresponding constitutive equations will be presented, after which the finite element formulation is discussed. To show the capability of the model to describe strain softening behaviour properly will be shown by means of quasi-static finite element simulations on an imperfect specimen.

2. Model formulation

In this section, Perzyna’s viscoplastic model formulation will be discussed. First, the constitutive equations will be given, after which the full Newton–Raphson procedure for the solution of the non-linear discretised equilibrium equations will be considered. Next, the capability of the model to describe strain softening behaviour properly will be shown by means of quasi-static finite element simulations on an imperfect specimen.

2.1. The constitutive equations

Analogous to the elastoplastic models, the total strain tensor \( \varepsilon \) is decomposed into an elastic part and a viscoplastic part (Zienkiewicz and Cormeau, 1974),
\[
\varepsilon = \varepsilon^e + \varepsilon^{vp},
\]
when initial strains are neglected. The relation between the Cauchy stress tensor \( \sigma \) and the elastic part of the strain tensor is simply defined by Hooke’s law:
\[
\sigma = 4D^e : \varepsilon^e \quad \text{with} \quad 4D^e = \frac{vE}{(1 + v)(1 - 2v)} \left( II + \frac{1 - 2v}{v} I^4 \right),
\]
where \( E \) is Young’s modulus, \( v \) Poisson’s ratio, \( I \) the second order unit tensor, and \( I^4 \) the fourth order unit tensor. This relation completely describes the reversible part of the deformation. The viscoplastic response of the material becomes manifest as soon as some specified combination of the stress components exceeds a characteristic scalar value. The occurrence of plastic yielding
depends on a yield condition \( F(\sigma, \kappa) \), where \( \kappa \) is a, yet to be defined, history parameter. Here, we will adopt the pressure-independent Von Mises yield criterion, which is defined by

\[
F(\sigma, \kappa) = \sqrt{3J_2} - \zeta(\kappa),
\]

with \( J_2 \) the second invariant of the deviatoric stress tensor, \( J_2 = (1/2)\sigma^d : \sigma^d \). The function \( \zeta(\kappa) \) defines the current yield stress and is taken to be linear in \( \kappa \),

\[
\zeta(\kappa) = \sigma^Y + h\kappa,
\]

with \( \sigma^Y \) the initial yield stress, \( h \) the hardening modulus and \( \kappa \) the equivalent viscoplastic strain given by

\[
\kappa = \int_0^t \dot{\kappa} \, dt \quad \text{with} \quad \dot{\kappa} = \sqrt{2 \dot{\varepsilon}^p : \dot{\varepsilon}^p}.
\]

Clearly, positive values of \( h \) correspond to hardening behaviour, whereas negative values indicate strain softening. The direction of the viscoplastic strain rate is defined as the gradient of the flow function, which is expressed by

\[
\dot{\varepsilon}^p = \dot{\lambda} \frac{\partial F(\sigma, \kappa)}{\partial \sigma} = \dot{\lambda} \mathbf{a}, \quad \text{where} \quad \dot{\lambda} = \gamma |[\phi(F)]|,
\]

with \( \gamma \) a fluidity parameter, \( \phi(F) \) a flow function, and \(|[\cdot]|\) referring to \(|[x]| = xH(x)\), with \( H(x) \) the Heaviside step function. Hence, viscoplastic strains are non-zero only if \( \phi(F) \geq 0 \), i.e. stress states on or outside the yield surface. Notice that this is in contrast with elastoplastic models, where only stress states inside or on the yield surface are legal. In this respect, the term ‘overstress’ model is evident. The amount of the viscoplastic strain rate is equal to the scalar value \( \gamma \phi(F) \). A power law is chosen for the flow function \( \phi(F) \) (Perzyna, 1971)

\[
\phi(F) = \left( \frac{F}{\sigma^Y} \right)^N,
\]

with \( N \) the rate-sensitivity parameter.

