The paper deals with the problem of robust synchronization of dynamical systems. The design procedure is based on the concept of observers with absolutely stable error dynamics. In the general case of nonlinear time-varying error dynamics the procedure requires exact knowledge of a Lyapunov function while in case of the linearizable error dynamics frequency domain conditions which ensure existence of such a function can be employed. Two examples are considered: synchronization of two Lorenz systems and Rössler systems.

1. Introduction

In recent years there has been considerable interest in the dynamics and control of systems exhibiting complex behavior. The number of papers related to this subject seems to grow at an almost exponential rate [Chen, 1998]. For a review of some of the prevailing research problems in the area the reader may consult the seminal papers in the November 1997 special issue of *IEEE Transactions on Circuits and Systems, Part I*.

In the 70s the problem of synchronization attracted great attention, in particular, in connection with the question of synchronization of satellites and of mechanical vibrators. Some mathematical aspects of the synchronization problem can be found in [Gurtovnik & Neimark, 1974]. Quite recently the interest to synchronization phenomena was revived by Pecora and Carroll [1990] who studied master–slave synchronization of dynamical systems with chaotic behavior. Often a master–slave formalism is taken. Given a particular dynamical system, the master, together with an identical (sub)system, the slave, the aim is to synchronize to the master system the complete response of the slave system, by driving the latter with a (scalar) signal derived from the master system. In this context synchronization is often considered to be a remarkable property when the master dynamics are chaotic and thus sensitive to initial condition variations. A promising application in private communication suggested in [Cuomo et al., 1993] uses such a chaotic master dynamics to mask a message and a synchronized slave system to recover the message.

The above master–slave viewpoint leaves some ambiguity as to what the actual slave system should be, given the master system. A naive, but often realistic approach, would be to consider the master dynamics (transmitter) as transmitting a signal to...
the slave dynamics (receiver) and the receiver is requested to recover the full state trajectory of the transmitter. The problem is of course only interesting if the signal received is not equal to the full state. In this situation the receiver has in principle the freedom to build any dynamical system. The receiver system could be a copy of the master system, but it need not be. The real requirement is that given the received signal the receiver dynamics will asymptotically synchronize to the transmitter dynamics. In thus allowing the receiver the freedom of which dynamical system to implement, we enlarge the class of master/slave systems that allow synchronization. Note that at this point we do not consider the actual physical realization of the new receiver’s dynamical system.

The problem just described is in fact the observer problem from control theory (see [Nijmeijer & Mareels, 1997] where the problem of synchronization is considered within this framework) although different approaches to define synchronization exist, see for example [Blekhman et al., 1997] where the concept of controlled synchronization is considered. In the present note we will interprete the problem of synchronization as an observer design problem.

At the same time since the synchronization of dynamical systems is useful in practice, there is a question of how to ensure robust synchronization. This problem can be considered as a problem of design of observers with robustly stable error dynamics. A fruitful framework for this design is the concept of absolute stability. In the paper we revisit this concept for the purposes of observer design and introduce the concept of observers with $L_p$-asymptotically absolutely stable error dynamics. This concept allows to design a class of observers for a given system and so the designer has freedom to choose a particular solution based on some optimization technique.

Therefore it is interesting to find a solution to the following problem: Given a transmitter, how to find a class of possible receivers such that any receiver from this class ensures the goal of synchronization. The problem statement means that once the class of admissible receivers is found in practical implementation the receiver can be realized with several degrees of uncertainties. However for the solution to this problem we will assume that the dynamics of the transmitter is given and therefore the uncertainty under consideration is not a traditional parametric uncertainty in transmitter or caused by some unmodeled dynamics in the observed system.

In general the error dynamics can be nonlinear and time-variant. In this paper this case is investigated by an example of master–slave synchronization of two Lorenz systems. We present sufficient conditions which ensure robust synchronization in the sense that they provide $L_2$-asymptotically absolutely stable error dynamics. It is shown that some known synchronization schemes including the Pecora–Carroll scheme and high-gain observers belong to a class of observers which satisfy several local quadratic constraints.

