On coupled gradient-dependent plasticity and damage theories with a view to localization analysis

René de Borst a,1, Jerzy Pamin b, Marc G.D. Geers c

a Koiter Institute Delft/Faculty of Aerospace Engineering, Delft University of Technology, P.O. Box 5058, 2600 GB Delft, The Netherlands
b Faculty of Aerospace Engineering, Delft University of Technology/Faculty of Civil Engineering, Cracow University of Technology, Poland
c Faculty of Mechanical Engineering, Eindhoven University of Technology/Faculty of Civil Engineering, Royal Military Academy, Brussels, Belgium

(Received 12 November 1998; revised and accepted 3 February 1999)

Abstract – Combinations of gradient plasticity with scalar damage and of gradient damage with isotropic plasticity are proposed and implemented within a consistently linearized format. Both constitutive models incorporate a Laplacian of a strain measure and an internal length parameter associated with it, which makes them suitable for localization analysis.

The theories are used for finite element simulations of localization in a one-dimensional model problem. The physical relevance of coupling hardening/softening plasticity with damage governed by different damage evolution functions is discussed. The sensitivity of the results with respect to the discretization and to some model parameters is analyzed. The model which combines gradient-damage with hardening plasticity is used to predict fracture mechanisms in a Compact Tension test.

gradient-dependent continuum / plasticity / damage / strain localization / finite elements

1. Introduction

The problem of strain localization driven by material instabilities has been thoroughly investigated, see for instance (de Borst et al., 1993) or the recent book edited by de Borst and van der Giessen (1998), and is by now rather well understood. If a material instability (Hill, 1958) is encountered in the deformation history of a body, the strains tend to localize in a number of narrow bands, while the remaining parts of the body unload. Within a classical, local continuum formulation this phenomenon is, for static problems, associated with the loss of ellipticity of the governing partial differential equations, and therefore, discretization methods used to solve them may yield meaningless results. To overcome this problem, some form of rate-dependent or nonlocal enhancement of the constitutive model must be adopted (de Borst et al., 1993). In other words, a continuum formulation equipped with an internal length parameter should be used.

As constitutive framework, a combination of plasticity and damage theories is physically appealing since a host of materials exhibit an interaction of inelastic mechanisms of microcrack or microvoid growth with plastic flow. The coupled models are more flexible and make it possible to reproduce a realistic elastic stiffness degradation, which is important for cyclic loading and extensive stress redistributions.

Within a local continuum format, plasticity and damage couplings have been analyzed in a small strain format by Simo and Ju (1987), Ju (1989), Hansen and Schreyer (1994), Doghri (1995), and in a large strain format by

1 E-mail: r.deborst@lr.tudelft.nl.

A gradient-regularized hardening plasticity theory has been coupled to damage in (Svedberg and Runesson, 1997). Localization characteristics of plasticity combined with elastic degradation have been analyzed in (Rizzi et al., 1995) leading to the conclusion that localization in the sense of ellipticity loss can appear even when both the plasticity and damage mechanisms are hardening, which means that the combination has a destabilizing effect.

This paper investigates coupled plastic-damage theories applied in numerical simulation of localization phenomena caused by material softening. To avoid the loss of ellipticity, a gradient enhancement of either the plasticity or the damage part of the model is adopted. The theories have a simple, isotropic format (intrinsic and induced anisotropy are neglected). The gradient-dependent plasticity enhancement is based on (Mühlhaus and Aifantis, 1991; de Borst and Mühlhaus, 1992; Pamin, 1994). The gradient damage model follows (Peerlings et al., 1996), although enhancements have recently been proposed (Geers, 1997; de Borst et al., 1998).

The paper focuses on the numerical response of the gradient-dependent coupled plasticity-damage models in localization problems under quasi-static loading conditions. Section 2 summarizes the two constitutive models, their incremental finite element formulation and other algorithmic aspects. Section 3 presents the results of one-dimensional numerical studies and Section 4 contains some simulations of fracture in a Compact Tension test of a fiber-reinforced composite material. Conclusions are gathered in Section 5.

### 2. Theory and algorithmic aspects

#### 2.1. Coupling of scalar damage and plasticity

In this paper attention is limited to small deformations and static problems. Hence the equilibrium and kinematic equations have the form:

\[ \mathbf{L}^T \mathbf{\sigma} + \mathbf{b} = 0, \tag{1} \]

\[ \mathbf{\varepsilon} = \mathbf{L} \mathbf{u}, \tag{2} \]

where \( \mathbf{L} \) is a differential operator matrix, \( \mathbf{\sigma} \) is the stress tensor in a vector form, \( \mathbf{b} \) is the body force vector, \( \mathbf{\varepsilon} \) is the strain tensor in a vector form, \( \mathbf{u} \) is the displacement vector and the superscript \( T \) is the transpose symbol. The stresses and displacements satisfy the relevant natural and essential boundary conditions.

We start the discussion of constitutive relations by assuming a form of coupling between plasticity and damage theories. We combine a plasticity theory formulated in stress space and an isotropic damage theory formulated in strain space. However, one of the two theories is made gradient-dependent (de Borst et al., 1995) in order to assure that numerical simulations of strain localization yield meaningful results.

Considering the damage evolution we distinguish the actual body with strains \( \mathbf{\varepsilon} \) and stresses \( \mathbf{\sigma} \) (i.e. the physical space) and its fictitious undamaged counterpart with stresses \( \mathbf{\hat{\sigma}} \) and strains \( \mathbf{\hat{\varepsilon}} \) (i.e. the effective space). The fictitious counterpart represents the undamaged “skeleton” of the body, and the stresses \( \mathbf{\hat{\sigma}} \) acting on it are called effective. We adopt the postulate that the strains observed in the actual body and in its undamaged representation are equal (Lemaitre, 1984; Simo and Ju, 1987; Ju, 1989):

\[ \mathbf{\varepsilon} = \mathbf{\hat{\varepsilon}}, \tag{3} \]
and that the stresses are related by means of a scalar damage measure $\omega$

$$\sigma = (1 - \omega)\hat{\sigma}. \quad (4)$$

We notice that, although damage and plastic processes can be coupled, plasticity is associated with the undamaged “skeleton” of the body, so we can write the elastic constitutive relation between the effective stresses and elastic strains as follows:

$$\hat{\sigma} = D^e \varepsilon^e, \quad (5)$$

where $D^e$ is the elastic stiffness operator. Combining Eqs (4) and (5) we obtain the coupling relation:

$$\sigma = (1 - \omega)D^e \varepsilon^e, \quad (6)$$

in which the damage measure $\omega$ accounts for the degradation of the elastic stiffness.