### 2.2. Finite element formulation

The partial differential equations for the viscoplastic model, combined with the equilibrium equations, will be solved numerically, using the finite element method. The method is based on the weighted residual formulation, that is obtained by multiplying the equilibrium equations with weighting functions \( \tilde{w} \), integrating over the domain \( \Omega \), integrating by parts and applying Cauchy’s theorem, which states that the stress vector \( t_i = \sigma_{ij} n_j \):

\[
\int_{\Omega} \frac{\partial w_i}{\partial x_j} \sigma_{ij} \, d\Omega = \int_{\Gamma} w_i t_i \, d\Gamma \quad \forall w_i.
\]

When applying the generally used matrix with differential operators \( \mathbf{L} \) and introducing the columns \( \mathbf{e}^T = [e_{11}, e_{22}, e_{12}] \), \( \sigma^T = [\sigma_{11}, \sigma_{22}, \sigma_{12}] \) and \( \mathbf{u}^T = [u_1, u_2] \), appropriate for the two-dimensional plane stress case, it follows that \( \mathbf{e} = \mathbf{L} \mathbf{u} \). Using this, accompanied with the definitions \( \mathbf{w}^T = [w_1, w_2] \) and \( f^T = [t_1, t_2] \), we obtain (8) in matrix notation

\[
\int_{\Omega} (\mathbf{L} \mathbf{w})^T \mathbf{\sigma} \, d\Omega = \int_{\Gamma} \mathbf{w}^T \mathbf{t} \, d\Gamma \quad \forall \mathbf{w}.
\]

This equation is interpolated, making use of an isoparametric formulation (Zienkiewicz and Taylor, 1989). Following Galerkin’s method, the interpolation functions \( \mathbf{N} \) for the displacements and the weighting functions are chosen identically

\[
\mathbf{u}(\xi, \eta) = \mathbf{N}^T(\xi, \eta) \mathbf{u}^e \quad \text{and} \quad \mathbf{w}(\xi, \eta) = \mathbf{N}^T(\xi, \eta) \mathbf{w}^e,
\]

with \( \mathbf{u}^e \) and \( \mathbf{w}^e \) columns containing the values of the degrees of freedom and the weighting functions at the nodes, respectively. Defining the matrix \( \mathbf{B} \) according to \( \mathbf{L} \mathbf{w} = \mathbf{B} \mathbf{w}^e \) and recalling that (9) holds for all \( \mathbf{w} \), we obtain the discretised equilibrium equations

\[
\int_{\Omega} \mathbf{B}^T \mathbf{\sigma} \, d\Omega = \int_{\Gamma} \mathbf{N}^T \mathbf{t} \, d\Gamma \quad \text{or} \quad \mathbf{f}^{\text{int}} = \mathbf{f}^{\text{ext}},
\]

with \( \mathbf{f}^{\text{int}} \) the internal force vector, and \( \mathbf{f}^{\text{ext}} \) the external load vector.

Because of the history and time dependent material behaviour, the value of the state variables at the end of the loading, cannot be determined
directly. Instead, an incremental-iterative procedure is adopted. Here, the total loading history will be applied in a number of small loading steps, also called increments, in order to calculate the deformation history accurately. Thus, the state of the material is determined at a finite number of discrete moments in time \( t_n \), with \( n = 0, 1, 2, \ldots \). From now on, the attention is focussed on the current increment, which starts at \( t = t_n \) and ends at \( t = t_{n+1} = t_n + \Delta t \). Furthermore, it is assumed that the state at time \( t = t_n \) satisfies the equilibrium equations. These equations can now be written as

\[
\int_{\Omega_{n+1}} B^T \sigma_{n+1} \, d\Omega = f_{n+1}^{\text{ext}}. \tag{12}
\]

The Newton–Raphson iteration procedure is used to determine a sequence of approximate solutions until (12) is satisfied to a certain degree of accuracy. The unknown current value of the stresses is written as \( \sigma_{n+1} = \sigma_{n+1}^* + \delta \sigma \), where \( \sigma_{n+1}^* \) is the estimation of the stresses and \( \delta \sigma \) the correction thereof. Hence, (12) becomes

\[
\int_{\Omega_{n+1}} B^T \delta \sigma_{n+1} \, d\Omega = f_{n+1}^{\text{ext}} - \int_{\Omega_{n+1}} B^T \sigma_{n+1}^* \, d\Omega = f_{n+1}^{\text{ext}} - f_{n+1}^{\text{int}} = r_{n+1}, \tag{13}
\]

in which \( r_{n+1} \) is the residual force vector which has to become zero to satisfy the equilibrium equations. This residual is dependent on \( \sigma_{n+1}^* \), which can be calculated in an accurate way by a so-called stress update algorithm, that will be developed below.