A particular interest in the paper is drawn to observers with linearizable time-invariant error dynamics. The problem in this case is divided into two steps. First we should find a coordinate transformation and/or output rescaling such that in the new coordinates the system admits an observer with linear time-invariant dynamics. At the second step we should check absolute stability of the error system. Necessary and sufficient condition which ensure existence of the solution to the first problem can be derived for example from [Tchon & Nijmeijer, 1993; Krener & Isidori, 1983; Krener & Respondek, 1985]. Sufficient conditions for the second problem which are formulated in terms of frequency domain inequalities, or, equivalently, in terms of linear matrix inequalities can be obtained by Kalman–Yakubovich–Popov lemma [Kalman, 1963; Yakubovich, 1962]. These conditions are sufficient in the case of local quadratic constraints, but also necessary in the case of integral quadratic constraints [Yakubovich, 1973; Megretsky & Treil, 1993].

Analysis of synchronization phenomena can be performed by the analysis of the stability of zero solution of the error dynamics. A possible approach to the problem is based on the concept of Lyapunov exponent. The well-known Lyapunov theorem asserts that under some conditions the trivial solution of nonlinear time-varying system is exponentially stable as soon as the Lyapunov exponents of the first approximation system are negative. Unfortunately conditions of the original Lyapunov theorem are too restrictive to employ this result in a general case. Later on these conditions were relaxed by Chetaev [1955] and Massera [1957], however, the statements of their theorems are not convenient to study synchronization. Amazingly, but in 1913, Latvian mathematician P. Bol introduced the concept of general exponent (or Bol exponent) and proved a theorem on uniform asymptotic stability of zero solution of nonlinear time-varying system...
This overlooked result is more general than the result due to Massera and involves computation of the Bol exponents which is a quite tricky problem in practice. A brief comparison between the concepts of Lyapunov and Bol exponents can be found in [Fradkov & Pogromsky, 1998]. Traditionally in control theory stability analysis of nonlinear \textit{time-varying} systems is performed by the second Lyapunov method and in this paper this method is extensively used for the problems of designing synchronizing systems.

As an example we consider the synchronization of two Rössler systems. First, it is shown that there exists a coordinate transformation which is well defined in the domain where solutions of the Rössler system are defined. This coordinate change transforms the Rössler system to a system (which is defined in the area where all solutions of the original system have infinite interval of existence) which has an observer with linear time-invariant error dynamics. Absolute stability of the error dynamics in this case is investigated by the circle criterion.

The goal of this paper is to apply results involving a famous theorem attributed to Kalman Yakubovich and Popov to the study of robust synchronization. In this way we further develop the observer perspective on the synchronization problem. Since not all readers may be completely familiar with basics from control, we have tried to give a concise but complete presentation on the subject.

The paper is organized as follows. First we define the concept of robust synchronization based on the concept of observer with absolutely stable error dynamics. This concept is illustrated by an example of synchronization of two Lorenz systems in Sec. 4. A particular case of the master–slave synchronization when the master system admits an observer with linear time-invariant dynamics is considered in Sec. 5.

In the paper we use the following nomenclature. The Euclidean norm in $\mathbb{R}^n$ is denoted as $|\cdot|$, $|x|^2 = x^\top x$, where $\top$ stands for the transpose operation. The $L_p$ norm, $1 \leq p \leq \infty$ is denoted as $\|\cdot\|_p$ with

$$\|y\|_p = \left( \int_0^\infty |y(t)|^p dt \right)^{1/p}.$$ 

If $\|y\|_p$ is finite for a function $y$ defined on $\mathbb{R}_+$ we will write $y \in L_p(\mathbb{R}_+)$. A function $V : X \rightarrow \mathbb{R}_+$ defined on a subset $X$ of $\mathbb{R}^n$, $0 \in X$ is positive definite if $V(x) > 0$ for all $x \in X \setminus \{0\}$ and $V(0) = 0$. It is radially unbounded if $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.