The damage $\omega$ is a function of a damage history parameter $\kappa^d$ that will be specified later

$$\omega = \omega(\kappa^d), \quad (7)$$

and it grows from zero to one as $\kappa^d$ grows from the damage threshold $\kappa_0$ to its ultimate value $\kappa_u$. The damage growth function can represent a uniaxial stress–strain relationship which can be validated on experimental measurements.

During damage evolution the effective stress rate is determined by differentiating Eq. (5):

$$\dot{\sigma} = D^e \dot{\varepsilon}^e, \quad (8)$$

and the stress rate by differentiating Eq. (6):

$$\hat{\sigma} = (1 - \omega)D^e \dot{\varepsilon}^e - \dot{\omega} \hat{\sigma}. \quad (9)$$

### 2.2. Gradient plasticity coupled to damage

To determine the damage model it remains to define a damage function which limits the elasto-plastic behaviour of the material in the strain space:

$$f^d = \varepsilon - \kappa^d = 0. \quad (10)$$

Suitable loading/unloading conditions are formulated as:

$$\kappa^d \geq 0, \quad f^d \leq 0, \quad \dot{\kappa}^d f^d = 0. \quad (11)$$

During the damage evolution the history parameter $\kappa^d$ is equal to the largest value of $\varepsilon$ reached in the loading history. The equivalent strain measure $\varepsilon$ can be defined as a strain energy release rate, cf. (Ju, 1989), or in a form more suitable for quasi-brittle materials (Mazars and Pijaudier-Cabot, 1989; de Vree et al., 1995; Peerlings et al., 1998).

We now revisit the gradient plasticity formulation (Mühlhaus and Aifantis, 1991; de Borst and Mühlhaus, 1992) and write the yield function, which depends on the Laplacian of an equivalent plastic strain measure $\kappa^p$, in the effective stress space:

$$f^p = \tilde{\sigma}(\tilde{\sigma}) - \sigma_y(\kappa^p) + g \nabla^2 \kappa^p = 0,\quad (12)$$
where \( \hat{\sigma} \) is a classical (e.g. Huber–von Mises or Rankine) yield function, \( \sigma_y \) is the yield strength (isotropic hardening/softening is assumed) and \( g \) is a positive gradient influence factor, which will be here assumed constant, although it can be made dependent for instance on the accumulated equivalent plastic strain (Pamin, 1994).

The yield function satisfies the Kuhn–Tucker conditions:
\[
\dot{\lambda} \geq 0, \quad f^p \leq 0, \quad \dot{\lambda} f^p = 0, \quad (13)
\]
in which \( \lambda \) is the plastic multiplier. The plastic multiplier determines the magnitude of the plastic strains according to the classical flow rule. Assuming the standard additive decomposition of strain rates into an elastic and a plastic part, the elastic strain rate is written as:
\[
\dot{\varepsilon}^e = \dot{\varepsilon} - \lambda \mathbf{m}(\hat{\sigma}), \quad (14)
\]
where \( \mathbf{m} \) is the plastic flow direction vector in the effective stress space. We also introduce the definitions of the vector \( \mathbf{n} \) normal to the yield function in the effective stress space, and of the softening modulus \( h \):
\[
\mathbf{n}(\hat{\sigma}) = \frac{\partial \hat{\sigma}}{\partial \sigma}, \quad h = \frac{\partial \sigma_y}{\partial \kappa^p}. \quad (15)
\]

The enhancement of the classical theory was made in order to preserve well-posedness of the governing equations for materials which do not comply with the material stability requirement (Hill, 1958), e.g. when a softening relation between stresses and strains is assumed \( h < 0 \). For a softening medium the factor \( g \) can be associated with an internal length parameter \( l \), e.g. in a one-dimensional analytical solution we have \( g = -hl^2 > 0 \) (de Borst and Mühlhaus, 1992). However, also for a hardening material the Laplacian term with \( g > 0 \) smooths the solution (de Borst and Pamin, 1996).

The finite element implementation is based on the following two weak-form equations governing respectively the static equilibrium and the plastic consistency:
\[
\int_V (L \mathbf{v})^T \mathbf{\sigma} \, dV = \int_V \mathbf{v}^T \mathbf{b} \, dV + \int_S \mathbf{v}^T \mathbf{t} \, dS, \quad (16a)
\]
\[
\int_V w f^p(\hat{\sigma}, \lambda, \nabla^2 \lambda) \, dV = 0, \quad (16b)
\]
where \( \mathbf{v} \) and \( w \) are suitable weighting functions, \( \mathbf{t} \) is the traction vector and equivalence of the plastic multiplier \( \lambda \) and the plastic strain measure \( \kappa^p \) is adopted for simplicity (de Borst and Pamin, 1996). Equation (16b), which requires the discretization of the \( \lambda \) field, is referred to the effective stress space and therefore does not change in presence of damage.