The equilibrium equation (13) is used to determine an iterative displacement field \( \delta u \). This is only possible when the iterative stress \( \delta \sigma \) is expressed in \( \delta u \), or equivalently, in \( \delta \varepsilon \),

\[
\delta \sigma = M \delta \varepsilon, \tag{14}
\]

where \( M \) is a material matrix, which follows from the chosen constitutive model. The obvious choice then is to use the so-called modular matrix, which relates the stress rate to the strain rate, \( \dot{\sigma} = C \dot{\varepsilon} \), and can be derived from the constitutive equations. However, Crisfield (1991) has pointed out that this matrix is inconsistent with the applied backward Euler time integration scheme (or any other integration scheme for that matter), which will destroy the quadratic convergence of the Newton–Raphson procedure. To restore this rapid convergence rate, a Jacobian matrix \( D^p \) has to be derived instead, which can be used to formulate the consistent tangential stiffness matrix \( K \). This can be accomplished by substituting \( \delta \sigma_{n+1} = D_n^p \delta u_{n+1} \) into (13). Henceforth, the iterative equilibrium Eq. (13) can be used to determine \( \delta u \), according to

\[
K_{n+1} \delta u = r_{n+1}. \tag{15}
\]

**Stress update algorithm:** As was already mentioned, an iterative stress update algorithm is required. To this end, the stresses at time \( t_{n+1} \) are written as

\[
\sigma_{n+1} = \sigma_n + D^e (\Delta \varepsilon - \Delta \varepsilon^p) = \sigma_n + D^e \Delta \varepsilon^p, \tag{16}
\]

where \( \sigma^e \) are the elastic trial stresses. To calculate the incremental value of the viscoplastic strains, \( \Delta \varepsilon^p \), a time integration scheme must be used. Here, we adopt the implicit backward Euler algorithm, from which (6) can be written as

\[
\Delta \varepsilon^p = \Delta \lambda a_{n+1}. \tag{17}
\]

The backward Euler time integration scheme involves a vector \( a_{n+1} \), that is defined at the end of the increment, and cannot be computed directly. Therefore, an inner-iterative loop at integration point level is employed in order to calculate the stresses in an accurate way. Following Crisfield (1997), we define the residuals

\[
s = \sigma_{n+1} - \sigma^e + \Delta \lambda D^e a_{n+1}, \tag{18}
\]

\[
g = \Delta \lambda - \Delta \gamma \| \phi(F) \|_{n+1}. \tag{19}
\]

Obviously, the iterative scheme should minimise these residuals. The residual vector \( s^{i+1} = s^i + \delta s \) is approximated by a first order Taylor series (clearly with fixed \( \sigma^e \))

\[
s^{i+1} = s^i + \delta \sigma_{n+1} + D^e a_{n+1} \delta \lambda + \Delta \lambda D^e \left( \frac{\partial a}{\partial \sigma} \right)_{n+1} \delta \sigma_{n+1} = 0, \tag{20}
\]

since it follows from (3) and (6) that \( \partial a / \partial \kappa = 0 \). Rearranging gives
\[ \delta \sigma_{n+1} = -Q^{-1}s - Ra_{n+1} \delta \lambda, \]

with

\[ Q = \left( I + \Delta \lambda D^T \left( \frac{\partial a}{\partial \sigma} \right)_{n+1} \right) \]
\[ R = Q^{-1} D^T. \]

The scalar residual \( g^{i+1} = g^i + \delta g \) is also approximated by a truncated Taylor series

\[ g^{i+1} = g^i + \delta \lambda - \Delta \gamma \phi_t^e a_{n+1}^T \delta \sigma_{n+1} - \Delta \gamma \phi_t^e F^e \delta \lambda, \]

where \( \phi_t^e = \partial \phi / \partial F \) and \( F^e = \partial F / \partial \kappa \). Elimination of \( \delta \sigma_{n+1} \) from (23) using (21) results in

\[ \delta \lambda = \frac{-g^i - \Delta \gamma \phi_t^e a_{n+1}^T Q^{-1}s}{1 + \Delta \gamma \phi_t^e a_{n+1}^T Ra_{n+1} - \Delta \gamma \phi_t^e F^e}. \]

This expression can then be used to update the stresses \( \sigma_{n+1} \) by substituting the value of \( \delta \lambda \) in (21).