\section{Robust Synchronization}

Consider the following dynamical system

$$\dot{x} = f(x)$$

with $x(t) \in \mathbb{R}^n$ and $f$ a smooth vector field. We assume that there is a set of initial conditions such that all solutions to (1) starting from this set are well defined on the infinite time interval $\mathbb{R}_+$. We denote the union of such sets as $\Omega \subseteq \mathbb{R}^n$, that is $\Omega$ is the set of all initial conditions for which solutions are well defined on the infinite time interval. Clearly, $\Omega$ is invariant under (1). The system defined by (2) is referred to as the transmitter. The transmitter generates a scalar signal

$$y = h(x)$$

with $h : \mathbb{R}^n \rightarrow \mathbb{R}$ a smooth map, which is sent to the receiver. The problem of designing synchronizing systems is to design the dynamics of a receiver which reconstructs the whole state vector $x(t)$. As it is easy to notice this problem is equivalent to the (full order) observer design problem, though different points of view on this problem are also possible. For example, in [Blekhman et al., 1997] a dual understanding of the synchronization problem is suggested and this problem is treated as a control problem.

The receiver dynamics can be described by the following system

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, y)$$

where $\tilde{x}(t) \in \mathbb{R}^n$ and $\tilde{f} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is smooth and such that the error $e(t) := x(t) - \tilde{x}(t)$ asymptotically vanishes as time goes to infinity for all $x(0) \in \Omega$ at least for sufficiently small $e(0)$. Additionally it is natural to require that the set $\{x, \tilde{x} : x \in \Omega, x = \tilde{x}\}$ is invariant under the dynamics given by (1) and (3). For our purposes it is convenient to rewrite (3) as follows

$$\dot{\tilde{x}} = f(\tilde{x}) + k(\tilde{x}, y)$$

where $k(\tilde{x}, y) := \tilde{f}(\tilde{x}, y) - f(\tilde{x})$. The vector field $k$ parametrized by $y$ can be understood as the \textit{output injection} and the synchronization problem is to design it in such a way that $e(t) \rightarrow 0$ as $t \rightarrow \infty$ provided that $x(0) \in \Omega$ and $e(0)$ arbitrary (global observer problem), or, additionally $|e(0)|$ is small enough (local observer problem).
The observer problem, that is the question how to design \( k(\tilde{x}, y) \) in (4) that yields \( e(t) \to 0 \) when \( t \to \infty \), is for general systems (1) and (2) unsolved. For particular classes of systems a solution has been obtained, see for example [Nijmeijer & Mareels, 1997] and references in there. In this note we address to the problem of design of robust observers which is motivated by the problem of robust synchronization.

The problem we consider in the present paper is to study “robust” synchronization. That is, the output injection \( k(\cdot) \) may contain uncertainties and we still wish to guarantee a successful synchronization through the observer (4). Another motivation comes from the fact that once one particular observer (4) is found for the system (1) and (2) it becomes a natural and important question to seek for an optimized design procedure. In other words, is it possible to find a class of observers for the system (1) and (2)? Depending on the required optimization, one then could seek the “best” observer in this class.

One of the fruitful approaches to the posed problem relies on the concept of absolute stability. The term robust stability also became in use. In this section we slightly modify the conventional definition to fit the problem statement discussed in the previous section.