Equations (16) are written for iteration \( i + 1 \) of the incremental-iterative algorithm. Now, the following decomposition is used:
\[
\sigma^{(i+1)} = \sigma^{(i)} + d\sigma, \quad \lambda^{(i+1)} = \lambda^{(i)} + d\lambda, \quad (17)
\]
and the yield function is developed in a truncated Taylor series around \( (\hat{\sigma}^{(i)}, \lambda^{(i)}) \) to obtain the following incremental equations:
\[
\int_V (L \mathbf{v})^T d\mathbf{\sigma} \, dV = f_{\text{ext}} - f_{\text{int}}, \quad (18a)
\]
\[
\int_V w [\mathbf{n}^T d\hat{\sigma} - h d\lambda + g \nabla^2 (d\lambda)] \, dV = - \int_V w f^p(\hat{\sigma}^{(i)}, \kappa^{p(i)}, \nabla^2 \kappa^{p(i)}) \, dV, \quad (18b)
\]
On coupled gradient-dependent plasticity and damage theories

with

\[ f_{\text{ext}} = \int_S \mathbf{v}^T \mathbf{d}S + \int_V \mathbf{v}^T \mathbf{b} \, dV, \quad (19a) \]
\[ f_{\text{int}} = \int_V (\mathbf{L} \mathbf{v})^T \mathbf{\sigma}^{(i)} \, dV. \quad (19b) \]

It is important to notice that, if the yield condition (12) is used in the classical return mapping algorithm to
distinguish elastic and plastic states, a \( C^1 \)-continuous interpolation of \( \lambda \) is unavoidable, otherwise \( \nabla^2 \lambda \) loses
meaning (Pamin, 1994; de Borst and Pamin, 1996). Therefore, the Laplacian term will not be removed from the
left-hand side of Eq. (18b) using Green’s formula, although this can be done if a homogeneous non-standard
boundary condition \( \nabla d\lambda \cdot \nu = 0 \) is assumed (\( \nu \) is the vector normal to the surface of the plastic part of
the body).

Now we substitute Eq. (14) into Eqs (9) and (8) and substitute their incremental forms into Eqs (18a) and
(18b), respectively:

\[ \int_V (\mathbf{L} \mathbf{v})^T [(1 - \omega^{(i)}) \mathbf{D}^e \mathbf{d} \mathbf{e} - \hat{\mathbf{\sigma}}^{(i)} \, d\omega] \, dV - \int_V (\mathbf{L} \mathbf{v})^T [(1 - \omega^{(i)}) \mathbf{D}^e \mathbf{m}^{(i)}] \, d\lambda \, dV = f_{\text{ext}} - f_{\text{int}}, \quad (20a) \]
\[ \int_V \mathbf{w}^T \mathbf{n}^{(i)} \mathbf{D}^e \mathbf{d} \mathbf{e} \, dV - \int_V \mathbf{w} [ (h^{(i)} + \mathbf{n}^{(i)} \mathbf{D}^{p(i)} \mathbf{d} \lambda + g \nabla^2 (d \lambda) ] \, dV \]
\[ = - \int_V \mathbf{w} \mathbf{f}^p (\hat{\mathbf{\sigma}}^{(i)}, k^{P(i)}, \nabla^2 \mathbf{e}^{P(i)}) \, dV. \quad (20b) \]

The increment of damage \( d\omega \) is non-zero only when damage grows, i.e. when Eq. (10) is satisfied, and it can
then be related to the total (elasto-plastic) strain increment as follows:

\[ d\omega = H \mathbf{s}^T \mathbf{d} \mathbf{e}, \quad (21) \]

where the following definitions have been used:

\[ H = \frac{d\omega}{d\mathbf{d} \mathbf{e}}, \quad \mathbf{s}^T = \frac{\partial \mathbf{\tilde{e}}}{\partial \mathbf{\tilde{e}}}. \quad (22) \]

It is noted that \( \mathbf{\tilde{e}} \) is here an equivalent measure of total strains. A similar measure of elastic strains loses sense
in case of softening plasticity and, in order that the gradient regularization be active for the considered coupled
model, the damage growth must depend on plastic strains.

Substituting Eq. (21) into Eq. (20a) and defining the elastic-damage tangent operator as follows:

\[ \mathbf{D}^{ed} = (1 - \omega^{(i)}) \mathbf{D}^e - H^{(i)} \hat{\mathbf{\sigma}}^{(i)} \mathbf{s}^{T(i)} \quad (23) \]

we obtain the following form of Eq. (20a):

\[ \int_V (\mathbf{L} \mathbf{v})^T \mathbf{D}^{ed} \mathbf{d} \mathbf{e} \, dV - \int_V (\mathbf{L} \mathbf{v})^T [(1 - \omega^{(i)}) \mathbf{D}^e \mathbf{m}^{(i)}] \, dV = f_{\text{ext}} - f_{\text{int}}. \quad (24) \]

Next we discretize Eqs (24) and (20b). The displacements \( \mathbf{u} \) and the plastic multiplier \( \lambda \) are interpolated as follows:

\[ \mathbf{u} = \mathbf{N} \mathbf{a}, \quad \lambda = \mathbf{h}^T \mathbf{A}, \quad (25) \]
where $N$ and $h$ contain interpolation polynomials for the displacements and the plastic multiplier, respectively, and $a$ and $A$ are arrays which contain their respective discrete nodal values. Consequently, we obtain for the strains $\varepsilon$ and the Laplacian of the plastic multiplier:

$$\varepsilon = Ba, \quad \nabla^2 \lambda = p^T A,$$

where $B = LN$ and $p = \nabla^2 h$. The respective weighting functions are interpolated similarly according to the Galerkin approach. Invoking the usual argument that the weighting functions are arbitrary, the discrete counterpart of Eqs (24) and (20b) is the following set of linear algebraic equations:

$$
\begin{bmatrix}
K_{aa} & K_{a\lambda} \\
K_{a\lambda} & K_{\lambda\lambda}
\end{bmatrix}
\begin{bmatrix}
da \\
da
\end{bmatrix} =
\begin{bmatrix}
f_{\text{ext}} - f_{\text{int}} \\
f_{\lambda}
\end{bmatrix},
$$

with the damage dependent matrices:

$$K_{aa} = \int_V B^T D^{ed} B \, dV,$$

$$K_{a\lambda} = -\int_V (1 - \omega(i)) B^T D^{p(i)} m^{(i)} h^T \, dV,$$

$$K_{\lambda\lambda} = \int_V \left[(h(i) + n^{(i)} D^{p(i)} m^{(i)}) hh^T - g h p^T \right] \, dV,$$

$$f_{\lambda} = \int_V h f^p(\sigma^{(i)}, \kappa^{p(i)}, \nabla^2 \kappa^{p(i)}) \, dV.$$ 

For an elastic process we take $n = m = 0$, so the off-diagonal matrices are zero, but the matrix $K_{\lambda\lambda}$ is non-singular, so that $dA = 0$ is obtained. Further details of finite element implementation can be found in (de Borst and Pamin, 1996).