**Tangential stiffness matrix:** The derivation of the consistent tangential stiffness matrix is based on the variation of (16) combined with (17)

\[ \delta \sigma_{n+1} = D^s \delta \varepsilon_{n+1} - D^s a_{n+1} \delta \lambda \]
\[ - \Delta \lambda D^T \left( \frac{\partial a}{\partial \sigma} \right)_{n+1} \delta \sigma_{n+1}. \]

Rearranging then gives

\[ \delta \sigma_{n+1} = R(\delta \varepsilon_{n+1} - a_{n+1} \delta \lambda). \]

Elimination of \( \delta \lambda \) can be achieved by writing (6) in incremental form,

\[ \Delta \gamma |[\phi(F)]_{n+1} = \Delta \lambda. \]

**Variation**

\[ \Delta \gamma \phi_t^e a_{n+1}^T \delta \sigma_{n+1} + \Delta \gamma \phi_t^e F^e \delta \lambda = \delta \lambda. \]

When substituting (26) into (28), we obtain a relation between \( \delta \lambda \) and \( \delta \varepsilon \)

\[ \delta \lambda = \frac{\Delta \gamma \phi_t^e}{1 + \Delta \gamma \phi_t^e a_{n+1}^T Ra_{n+1} - \Delta \gamma \phi_t^e F^e} a_{n+1}^T R \delta \varepsilon_{n+1}. \]

Elimination of \( \delta \lambda \) from (26) using (29) gives the relation for the Jacobian matrix \( D^p \):

\[ \delta \sigma_{n+1} \]
\[ = \left( R - \frac{1}{(\Delta \gamma \phi_t^e)^{-1} + a_{n+1}^T Ra_{n+1} - F^e} \right) a_{n+1}^T R \delta \varepsilon_{n+1}. \]

The stress-update algorithm and the consistent tangential stiffness matrix described above, was also found by Perić (1993).

**2.3. Strain softening simulations**

Quasi-static finite element simulations on an imperfect specimen will be carried out. To check the mesh-objectivity required to properly model strain softening, the response of the simulations must indeed converge to a unique solution upon refining the mesh. To this end, a tensile specimen is used which is shaped in such a way that in the center, a stress concentration is initiated, resulting in localisation of the deformation. The specimen and the applied boundary conditions are shown in Fig. 1. The following values for the material parameters are chosen: \( E = 10000 \) (MPa), \( v = 0.3 \) (-), \( \gamma = 25 \) (MPa), \( h = -1000 \) (MPa), \( \gamma = 10 \) (1/s) and \( N = 1 \) (-). The prescribed strain rate is \( \dot{\varepsilon} = 1 \) s\(^{-1}\). To check the influence of the spatial discretisation of the specimen on the results of the simulations, several meshes will be used which are shown in Fig. 2.

As can be inferred from this figure, due to symmetry of the specimen and the loading conditions, only a quarter of the specimen has to be

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**Fig. 1.** The tensile specimen with boundary conditions.
The resulting force–displacement curves for the different meshes are shown in Fig. 3. From this, we may conclude that the viscoplastic model can indeed be used to describe strain softening material behaviour in a proper way. Of course, when the viscosity is reduced to zero (i.e. $\gamma \rightarrow \infty$), a rate-independent plasticity model is recovered, that is not able to describe strain softening.

To illustrate the typical characteristics of the model, several local stress–strain curves are shown in Fig. 4. The left picture shows the behaviour during continuous loading, obviously caused by the stress concentration in the center of the specimen. The picture in the center clearly illustrates the loading and subsequent elastic unloading behaviour which appears to be properly captured by the model. The resulting force–displacement curves for the different meshes are shown in Fig. 3. From this, we may conclude that the viscoplastic model can indeed be used to describe strain softening material behaviour in a proper way. Of course, when the viscosity is reduced to zero (i.e. $\gamma \rightarrow \infty$), a rate-independent plasticity model is recovered, that is not able to describe strain softening.
model. The right curve shows the elastic response, since in this part of the specimen, stresses simply are not high enough to give rise to any viscoplastic deformation.

3. Homogenisation of the viscoplastic model

The purpose of the homogenisation process is to model a heterogeneous material by means of a unique continuous medium (Maugin, 1992). In that way, it is possible to establish a constitutive equation which can be used to describe the mechanical behaviour of the homogeneous equivalent continuum, which ‘best’ represents the response of the real heterogeneous material (Fig. 5). The effective properties of a heterogeneous material define the relations between averages of field variables such as stress and strain when their spatial variation is statistically homogeneous, that is, the effective properties define the constitutive equations of the equivalent homogeneous continuum.