The vector function \( k \) defines the relation between the measured output \( y \) and the input \( u \) given the variables \( \tilde{x} \). Suppose that the function \( k(\cdot) \) is uncertain. In case a nominal function \( k_0(\cdot) \) is available that solves the observer problem one may proceed to seek for admissible “perturbations” of \( k_0(\cdot) \) that yield an admissible observer. This reflects a class of possible uncertainties in the output injection \( k(\cdot) \) that guarantee asymptotic convergence of the error dynamics. Our goal is to investigate methods for finding such classes of perturbations. Without loss of generality one can take

\[
k(x, \tilde{x}, t) = k_0(\tilde{x}, y) + u(t) \tag{5}
\]

where \( k_0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is the nominal output injection, \( k : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \) is the uncertain output injection, \( u : \mathbb{R}_+ \to \mathbb{R}^n \) is the uncertainty. Thus we have decomposed our uncertain system into nominal system and uncertainty system. The uncertain system is the nominal system augmented by the uncertainty. Our purpose is to find a class of possible uncertainties which solve the observation problem. This class can be defined for example by a local constraint

\[
w(x, \tilde{x}, u) \leq 0 \tag{6}
\]

where \( w : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is assumed to be well defined at least on the region of interest. The relation (6) describes an uncertainty in the output injection \( k(\cdot) \) in a static form. Typically one may think of \( w \) as for instance being the time derivative of a Lyapunov function with respect to the (perturbed) error dynamics and where the Lyapunov function itself corresponds to the nominal system.

Let us clarify how the constraint (6) can describe an uncertainty in the output injection by a simple example. Suppose that subtracting (6) can be rewritten as follows

\[
\dot{e}(t) = F(x(t), \tilde{x}(t)) - u(t)
\]

where the dynamics \( F(x, \tilde{x}) \) corresponds to the nominal output injection and \( u(t) \in \mathbb{R}^n \) is the uncertainty. In the simplest case the uncertainty can be characterized by time-dependent perturbations of the nominal output injection:

\[
u(t) = \Delta(t)k_0(\tilde{x}(t), y(t))
\]

where \( \Delta(t) \) is the uncertainty matrix which satisfies \( |\Delta(t)| \leq C \) for all \( t \in \mathbb{R}_+ \), where \( |\cdot| \) denotes the standard induced matrix norm and \( C \) is some given constant. In this case the relation \( |\Delta(t)| \leq C \) can be rewritten as follows

\[
|u|^2 - C^2|k_0(\tilde{x}, y)|^2 \leq 0 \tag{7}
\]

which in fact is of the form (6). Notice that for any given \( C > 0 \) the constraint (7) defines a class of uncertain output injections of the form (5) and the problem is to prove that for given \( C > 0 \) all corresponding observers ensure synchronization. It is intuitively clear that this is the case not for all \( C \), but small enough and in the sequel we will present a method to derive bounds on the uncertainty affecting the system.

A more general description of uncertainty involves an integral constraint and allows for \( u(t) \) to depend dynamically on \( x(t) \) and \( \tilde{x}(t) \). Given the functions \( u : \mathbb{R}_+ \to \mathbb{R}^n \), \( x : \mathbb{R}_+ \to \mathbb{R}^n \), \( \tilde{x} : \mathbb{R}_+ \to \mathbb{R}^n \), \( w : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \). We say that the functions \( u, x, \tilde{x} \) satisfy the integral constraint

\[
\int_0^T w(x(t), \tilde{x}(t), u(t))dt \leq \gamma \tag{8}
\]
if there exist a positive number $\gamma$ and a sequence of time instants $\{T_j\}$, $T_j \geq 0$, $j = 1, 2, \ldots$ such that $T_j \to \infty$ as $j \to \infty$ and (8) holds for all $j$. It is clear that (6) is a particular case of (8). Indeed, (6) implies (8) for all numbers $T_j \geq 0$ and $\gamma \geq 0$.

Now we are prepared to give a definition of an observer with absolutely stable error dynamics.