2.3. Gradient damage coupled to hardening plasticity

In the second combination of the theories the plasticity formulation remains standard. The yield condition is formulated in the effective stress space

$$f^p = \tilde{\sigma}(\tilde{\sigma}) - \sigma_0(\kappa^p) = 0.$$

The loading/unloading conditions (13), the flow rule in Eq. (14) and the definitions (15) remain valid. In the numerical examples $\tilde{\sigma}$ is the Huber–von Mises yield function, but gradient damage can be coupled in a similar way to other yield functions, for instance to Drucker–Prager plasticity in order to incorporate a difference between the uniaxial and biaxial compressive strength and a possible non-normality of plastic flow. It is noted that in this combination the unstable material behaviour is caused by damage and the plasticity models are hardening.

Following (Peerlings et al., 1996), the damage evolution is now governed by the following damage function

$$f^d = \tilde{\varepsilon} - \kappa^d = 0,$$
On coupled gradient-dependent plasticity and damage theories

where the averaged (nonlocal) strain measure \( \tilde{\varepsilon} \) satisfies the following Helmholtz equation:

\[
\tilde{\varepsilon} - c \nabla^2 \tilde{\varepsilon} = \tilde{\varepsilon}.
\] (32)

The loading/unloading conditions (11) still apply. If it is assumed that the equivalent strain measure is a function of elastic strains only \( \tilde{\varepsilon} = \tilde{\varepsilon}(\tilde{\varepsilon}) \), the two theories become uncoupled (Ju, 1989). If a coupling exists, which seems more relevant, for instance plastically induced damage in metals, plastic strains also contribute to \( \tilde{\varepsilon} \). In this paper we consider both possibilities. The parameter \( c > 0 \) has a unit of length squared and is related to an internal length scale. It is assumed here to be constant, although, with some modifications, it can be made a function of \( \tilde{\varepsilon} \) or \( \tilde{\varepsilon} \) (Geers, 1997).

The finite element implementation is based on the equilibrium equation (16a) and a weak-form of Eq. (32) obtained using Green’s formula and the non-standard boundary condition:

\[
\int_{V} [w \tilde{\varepsilon} + c(\nabla w)^T \nabla \tilde{\varepsilon}] \, dV = \int_{V} w \tilde{\varepsilon} \, dV.
\] (33)

In the ensuing two-field formulation the average strain measure must be discretized in addition to the displacements, but \( C^1 \)-continuity now suffices for all shape functions. The coupling to plasticity influences only the equilibrium equation (16a), while Eq. (33) is exactly the same as for pure gradient damage.

We write Eqs (16a) and (33) for iteration \( i + 1 \) of the incremental-iterative algorithm and decompose the stress vector, the strain measure and its averaged version as

\[
\sigma^{(i+1)} = \sigma^{(i)} + \text{d} \sigma, \quad \tilde{\varepsilon}^{(i+1)} = \tilde{\varepsilon}^{(i)} + \text{d} \tilde{\varepsilon}, \quad \tilde{\varepsilon}^{(i+1)} = \tilde{\varepsilon}^{(i)} + \text{d} \tilde{\varepsilon}
\] (34)

to obtain:

\[
\int_{V} (Lw)^T \text{d} \sigma \, dV = f_{\text{ext}} - f_{\text{int}},
\] (35a)

\[
- \int_{V} w \, d \tilde{\varepsilon} \, dV + \int_{V} [w \, d \tilde{\varepsilon} + c(\nabla w)^T \nabla (d \tilde{\varepsilon})] \, dV = \int_{V} w \tilde{\varepsilon}^{(i)} \, dV - \int_{V} [w \tilde{\varepsilon}^{(i)} + c(\nabla w)^T \nabla \tilde{\varepsilon}^{(i)}] \, dV.
\] (35b)

Before we proceed with the discretization of the above equations, \( \text{d} \sigma \) has to be related to \( \text{d} \tilde{\varepsilon} \). To achieve this we first invoke the local plastic consistency condition \( \dot{\tilde{\varepsilon}} = 0 \) in order compute \( \dot{\lambda} \) for the classical theory formulated in the effective stress space and substitute it into Eq. (14):

\[
\dot{\varepsilon}^e = \dot{\varepsilon} - \frac{1}{h} m \gamma^T \sigma.
\] (36)

We then substitute \( \dot{\varepsilon}^e \) into Eq. (9). After some rearrangements and with the help of Sherman–Morrison formula, we obtain the following relation, written now in an incremental form:

\[
\text{d} \sigma = (1 - \omega) D^{\text{ep}} \text{d} \varepsilon - \text{d} \omega \sigma,
\] (37)

with the classical elasto-plastic matrix

\[
D^{\text{ep}} = D^e - \frac{D^e m n^T D^e}{h + n^T D^e m}.
\] (38)

When Eq. (31) holds, the damage increment \( \text{d} \omega \) can be computed as

\[
\text{d} \omega = H \text{d} \tilde{\varepsilon},
\] (39)
with $H$ defined in Eq. (22)_1, while the increment of the strain measure $d\tilde{\varepsilon}$ is computed as:

$$d\tilde{\varepsilon} = s^T d\varepsilon,$$

with $s^T$ defined in Eq. (22)_2. When $\tilde{\varepsilon}$ is a measure of elastic strains only, the computation of $s^T$ becomes slightly more intricate, since it then equals:

$$s^T = \frac{d\tilde{\varepsilon}}{d\varepsilon} d\varepsilon = \frac{d\tilde{\varepsilon}}{d\varepsilon} (D^e)^{-1} D^{ep}.$$

Equation (39) is substituted into Eq. (37) and next Eqs (37) and (40) are substituted into Eqs (35a) and (35b), respectively, to obtain:

$$\int_V (L\v) (1 - \omega^{(i)}) D^{ep} d\varepsilon dV - \int_V (L\v) H^{(i)} \hat{\sigma}^{(i)} d\tilde{\varepsilon} dV = f_{ext} - f_{int},$$

$$- \int_V w s^{(i)} d\varepsilon dV + \int_V [w d\tilde{\varepsilon} + c(\nabla w)^T \nabla (d\tilde{\varepsilon})] dV$$

$$= \int_V w \dot{\varepsilon}^{(i)} dV - \int_V [w \dot{\varepsilon}^{(i)} + c(\nabla w)^T \nabla \dot{\varepsilon}^{(i)}] dV.$$