The homogenisation strategy which is discussed in this section, can be outlined as follows. The first step is defining the relations between the microscopic and macroscopic state variables, e.g. stresses and strains. By using these definitions, it is then possible to derive microscopic boundary conditions which can be applied to the representative volume element (RVE). These boundary conditions are formulated as displacements related to the macroscopic deformation quantities. It is important to observe that the deformation of the RVE does not follow from any periodicity considerations. This is contradictory to the generally employed homogenisation techniques, which are based on periodicity demands of the microstructure (e.g. Auriault, 1991; Ghosh et al., 1995, 1996; Kruch et al., 1998; Smit et al., 1998). The response of the RVE is subsequently calculated by finite element simulations. Then, the macroscopic state variables can be calculated, using the micro–macro relations. Since the prescribed values of the macroscopic strains and strain rates are known, and subsequently, the macroscopic state quantities can be calculated, it is possible to determine an effective parameter set for the constitutive equations describing the mechanical behaviour of the homogeneous equivalent continuum.

Fig. 5. Graphical representation of the homogenisation process.
3.1. Consistent boundary conditions

The macroscopic strain tensor is defined as the average of the corresponding quantity defined on the RVE,

\[ \varepsilon(x) = \langle \varepsilon \rangle(x), \quad x \in \Omega, \]  

(31)

in which \( \langle \cdot \rangle \) denotes the averaging operator. In the deterministic theories (Hill, 1965; Maugin, 1992), the average is taken over the region \( R \), occupied by the RVE,

\[ \langle \varepsilon \rangle(x) = \frac{1}{V} \int_{R} \varepsilon(x, y) \, dy, \quad \text{with} \quad V = \int_{R} dy. \]  

(32)

The macroscopic strain tensor will be applied to the RVE through appropriate boundary conditions. These boundary conditions have to be consistent with the definition (32). The consistency conditions are a direct consequence of the definition of the linear strain tensor and the divergence theorem,

\[ \varepsilon_{ij}(x) = \frac{1}{2V} \int_{\partial R} \left( u_i(x, y)n_j(x, y) + u_j(x, y)n_i(x, y) \right) ds, \]  

(33)

where \( n(x, y) \) is the outward normal vector on \( \partial R \). When we assume a uniform strain field as boundary conditions,

\[ u_i(x, y) = u_0(x) + \varepsilon_{ij}(x)y_j \quad \text{on} \ \partial R, \]  

(34)

it is easily verified that these boundary conditions are consistent with (33).

3.2. Determination of the effective parameters

The effective parameters for the description of the mechanical behaviour of the equivalent homogeneous continuum, now have to be fitted onto finite element calculations performed on the RVE. For simplicity, the RVE which we will use and will be discussed here, represents a regular cubic hole distribution. The finite element mesh is shown in Fig. 6. The characteristic size of the RVE is set to \( a = 1 \) (mm). The radius of the hole is \( r = 0.25 \) (mm), which corresponds to an initial void volume fraction of \( f = 20\% \). Notice from Fig. 6, that the hole is also discretised into a number of finite elements. This precludes the necessity of prescribing additional boundary conditions on the inner edge, caused by the hole. The inclusion is assumed isotropic elastic with a modulus of \( E = 1 \) (MPa) and \( v = 0.3 \). These values are based on a comparison of the global mechanical response of the ‘filled’ RVE and an RVE with an actual hole. The force–displacement curves of the two RVEs, obviously obtained from identical boundary conditions, were identical using these values. An additional advantage of discretising the hole is that also arbitrary inclusions with different mechanical properties can be modelled using the same finite element mesh.