**Definition 2.1.** The system (3) is called an observer for the system (1) and (2) with globally $L_p$-asymptotically absolutely stable dynamics with respect to the integral constraint (8) if for any initial conditions $x(0) \in \Omega$, $\tilde{x}(0) \in \mathbb{R}^n$ and for any admissible output injections satisfying (8) the corresponding solution $\tilde{x}(t)$ to (3) is unique and well defined on the infinite time interval $\mathbb{R}_+$ and there exist non-negative functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha(0) = 0$, $\beta(0) = 0$ such that the following inequality holds

$$
\int_0^\infty |e(t)|^p \, dt \leq \alpha(|e(0)|) + \beta(\gamma) \tag{9}
$$

and additionally

$$
\lim_{t \to \infty} e(t) = 0. \tag{10}
$$

If the initial mismatch $|x(0) - \tilde{x}(0)|$ is allowed to be small the observer (3) is referred to as a local observer and the corresponding synchronization problem is referred to as the local synchronization problem.

The constraint (8) describes the so-called unstructured uncertainty and it can be too conservative when there is additional information about the system uncertainty. To describe a structured uncertainty one can use a set of constraints of the form (8) for several functions $w_s$, $s = 1, \ldots, r$. We say that the functions $u, x, \tilde{x}$ satisfy the integral constraints

$$
\int_0^{T_i} w_s(x(t), \tilde{x}(t), u(t)) \, dt \leq \gamma_s, \quad s = 1, \ldots, r \tag{11}
$$

where $w_s : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $s = 1, \ldots, r$ are some functions if there exist positive numbers $\gamma_s$ and a sequence of time instants $\{T_j\}$, $T_j \geq 0$, $j = 1, 2, \ldots$ such that $T_j \to \infty$ as $j \to \infty$ and (11) holds for all $j$. In case of multiple constraints a definition of an observer with absolutely stable dynamics can be given similarly to Definition 2.1 with (9) replaced by

$$
\int_0^\infty |e(t)|^p \, dt \leq \alpha(|e(0)|) + \sum_{s=1}^r \beta_s(\gamma_s), \quad s = 1, \ldots, r \tag{12}
$$

In this case the analysis of robust stability is based on some extension of the so-called $S$-procedure Theorem. $S$-procedure theorems allow to convert the problem of absolute stability with multiple constraints into the problem of absolute stability with one constraint. The key idea is to consider a new integral constraint

$$
\int_0^{T_j} w_r(x(t), \tilde{x}(t), u(t)) \, dt \leq \gamma_r, \tag{13}
$$

where

$$
w_r(x, \tilde{x}, u) = \sum_{s=1}^r \tau_s w_s(x, \tilde{x}, u)$$

with $\tau_s \geq 0$, $s = 1, \ldots, r$, is a convex combination of the $w_s(\cdot)$, $s = 1, \ldots, r$. If it is possible to find a set of numbers $\tau_s$ that the system is absolutely stable with such composed constraint then it is absolutely stable with respect to the set of constraints (11). This assertion directly follows from the definition of absolute stability. The converse statement is also true under some regularity conditions. Namely, if the system is absolutely stable with respect to the set of constraints (11) then there are some $\tau_s \geq 0$, $\tau_1 + \ldots + \tau_r > 0$ such that the system is absolutely stable with respect to the constraint defined by $w_r$. Various modifications of this statement are called $S$-procedure theorems. These theorems allow to establish necessity of the conditions of absolute stability. As one can see the idea to compose a unique constraint $w_r$ is very close to the solution of the problem of constraint optimization and from this point of view one can interpret the numbers $\tau_1, \ldots, \tau_r$ as Lagrange multipliers. One of the most general versions of the $S$-procedure for $\mathcal{L}_2$-absolute stability for nonlinear systems can be found in [Savkin & Petersen, 1995], see also [Fradkov & Polushin, 1997] where a version of the $S$-procedure is presented for averaged functionals.

In the next section we will demonstrate how in a particular case the above setting becomes operational for robust synchronization of two Lorenz systems.