The displacements and the averaged strain measure are now discretized as follows:

$$u = Na, \quad \tilde{\varepsilon} = h^T E,$$

so that

$$\varepsilon = Ba, \quad \nabla \tilde{\varepsilon} = q^T E,$$

with $q^T = \nabla h^T$. The respective weighting functions are discretized similarly, so that for arbitrary weighting functions Eqs (42) are represented by the following matrix equation:

$$\begin{bmatrix} K_{aa} & K_{a\tilde{\varepsilon}} \\ K_{\tilde{\varepsilon}a} & K_{\tilde{\varepsilon}\tilde{\varepsilon}} \end{bmatrix} \begin{bmatrix} da \\ d\varepsilon \end{bmatrix} = \begin{bmatrix} f_{ext} - f_{int} \\ f_e - f_f \end{bmatrix},$$

with the plasticity dependent submatrices now defined as follows:

$$K_{aa} = \int_V (1 - \omega^{(i)}) B^T D^{ep} B dV,$$

$$K_{a\tilde{\varepsilon}} = - \int_V H^{(i)} B^T \hat{\sigma}^{(i)} h^T dV,$$

and the other matrices similar to the gradient damage formulation of (Peerlings et al., 1996):

$$K_{\tilde{\varepsilon}a} = \int_V h s^{(i)} B dV,$$

$$K_{\tilde{\varepsilon}\tilde{\varepsilon}} = \int_V (h h^T + c q q^T) dV,$$

$$f_e = \int_V h \dot{\varepsilon}^{(i)} dV, \quad f_f = K_{\tilde{\varepsilon}\tilde{\varepsilon}} E^{(i)}.$$

In the absence of damage growth $K_{a\tilde{\varepsilon}} = 0$, so that the equilibrium equations in identity (45) are then uncoupled from the averaging equations. For further details of gradient damage implementation the reader is referred to (Peerlings et al., 1996).
1. Solve Eq. (45) for increments of nodal displacements and averaged strain [or Eq. (27) for increments of nodal displacements and plastic multiplier]

2. At integration point compute strain increments $\Delta \epsilon$, $\Delta \hat{\epsilon}$ [or $\Delta \lambda$] and update total strains $\epsilon$

3. Resolve plasticity in effective stress space:
   trial stress $\hat{\sigma} = \hat{\sigma}_0 + D^e \Delta \epsilon$, $\hat{\sigma}_0 = \sigma_0/(1 - \omega_0)$
   if $f^p > 0$ – Eq. (30) [or (12)] then $\hat{\sigma} = \hat{\sigma}_1 - \Delta \lambda \cdot D^e m(\hat{\sigma})$,
   else $\hat{\sigma} = \hat{\sigma}_1$

4. Resolve damage:
   strain measure $\hat{\epsilon}(\epsilon)$ or $\hat{\epsilon}(\epsilon^e)$
   if $f^d > 0$ – Eq. (31) [or (10)] then $\kappa^d = \hat{\epsilon}$ [or $\hat{\epsilon}_e$], $\omega = \omega_0(\kappa^d)$,
   else $\omega = \omega_0$

5. Transform stress $\sigma = (1 - \omega) \hat{\sigma}$

6. Compute out-of-balance forces and tangent matrices according to Eqs (46), (47) [or Eqs (28), (29)], check convergence and, if necessary, return to 1

**Box 1.** Stress update algorithm for gradient-damage coupled to plasticity or, with modifications in square brackets, for gradient-plasticity coupled to damage.

### 2.4. Algorithm

Box 1 presents the algorithm of incremental-iterative finite element computations, which is quite similar for both combinations of the local and gradient-dependent theories. In the algorithm the index 0 denotes the beginning of the current increment (“iteration 0”), i.e. quantities stored at the end of previous increment. The stress vector $\sigma_0$ is first mapped to the effective stress space and the trial stress $\sigma_1$ is computed assuming that the whole strain increment $\Delta \epsilon$ is elastic (elastic predictor). A plastic corrector algorithm is then applied and, finally, the updated effective stress $\hat{\sigma}$ is mapped back using the updated damage variable. In fact, if the damage measure is based on total strains $\hat{\epsilon}(\epsilon)$, items 3 and 4 of the algorithm can be reversed.

It is emphasized that, since the “total-incremental” stress update algorithm is used for plasticity, the so-called consistent linearization requires a correction of the stiffness operator $D^e$ in order to achieve a quadratic convergence rate. In the considered models the correction is limited to replacing the matrix $D^e$ by a matrix $H$ computed in the current iteration as (de Borst and Groen, 1994):

$$ H = \left( I + \Delta \lambda \cdot D^e \frac{dm}{d\sigma} \right)^{-1} D^e, \quad (48) $$

with the unit matrix $I$, in Eqs (23), (28b), (29a,b) and (38).

The algorithm in Box 1 holds for general three-dimensional and plane strain conditions as well as for the one-dimensional case. However, since the algorithm is driven by strains, the plane stress condition $\sigma_{zz} = 0$...
yields some additional difficulty (de Borst, 1991). Three approaches can essentially be used for the present models:

1. The coupled constitutive model is resolved in three-component space of strains \((\varepsilon_{xx}, \varepsilon_{yy}, 2\varepsilon_{xy})\) and stresses \((\sigma_{xx}, \sigma_{yy}, \sigma_{xy})\), and \(\varepsilon_{zz}\) is derived from elastic and plastic parts of in-plane normal strains, but the plastic return mapping is not radial, which means that iterations are required to satisfy the yield condition and large strain increments must be avoided, otherwise these iterations may fail;

2. The plastic counterpart of the model is resolved in four-component (plane strain) space with the condition \(\sigma_{zz} = 0\), which allows for a radial return mapping, see (de Borst, 1991) for the general derivation and (Pamin, 1994) for the gradient plasticity case, and the damage counterpart is considered in three-component space;

3. The expansion-compression idea from (de Borst, 1991) is applied to the coupled model, i.e. strains are expanded to four-component space, constitutive relations for plane strain are used, and finally stresses are compressed to three-components.