The viscoplastic model is applied both on the macrolevel and on the microlevel. However, on the microlevel, the material parameters are known (Table 1), and on the macrolevel, the parameters must follow from our homogenisation procedure. Furthermore, a thoroughly motivated selection of a constitutive model at the macroscopic level remains difficult. In literature, several macroscopic models are suggested for the description of voided materials (e.g. Gurson, 1977; Becker and Needleman, 1986; Haghi and Anand, 1992; Garajeau and Suquet, 1997). In these models, pressure dependent yield surfaces follow from microstructural con-
considerations. In this paper, we will confine ourselves to pressure independent yield at the macroscopic level, hereby not providing plastic compressibility. The strategy of determining the effective properties for the equivalent homogeneous continuum is divided into three steps: (i) effective elastic properties, resulting in values for $E$ and $\nu$; (ii) effective plastic properties, giving values for $\sigma_Y$ and $\dot{\gamma}$; (iii) and effective viscous properties, yielding $c$ and $N$. These three steps will be elucidated in the following. Using the averaging operator introduced in (32), the following micro–macro relations are defined

$$
\Sigma(x) = \frac{1}{V} \int_{R} \sigma(x, y) \, dy, \quad \Sigma_{eq} = \sqrt{\frac{3}{2} \Sigma^d \cdot \Sigma^d}, \quad (35)
$$

$$
\varepsilon(x) = \frac{1}{V} \int_{R} \varepsilon(x, y) \, dy, \quad (36)
$$

$$
\dot{\varepsilon}(x) = \frac{1}{V} \int_{R} \dot{\varepsilon}(x, y) \, dy. \quad (37)
$$

In the following paragraphs, the uniform boundary conditions (34) are applied under tensile conditions by prescribing displacements on $y_1 = \pm a/2$. Henceforth, the applied constant macroscopic strain rate is $\dot{\varepsilon}_{11} = 1 \text{ s}^{-1}$. Because of geometric and loading symmetry, only one quarter of the RVE has to be modelled. In addition, the vertical displacements on the boundaries $y_2 = \pm a/2$ are prescribed in such a way that the boundaries should remain straight during loading, while the resulting force in $y_2$-direction equals zero (Idesman et al., 1995). This can be easily implemented in the finite element program by applying nodal constraints (e.g. Bathe, 1982).

### 3.2.1. Effective elastic properties

Since the boundary conditions represent a tensile configuration, we assume that the following holds: $\Sigma_{22} = 0$, $\Sigma_{12} = 0$ and $\dot{\varepsilon}_{12} = 0$. From the averaged elastic constitutive Eq. (2), the following expressions for the effective elastic properties are readily obtained

$$
\tilde{\Sigma} = \frac{\Sigma_{eq}}{\Sigma_{eq}}, \quad \tilde{\varepsilon} = \frac{\varepsilon_{eq}}{\varepsilon_{eq}}, \quad \tilde{\dot{\varepsilon}} = \frac{\dot{\varepsilon}_{eq}}{\dot{\varepsilon}_{eq}}. \quad (38)
$$

Of course, the resulting equivalent stresses in the RVE are not allowed to exceed the local yield limit of the material. The effective values are obtained as $\tilde{E} = 6191$ (MPa) and $\tilde{\nu} = 0.27$ (−). Ju and Tseng (1996) found $\tilde{E} = 6547$ (MPa) and $\tilde{\nu} = 0.28$ (−) for a given volume fraction of 20%.

### 3.2.2. Effective plastic properties

For determining the effective initial yield stress $\sigma_Y$, we assume that at the instant of time at which local yielding occurs in any point $y$ in the RVE, the effective initial yield stress is reached. At this time instant, it holds that

$$
\sigma_Y^e = \Sigma_{eq}. \quad (39)
$$

The time instant can be calculated accurately by increment refinement. Using the boundary conditions (34) and prescribing only the macroscopic normal strain in the $y_1$-direction including the specified nodal ties applied on the horizontal edges, the converged value of the effective yield stress was obtained as $\sigma_Y^e = 8.1$ (MPa). Further increment refinement did not alter this value.

When averaging Eqs. (3) and (4),

$$
\Sigma_{eq} - \sigma_Y^e - \tilde{h} \cdot \kappa = 0, \quad (40)
$$

an expression can be formulated for determining the effective hardening coefficient

$$
\tilde{h} = \frac{\partial \Sigma_{eq}}{\partial \kappa}. \quad (41)
$$

For the purpose of obtaining the value for $\tilde{h}$, we again perform a tensile test on the RVE (Fig. 6).
Contradictory to the loading of the elastic simulation, the value of the imposed strain $\varepsilon_{11}$ is such that also viscoplastic deformations are allowed. By means of the proposed micro–macro relations, the averaged equivalent stress $\Sigma_{eq}$ in the RVE and the averaged equivalent viscoplastic strain $\mathcal{K}$ can be calculated. This is shown in Fig. 7(a). The effective hardening modulus is obtained by using a standard Marquardt–Levenberg least-squares fitting procedure: $\dot{h} = -1026$ (MPa). In Fig. 7(b), the calculated and fitted plots of $\Sigma_{eq}$–$\mathcal{K}$ are shown.