3. An Example: The Lorenz System

The well-known Lorenz system is given by the equations

$$
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= -xz + rx - y \\
\dot{z} &= xy - bz.
\end{align*}
$$

\[\text{(13)}\]
For suitable parameters $\sigma, r, b$ this system exhibits chaotic behavior. Suppose that the transmitter sends to the receiver the output signal $x$, that is

$$h(x, y, z) = x$$ \hfill (14)

The problem is to design an observer with absolutely stable error dynamics which asymptotically reconstructs the rest of the system state variables.

First, recall that all the solutions of the Lorenz system (13) are ultimately bounded and therefore well defined on $\mathbb{R}_+$ provided that $\sigma > 0$, $r > 0$, $b > 0$, i.e. $\Omega = \mathbb{R}^3$. Consider the following scalar function

$$W(x, y, z) = x^2 + y^2 + (z - \sigma - r)^2.$$ \hfill (15)

Its time derivative is given as

$$\dot{W}(x, y, z) = -2\sigma x^2 - 2y^2 - 2b \left( z - \frac{\sigma + r}{2} \right)^2 + b \frac{(\sigma + r)^2}{2}.$$  

It is seen that $\dot{W} = 0$ determines an ellipsoid outside which the derivative of $W$ is negative. If $K$ satisfies

$$K = \sqrt{\frac{1}{4} + \frac{b}{4} \max \left\{ \frac{1}{\sigma}, 1 \right\}},$$ \hfill (16)

then inside the sphere

$$x^2 + y^2 + (z - \sigma - r)^2 = K^2(\sigma + r)^2$$ \hfill (17)

we have $\dot{W} \leq 0$ and therefore all trajectories of the Lorenz system end up inside of the sphere (17).

We seek an observer for (13) and (14) in the following form

$$\begin{align*}
\dot{x} &= \sigma(y - x) + k_{01}(x, y, z, x) \\
\dot{y} &= -x \bar{z} + r \bar{x} - \bar{y} + k_{02}(x, y, z, x) \\
\dot{z} &= \bar{x} \bar{y} - b \bar{z} + k_{03}(x, y, z, x)
\end{align*}$$ \hfill (18)

where the functions $k_{0i}, i = 1, 2, 3$ are such that (18) is an observer for the system (13) and (14). Our problem is to specify the class of $k_{0i}$’s such that (18) yields $L_2$—absolutely stable error dynamics. This class can be specified for example by a relation in the form (6) which is a particular case of the constraint (8). Thus the problem is to find such a function $w(x, y, z, \bar{x}, \bar{y}, \bar{z}, u_1, u_2, u_3)$ where $u_1$, $u_2$, $u_3$ define the uncertainty in output injection so that $w \leq 0$ implies $e_x(t) \to 0$, $e_y(t) \to 0$, $e_z(t) \to 0$ as $t \to \infty$ and $e_x \in L_2(\mathbb{R}_+)$, $e_y \in L_2(\mathbb{R}_+)$, $e_z \in L_2(\mathbb{R}_+)$, where $e_x = x - \bar{x}, e_y = y - \bar{y}, e_z = z - \bar{z}$.

To solve the above problem rewrite the error dynamics in the following form

$$\begin{align*}
\dot{e}_x &= \sigma(e_y - e_x) - k_{01}(\bar{x}, \bar{y}, \bar{z}, x) \\
\dot{e}_y &= -xe_z - \bar{z}e_x + re_x - e_y - k_{02}(\bar{x}, \bar{y}, \bar{z}, x) \\
\dot{e}_z &= xe_y + \bar{y}e_x - be_x - k_{03}(\bar{x}, \bar{y}, \bar{z}, x)
\end{align*}$$ \hfill (19)

and consider the following quadratic radially unbounded scalar function

$$V(e_x, e_y, e_z) = \frac{1}{2} \left( \frac{1}{\sigma} e_x^2 + e_y^2 + e_z^2 \right).$$ \hfill (20)