We now briefly demonstrate the third idea for the combination of gradient damage and plasticity. Taking Eq. (37) as a starting point we can write:

\[
\sigma^{(i+1)} = \sigma^{(i)} + (1 - \omega^{(i)}) \mathbf{D}^{\text{ep}}^{(i+1)} \mathbf{d}\varepsilon - d\omega \hat{\sigma}^{(i)}. \tag{49}
\]

We now drop index \((i + 1)\), define a matrix \(\mathbf{D}\) as:

\[
\mathbf{D} = (1 - \omega^{(i)}) \mathbf{D}^{\text{ep}} \tag{50}
\]

and write Eq. (49) for the four-component situation partitioning all matrices as follows:

\[
\begin{bmatrix}
\sigma^c \\
\sigma_{zz}
\end{bmatrix} = \begin{bmatrix}
\sigma^{c(i)} \\
\sigma_{zz}^{(i)}
\end{bmatrix} + \begin{bmatrix}
\mathbf{D}_{xx} & \mathbf{D}_{xz} \\
\mathbf{D}_{zx} & \mathbf{D}_{zz}
\end{bmatrix} \begin{bmatrix}
\mathbf{d}\varepsilon^c \\
\mathbf{d}\varepsilon_{zz}
\end{bmatrix} - d\omega \begin{bmatrix}
\hat{\sigma}_z^{c(i)} \\
\hat{\sigma}_{zz}^{(i)}
\end{bmatrix}, \tag{51}
\]

where index \(c\) denotes the matrices associated with the “compressed” plane stress space. Next, we enforce the plane stress condition

\[
\sigma_{zz} = \sigma_{zz}^{(i)} + D_{xz} \mathbf{d}\varepsilon^c + D_{zz} \mathbf{d}\varepsilon_{zz} - d\omega \hat{\sigma}_{zz}^{(i)} = 0, \tag{52}
\]

which makes it possible to compute the increment of the normal strain in \(z\)-direction as:

\[
\mathbf{d}\varepsilon_{zz} = -\left(\sigma_{zz}^{(i)} + D_{xz} \mathbf{d}\varepsilon^c - d\omega \hat{\sigma}_{zz}^{(i)}\right)/D_{zz}. \tag{53}
\]

It is noted that the damage increment \(d\omega\) can be computed from the increment of nodal average strain values, cf. Eqs (39) and (43). The stress update for the plane stress situation is then written as:

\[
\sigma^c = \left(D_{xx} - \frac{D_{xz} D_{xx}}{D_{zz}}\right) \mathbf{d}\varepsilon^c + \sigma^{c(i)} + \frac{D_{xz}}{D_{zz}} \sigma_{zz}^{(i)} - d\omega \left(\hat{\sigma}_z^{c(i)} + \frac{D_{xz}}{D_{zz}} \hat{\sigma}_{zz}^{(i)}\right), \tag{54}
\]

and is used to set up the equilibrium equation (42a).

3. One-dimensional numerical studies

The implementation of the theories will first be applied to a uniaxial problem shown in figure 1 in order to illustrate the basic properties of the models. The equivalent strain measure \(\hat{\varepsilon}\) is in this section equal to the
On coupled gradient-dependent plasticity and damage theories

Figure 1. Tensile bar with imperfection for localization studies.

absolute value of the axial strain $\varepsilon$ (full coupling) or its elastic part $\varepsilon^e$ (uncoupled damage-plasticity). The damage growth function (7) is either linear

$$\omega(\kappa^d) = \frac{\kappa^d - \kappa_0}{\kappa_u - \kappa_0},$$

which means that, although eventually the stress drops to zero, there is a hardening stage in the stress–strain relationship, or it represents a linear softening stress–strain diagram for uniaxial uniform strain fields

$$\omega(\kappa^d) = \frac{\kappa_u (\kappa^d - \kappa_0)}{\kappa_u \kappa^d - \kappa_0}$$

The former case will further be called “linear damage”, while the latter case will just be called “damage”.

3.1. Gradient plasticity coupled to scalar damage

We analyze a bar in tension shown in figure 1. The bar has a geometrical imperfection in order to initiate strain localization, namely a 10% smaller cross section is adopted over the central $d = 10$ mm. The derivative of the plastic multiplier is set to zero at both ends of the bar to allow for reduced integration of $C^1$-continuous three-noded gradient plasticity elements (Pamin, 1994; de Borst and Pamin, 1996). Two meshes with 80 and 320 elements are used to examine the mesh sensitivity of the simulations. In the calculations the length of the bar is $L = 100$ mm, Young’s modulus is $E = 20000$ N/mm$^2$, the tensile strength $\sigma_y = 2$ N/mm$^2$. Linear softening with $h = -500$ N/mm$^2$ and the gradient constant $g = 12500$ N are assumed, so that the width of the localization zone predicted by gradient plasticity is $w = 31.4$ mm (de Borst and Mühlhaus, 1992). The solutions obtained for pure gradient plasticity are compared with the coupled model, in which $\bar{\varepsilon}$ is the absolute value of the total strain.

Figures 2, 3 present the results obtained when the damage growth function from Eq. (56) is used with $\kappa_o = 0.001$ and $\kappa_u = 0.1$. The load-displacement diagrams in figure 2a show the computed relations between the stress at the right end of the bar $\sigma$ and the elongation of the bar $u_{end}$. Figure 2b presents the damage evolution in the bar. Figures 3a, b compare the distributions of plastic strains in the loading history for pure gradient plasticity and for its coupling to damage.

The two meshes give similar results. Fast convergence has been observed in the calculations even when the strain in some elements exits the softening branch, i.e. when the load-displacement diagrams bend upwards and the localization zone starts broadening. For the coupled model the plastic strains are much more localized due to interaction with damage, but a spurious tendency to localize in the smallest possible volume is not observed. Apparently, the gradient regularization is active in spite of adding the second destabilizing component in the constitutive description.
Figure 2. (a) Mesh sensitivity for gradient plasticity with damage. (b) Damage evolution for 80/320 elements.