It should be remarked that after reaching the initial yield point, the mechanical behaviour becomes rate dependent. This implies that the stress level during viscoplastic loading is rate dependent. To investigate the influence of the imposed strain rate on the value of the effective hardening modulus, we performed tensile tests on the RVE with three different imposed strain rates, $\dot{\varepsilon}_{11} = 0.5 \, \text{s}^{-1}, 1.0 \, \text{s}^{-1}$ and $2.0 \, \text{s}^{-1}$. The evolution of the averaged equivalent stress as function of the averaged equivalent viscoplastic strain for these three cases is shown in Fig. 8. From this, we may conclude that the value of $\dot{h}$ is nearly independent of the imposed strain rates. Actual fitting revealed that deviations were well within 10%. It will be obvious that the ideal loading case for the determination of $\dot{h}$ would be a very low value of the imposed strain rate, since in that case, the stresses should be nearly on the yield surface instead of outside.

3.2.3. Effective viscous properties

The two parameters that are not yet determined, are the effective viscosity parameter $\dot{\gamma}$ and the flow constant $\dot{N}$. To obtain these values, it is proposed to use the averaged response of the RVE obtained from a tensile test, using the boundary conditions (34). The constitutive Eqs. (1), (2) and (6) for the averaged state variables in the viscoplastic regime under tensile conditions read

![Fig. 7. (a) The averaged equivalent stress as function of the averaged equivalent viscoplastic strain and (b) the fitted and calculated plot of $\Sigma_{eq}$–$\mathcal{K}$.](image1)

![Fig. 8. The averaged equivalent stress as function of the averaged equivalent viscoplastic strain for different imposed strain rates.](image2)
The effective constitutive parameters for the homogeneous equivalent continuum

<table>
<thead>
<tr>
<th>Elastic (MPa)</th>
<th>Plastic (MPa)</th>
<th>Viscous (1/s)</th>
<th>Viscous (–)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6191</td>
<td>0.27</td>
<td>8.1</td>
<td>–1026</td>
</tr>
<tr>
<td></td>
<td>0.65</td>
<td>2.62</td>
<td></td>
</tr>
</tbody>
</table>

This equation will be used to obtain the effective values \( \gamma \) and \( N \). Note that the corresponding values only depend on the viscoplastic response of the RVE, according to (49). Again, by applying a standard Marquardt–Levenberg least-squares algorithm, the fitted parameters are obtained: \( \gamma = 0.65 (1/s) \) and \( N = 2.62 (–) \). For convenience, the effective properties are given in Table 2. In Fig. 9, the right-hand side of Eq. (49) is depicted as the fitted part, whereas the left-hand side of Eq. (49) is plotted as the fitted part. Also from this figure, it can be seen that the fitted curve does not match the calculated curve exactly, indicating a deficiency in the macroscopic model. It is believed that the assumption of macroscopic incompressible plastic flow, as well as the constant viscoplastic material parameters are the main cause for this observation.

Another possibility of determining the effective viscoplastic parameters could be to perform tensile tests on the RVE with a wide range of imposed strain rates, resulting in a large database of sample points. This database could then be used with an
estimation algorithm to obtain values for the effective parameters. Such strategies are commonly used in mixed numerical–experimental approaches (e.g. Meuwissen, 1998; Geers et al., 1998).

4. Numerical validation

In this section, a uniaxial tensile test, a simple shear test, a biaxial test and a cyclic (tension–compression) test will be performed on a perforated plate, depicted in Fig. 10(a). Results obtained from direct calculations, where the complete heterogeneous specimen is discretised, are confronted with results from calculations in which the obtained homogenised parameter set is exploited. The applied meshes for the simulations are shown in Fig. 10(b) and (c). The shear test, the biaxial tensile test and the cyclic test are interesting since the loading history is different from the uniaxial tensile loading on the RVE from which the effective properties have been extracted. We have chosen $B_0 = 7a$ and $L_0 = 12a$. The thickness of the plate is taken as 0.001 (mm), justifying the plane stress case.