Calculating the time derivative of (20) along (19) yields

$$\dot{V} = -k_{01}e_x - (k_{02} + \bar{z}e_x + (r + 1)e_x) e_y - (k_{03} - \bar{y}e_x) e_z - e_x^2 - e_y^2 - be_z^2.$$ \hfill (21)

Now it is obvious that if the functions $k_{01}, k_{02}, k_{03}$ satisfy the inequality $\dot{V} < 0$ for all nonzero $e_x, e_y, e_z$ then by virtue of the standard Lyapunov argument it follows that $e_x(t) \to 0$, $e_y(t) \to 0$, $e_z(t) \to 0$ as $t \to \infty$. Moreover, integrating the inequality $\dot{V} \leq 0$ immediately yields

$$\int_0^\infty |e(t)|^2 dt \leq 2 \min\{1, \sigma\} V(e(0))$$

with $e = (e_x, e_y, e_z)^T$, which is a special case of the inequality (9), or in other words $e \in L_2(\mathbb{R}_+)$. Now let us demonstrate how to design different observers for the Lorenz system.

**Case 1.** Notice that a particular receiver (18) is obtained by taking

$$\begin{align*}
k_{01}(\bar{x}, \bar{y}, \bar{z}, x) &= 0 \\
k_{02}(\bar{x}, \bar{y}, \bar{z}, x) &= -\bar{z}(x - \bar{x}) + (r + 1)(x - \bar{x}) \\
k_{03}(\bar{x}, \bar{y}, \bar{z}, x) &= \bar{y}(x - \bar{x})
\end{align*}$$ \hfill (22)

which transforms (21) into the following inequality

$$\dot{V} = -e_x^2 - e_y^2 - be_z^2$$

and therefore the receiver (22) asymptotically reconstructs the state variables of the transmitter.
Now consider the functions $k_{0i}, i = 1, 2, 3$ as the nominal output injections and suppose that the real output injections are uncertain:

$$
\begin{align*}
  k_1 &= k_{01}(\tilde{x}, \tilde{y}, \tilde{z}, x) + u_1(t) \\
  k_2 &= k_{02}(\tilde{x}, \tilde{y}, \tilde{z}, x) + u_2(t) \\
  k_3 &= k_{03}(\tilde{x}, \tilde{y}, \tilde{z}, x) + u_3(t)
\end{align*}
$$

where $u_1, u_2, u_3$ describes the perturbations of the nominal output injection.

Notice that the right-hand side of (21) is a quadratic function. Denoting

$$
w(e_x, e_y, e_z, \tilde{x}, \tilde{y}, \tilde{z}, u_1, u_2, u_3) = -(k_{01} + u_1)e_x - (k_{02} + u_2 + \tilde{z})e_x - (r + 1)e_y - (k_{03} + u_3 - \tilde{y})e_z - e_x^2 - e_y^2 - be_z^2 + (e_x e_y e_z)P(e_x e_y e_z)^T
$$

where $P = P^\top > 0$ is some positive definite matrix we obtain that the local constraint $w(e_x, e_y, e_z, \tilde{x}, \tilde{y}, \tilde{z}, u_1, u_2, u_3) \leq 0$ specifies the class of observers with $L_2$-absolutely stable error dynamics. Therefore the inequality $w \leq 0$ can be interpreted as a local constraint of the form (6) which determines a class of admissible uncertain output injections $k_1, k_2, k_3$ which ensure the existence of the observer (18) for the Lorenz system (13) and (14) with $L_2$-asymptotically absolutely stable error dynamics.