Figure 4a shows load-displacement diagrams obtained with the linear damage growth function, Eq. (55), with \( \kappa_u = 0.01 \) and compares diagrams obtained for two damage thresholds \( \kappa_0 = 0.0001 \) and \( \kappa_0 = 0.001 \). For the first threshold value the damage and plastic processes are activated simultaneously, i.e. \( \kappa_0 = 0.0001 \) is reached for the same load as \( \sigma_y \). For the linear relation \( \omega(\kappa^d) \) the damage growth is initially much slower, but the ultimate value \( \kappa_u = 0.01 \) associated with \( \omega = 1 \) and \( \sigma = 0 \) is reached for smaller extension of the
The results for linear damage are not very sensitive to the value of $\kappa_0$. However, for the other damage growth function completely different results are obtained when the two unstable components of the model are triggered at the same moment. For the assumed imperfection (8 weaker elements) the strains and damage then localize in just two elements, which is associated with a snap-back behaviour, and when stresses drop below $\sigma = 0.5 \text{ N/mm}^2$ the localization zone starts to expand.
For hardening gradient plasticity we preserve the gradient constant $g = 12500$ N, but we adopt a linear hardening with $h = 500$ N/mm$^2$. We employ a discretization with 80 elements. Figure 4b shows load-displacement diagrams obtained for pure gradient-plasticity and for the coupling with three values of the damage threshold. If only plasticity is considered or the beginning of the damage process occurs later than the onset of plasticity we first observe yielding in the imperfect part of the bar and then, due to hardening, a
plastic state is reached in the whole bar. However, once damage starts, softening is observed and localization of both damage and plastic processes takes place in a zone whose size depends on the damage threshold ($w \approx 15/23$ mm for $\kappa_0 = 0.0005/0.001$, respectively). For $\kappa_0 = 0.0001$ a fast growth of damage is first observed in two central finite elements (symmetry of deformation is imposed) and then the damage zone gradually expands. Although the regularization related to plasticity seems to be active for the softening caused by damage unless both processes begin simultaneously, the dependence of results on the damage threshold seems excessive also for the hardening case. Therefore, this coupled model will not be used for two-dimensional simulations.

3.2. Gradient damage coupled to hardening plasticity

We analyze the same bar as in the preceding section (figure 1) with unchanged length, size of the imperfection and Young’s modulus. Three meshes with 20, 80 and 320 $C^0$-continuous three-noded gradient damage elements are used. The elements have a quadratic and linear interpolation of the displacement and averaged strain, respectively. The gradient constant $c = 4 \text{ mm}^2$ is assumed. The damage growth function representing linear softening, Eq. (56), with $\sigma_0 = 0.0001$ and $\kappa_u = 0.0125$ is employed.

The total strain is first adopted as the strain measure $\varepsilon = \varepsilon$. In figures 5, 6 the solutions obtained for pure gradient damage are compared with coupling to hardening plasticity with the following parameters: $\sigma_y = 5 \text{ N/mm}^2$ and $h = E$. Now the beginning of the plastic process is slightly delayed with respect to the damage threshold.

Figure 5a shows a comparison of load-displacement diagrams and figure 5b an evolution of the averaged strain $\varepsilon$. It is observed that, since the Helmholtz partial differential equations underlying the gradient plasticity and gradient damage models differ in the sign of the Laplacian term, the solution of the regularizing equation has a cosine shape for the former theory and is exponential for the latter.

The damage growth for the two models is shown in figure 6. After the plastic process has started, no more expansion of the damage zone is observed unlike in the absence of plasticity. Similar diagrams and profiles are obtained for 80 and 320 elements, so the numerical results converge upon mesh refinement to a unique, physically meaningful solution. This is expected, since the only unstable part of the constitutive model, i.e. the damage growth, is regularized by the gradient enhancement.

The distribution of plastic strains for $h = E$ is shown in figure 7a. Plastic strains first develop in a volume larger than the imperfect part of the beam, which is driven by interaction with nonlocal damage, but in the final stage the deformation localizes in the smallest possible volume (two integration points). The physical explanation is that the plastic strains together with damage finally evolve into a crack of zero width at midspan of the bar. Up to this moment, localization is limited during the whole deformation history.

The value adopted in this test for the hardening modulus $h$ may seem unrealistically large, but we note that plasticity now represents the behaviour of the material “skeleton” (the fictitious body, see Section 2.1) and the evolution of microcracks and microvoids which causes the strength and stiffness degradation of the material is modelled by the damage component. The sensitivity of the results with respect to the value of $h$ is shown in figure 7a. It is noted that a simultaneous beginning of plasticity and damage ($\sigma_y = 2 \text{ N/mm}^2$) is now assumed and comparison with figure 5a shows that the results are now much less sensitive to the plasticity threshold.

Figure 8 presents the results of the same test with plasticity parameters $\sigma_y = 5 \text{ N/mm}^2$ and $h = E$, but with the strain measure equal to the elastic strain $\varepsilon = \varepsilon^E$. As shown in figure 8a the obtained results exhibit an increased ductility of the model. This is because the elastic strains, which now govern the damage growth, increase much slower than the total strains in the plastic regime. The damage distribution is similar to the case with no coupling to plasticity, shown in figure 6a. This is because for this assumption plasticity and damage
Figure 5. (a) Load-displacement diagrams for gradient damage coupled to hardening plasticity ($\tilde{\varepsilon} = \varepsilon$). (b) Evolution of averaged (nonlocal) strain $\tilde{\varepsilon}$.

contribute to the response without mutual interaction. Also, the plastic strain evolution shown in figure 8b is quite similar to the former case with $\tilde{\varepsilon} = \varepsilon$. 
On coupled gradient-dependent plasticity and damage theories

4. Compact-Tension test

The Compact-Tension (CT) test is a plane stress configuration which has been used in standard experiments of (fatigue) fracture of metals and composites, cf. (Geers, 1997). The specimen is placed on two loading pins, which exert vertical tensile forces under deformation control. A symmetric half of the original specimen is
Figure 7. Gradient damage coupled to plasticity ($\dot{e} = \varepsilon$): (a) plastic strain evolution for $h = E$, (b) sensitivity to hardening modulus $h$ for $\sigma_y = 2.0$ N/mm$^2$. Plotted in figure 9 together with the finite element mesh employed in present computations. Gradient damage elements with quadratic interpolation of displacements and linear interpolation of the averaged strain measure and a $2 \times 2$ Gauss integration are used. The action of the pins is modelled by inelastic spring elements (stiff in compression and compliant in tension).
The Mazars’ definition of the equivalent strain measure is adopted (Mazars and Pijaudier-Cabot, 1989):

\[
\bar{\varepsilon} = \left[ \sum_{i=1}^{3} \left( \langle e_i \rangle \right)^2 \right]^{1/2},
\]  