4.1. Uniaxial tensile test on a perforated plate

The displacements are prescribed in the $x_2$-direction. In addition, a geometric imperfection has been used to give a heterogeneous deformation pattern, and is given by the expression (Timmermans, 1997)

$$B_1(x_2) = B_0 \left\{1 - \frac{1}{2}(1 - \xi) \left[ \cos \left(\frac{\pi x_2}{\xi B_0} + 1\right)\right]\right\},$$

$$0 \leq x_2 \leq \zeta B_0,$$

with $B_1$ the width over the imperfection, $B_0$ the initial width of the plate, $\xi = 0.98$ a measure of the imperfection ($\xi = B_1/B_0$) and $\zeta = 0.5$ defining the imperfection length $\zeta B_0$.

Fig. 10. (a) The perforated specimen, (b) the mesh used for the homogenised simulations, and (c) the mesh used for the direct simulations.
A comparison of the two simulations only is allowed concerning deformation quantities that are defined at the edge of the specimens. Thus, the resulting force–displacement curves can be used for this purpose, and are depicted in Fig. 11. It can be observed that a good agreement is obtained.

4.2. Simple shear test on a perforated plate

For this test, the displacements on the upper edge are prescribed in the \( x_1 \)-direction. In addition, the vertical displacements on this edge are suppressed. The resulting force–displacement curves of the direct and homogenised simulations are depicted in Fig. 12. Due to the deviatoric nature of the shear test, the resemblance appears quite excellent, which clearly resides from the fact that our homogenised model does not incorporate plastic volume variance. However, at larger deformations, the influence of the void growth becomes more pronounced, thus resulting in a larger difference between the simulations. Also from this result, one can conclude that the isotropy assumption for the equivalent continuum can be justified.

4.3. Biaxial tensile test on a perforated plate

Here, displacements in the \( x_1 \)-direction are prescribed on the right edge, and on the upper edge, displacement in the \( x_2 \)-direction are specified. The resulting force–displacement curves of the direct and homogenised simulations are depicted in Fig. 13. Again, the difference between the two simulations can be explained by the volume invariance assumption of the macroscopic model and the constant material parameters.

4.4. Cyclic tension–compression test on a perforated plate

In this case, alternating positive and negative displacements in \( x_2 \)-direction are prescribed. The
global force–displacement curves are shown in Fig. 14. Also for this loading scenario, a good agreement is obtained.

5. Discussion

A homogenisation method is proposed that provides a way to devise constitutive equations for heterogeneous strain softening materials. At the microstructural level, a viscoplastic model is used to capture the mechanical behaviour of the representative volume element. For simplicity, we have assumed a regular cubic hole stacking, which defines the geometry of the RVE. This assumption is by no means a limitation of the proposed method. Exactly the same reasoning could be followed when using more complex microstructures, such as viscoplastic inclusions with arbitrary geometry. The only factor that differs when considering complex microstructures, will be the computational time for the microscopic simulations.

For the homogeneous equivalent medium at the macrolevel, also a viscoplastic constitutive model is exploited, with yet unknown model parameters. The values for these effective parameters have been obtained by separating the three characteristics of the model, namely the elastic, plastic and viscous parts. The parameters were fitted onto finite element calculations on the RVE by applying the appropriate boundary conditions, which followed from the micro–macro relations. The fitting process of the viscous parameters revealed that the chosen macroscopic model is not adequate enough for describing the mechanical behaviour of the considered material. In fact, the viscoplastic parameters may be functions of a deformation characteristic (for instance, the equivalent viscoplastic strain rate). Moreover, the assumption of pressure independent yield appears to be a restriction.

Nevertheless, despite these simplifications, a good agreement was obtained between the responses of the homogenised plate and the perforated plate. The simple shear test revealed the influence of the absence of any pressure dependent yield criterion in the homogenised model. In addition, the obtained agreement between the results of the direct and homogenised simulations of the considered loading cases, suggests that the isotropy assumption at the macrolevel is justifiable. The occurring stress gradients caused by the holes indicate that the average only of the state variables may not be sufficient to describe the statistical information of the RVE accurately. Therefore, additional quantities in terms of the variations of the state quantities could be considered (Van der Sluis et al., 1999). Clearly, this will result in a higher accuracy of the statistical information of the state of the RVE, yielding a better approximation of the effective parameter set.

References