(57)
where $\varepsilon_i$ are the principal strains, $\langle \varepsilon_i \rangle = \varepsilon_i$ if $\varepsilon_i > 0$ and $\langle \varepsilon_i \rangle = 0$ otherwise. In order to follow the experimental tensile fracture of the specimen and avoid singularities caused by damage equal to one, the damage growth function (7) now represents exponential softening

$$\omega(\kappa^d) = 1 - \frac{\kappa_0}{\kappa^d} (1 - \alpha + \alpha e^{-\eta(\kappa^d - \kappa_0)}),$$

for which $\alpha$ and $\eta$ are additional parameters.

In figure 10 the obtained diagrams are compared with the experimental results, given in (Geers, 1997) for a composite specimen with randomly distributed glass-fibres. The displacement shown is measured between two markers placed 25 mm from the left edge and 14 mm from the symmetry axis of the specimen. Three cases are analyzed: pure gradient damage, gradient damage coupled to plasticity by the equivalent strain measure based on total strains $\bar{\varepsilon}(\varepsilon)$ and the uncoupled gradient damage-plasticity with $\bar{\varepsilon}(\varepsilon^e)$. The following model parameters are adopted: Young’s modulus $E = 3200$ N/mm$^2$, Poisson’s ratio $\nu = 0.28$, damage threshold $\kappa_0 = 0.016$, damage growth function parameters $\alpha = 0.99$, $\eta = 10$, yield strength $\sigma_y = 100$ N/mm$^2$, hardening modulus $h = 3200$ N/mm$^2$, gradient influence factor $c = 2$ mm$^2$. Huber–von Mises plasticity is used and the value of $\sigma_y$ is chosen such that the damage and plastic processes commence almost simultaneously.

With the adopted parameters the response of the coupled model fits the experimental results. However, it is emphasized that the other two models can also be easily tuned to fit the experimental results obtained for monotonic loading. The crucial point is that localization is limited and that for both damage-plasticity combinations unloading reveals permanent deformations. For a similar deformation, the model with $\bar{\varepsilon}(\varepsilon^e)$ exhibits considerably larger ductility and smaller stiffness degradation.

Figure 11a shows the incremental deformation pattern obtained for the coupled model at the final point on the curve in figure 10. It shows the redistribution of damage within a zone whose size is related to the gradient influence factor $c$, which is typical for nonlocal continuum modelling. The next pattern is plotted for a local continuum model (classical Rankine plasticity has been used for this purpose). Localization in just one row of elements is then observed, which would lead to spurious mesh sensitivity of results. From the
physical viewpoint, the most realistic simulations would be achieved if, after a regularized localization stage as in figure 11a, a crack failure mode like in figure 11b was predicted (Geers, 1997).

Finally, figures 12a, b compare final damage profiles for the gradient damage and the coupled model, figure 12c presents an averaged strain distribution for the latter model, and figure 12d shows equivalent plastic strains. The width of the damage zone is similar for both cases. Also, for the third case where the equivalent strain depends only on elastic strains, the width of the damage zone does not differ significantly. In spite of the presence of more localized plastic strains in the experimental crack zone, the damage distribution driven by nonlocal strains does not evolve into a crack.

5. Final remarks

Two gradient dependent couplings of plasticity and damage have been proposed. They are formulated in such a way that the coupling influences only the equilibrium equations, while the second differential equation which serves the purpose of regularization preserves the original format of gradient plasticity (de Borst and Mühlhaus, 1992) and gradient damage (Peerlings et al., 1996), respectively. For both couplings consistently linearized two-field finite element formulations are used.

Although softening gradient plasticity with the Rankine failure function is capable of reproducing to some extent the decohesive material failure which usually underlies damage, the lack of elastic stiffness degradation makes the model unacceptable for cyclic loading. However, coupling of this theory to damage suffers from two drawbacks: the results are excessively sensitive to the damage threshold, and, more importantly, it is difficult to find physical grounds for a combination of two softening mechanisms even if the regularization of only the plastic component of the model leads to mesh-insensitive results. The combination of hardening gradient plasticity and damage, considered also by Svedberg and Runesson (1997), requires further study.
On the other hand, the combination of gradient damage with hardening plasticity produces promising results: the physical interpretation of the model components is convincing, there are only a few model parameters to determine, elastic stiffness degradation is reproduced and spurious mesh sensitivity is avoided. However, as has been noticed by Geers (1997), the damage predictions of the model with a constant gradient influence factor $c$ and a damage evolution law of the type (58) are not realistic in two-dimensional configurations, since the damage zone broadens and does not evolve into a crack when failure is approached. The coupling to plasticity does not solve this problem, so the gradient-damage model must be improved, for instance by making the gradient influence factor variable, see (Geers, 1997). Alternatively, a damage evolution law which does not attain the limit $\omega = 1$ asymptotically can be used, but this inevitably necessitates appropriate remeshing procedures.

Further research should also encompass a deeper insight into the thermodynamic basis of gradient models, cf. (Svedberg and Runesson, 1997; Polizzotto and Borino, 1998), leading also to a better understanding of damage measures which should be used for different materials, and possible extensions to anisotropic damage models. A parametric study of the models for monotonic and cyclic loading, which involves the simulation of crack closure effects, is also envisaged.
Figure 12. Fracture zone of CT test: (a) final damage distribution for pure gradient damage (a); final damage (b), averaged strain (c) and equivalent plastic strain (d) distribution for gradient damage coupled to plasticity.

Acknowledgments

Financial support of the von Humboldt Foundation and the Max-Planck Society through the Max-Planck Research Award 1996 to the first author and of the Polish Committee for Scientific Research (the grant PB 7 T07A 03612) to the second author are gratefully acknowledged.

References