**Case 2.** A slight modification of the receiver (22) can be obtained as

$$
\begin{align*}
  k_{01}(\tilde{x}, \tilde{y}, \tilde{z}, x) &= 0 \\
  k_{02}(\tilde{x}, \tilde{y}, \tilde{z}, x) &= -\tilde{z}(x - \tilde{x}) + r(x - \tilde{x}) \\
  k_{03}(\tilde{x}, \tilde{y}, \tilde{z}, x) &= \tilde{y}(x - \tilde{x})
\end{align*}
$$

which results in the well-known Pecora–Carroll synchronizing scheme [Pecora & Carroll, 1990]. In this case one readily obtains

$$
\dot{V} = -e_x^2 + e_x e_y - e_y^2 - be_z^2 = -\left(e_x - \frac{1}{2} e_y\right)^2 - \frac{3}{4} e_y^2 - be_z^2
$$

Substituting (25) in (24) one can obtain the local constraint (6) on the admissible uncertainty $u_1, u_2, u_3$ in the form of perturbations of the nominal output injection resulting in the observer with absolutely stable error dynamics.

**Case 3.** Now consider the receiver

$$
\begin{align*}
  k_{01}(\tilde{x}, \tilde{y}, \tilde{z}, x) &= \lambda(x - \tilde{x}) \\
  k_{02}(\tilde{x}, \tilde{y}, \tilde{z}, x) &= 0 \\
  k_{03}(\tilde{x}, \tilde{y}, \tilde{z}, x) &= 0
\end{align*}
$$

After simple calculations one may obtain that

$$
\dot{V} = -(\lambda/\sigma + 1)e_x^2 - e_y^2 - be_z^2 - (z - (r + 1))e_x e_y - ye_x e_z
$$

and using (ultimate) boundedness of the variables $z, y$ of the transmitter it is not difficult to notice that there exists a number $\lambda$ such that the receiver (26) asymptotically reconstructs the state variables of the transmitter as soon as $\lambda \geq \tilde{\lambda}$. Indeed, since all trajectories end inside of sphere (17) one obtains that

$$
\lim_{t \to \infty} |z(t)| \leq (K + 1)(\sigma + r)
$$

and therefore it follows from (27) that as soon as $\lambda \geq \tilde{\lambda}$, where

$$
\tilde{\lambda} = \sigma \left(1 + r + (K + 1)(\sigma + r)^2 + K^2(\sigma + r)^2/b\right)
$$

and the value $K$ is from (16), $\dot{V}$ is eventually non-positive which implies asymptotic synchronization.

As before, substitution of (26) into (24) yields a local constraint (6) which describes admissible deviations of the nominal output injection.

### 4. Systems with Linearizable Error Dynamics

The design of the Lorenz synchronization of Sec. 3 is based on the choice of a suitable Lyapunov function. In this regard the proposed design is perhaps conservative in that other Lyapunov functions may give other (better!) results. At the same time a systematic procedure is known if the error dynamics can be described by the system which has a linear time-invariant part and a nonlinear one which depends only on measurable variables. In this case there exist constructive (frequency domain) necessary and sufficient conditions which ensure the existence of a quadratic form which can be taken as
a Lyapunov function [Yakubovich, 1962]. We con-
consider this case in the present section. The idea is to
consider systems that may give rise to linear time-
invariant error dynamics perhaps via an appropriate
change of coordinates \( \xi = \phi(x) \) and/or rescaling
of the output variables \( \eta = \psi(y) \) such that in the
new coordinates we have a system description of the form:
\[
\begin{align*}
\dot{\xi} &= A\xi + g(\eta, t) \\
\eta &= C\xi
\end{align*}
\] (28)
Assume the matrix pair \((A, C)\) is detectable that
means we can construct an observer for (28) in the
following form
\[
\begin{align*}
\dot{\tilde{\xi}} &= A\tilde{\xi} + g(\eta, t) + K(\eta - \tilde{\eta}) \\
\tilde{\eta} &= C\tilde{\xi}
\end{align*}
\] (29)
with \(K\) an appropriate gain matrix which can be
chosen such that the matrix \(A - KC\) is Hurwitz. In
this case the error dynamics is
\[
\dot{e} = (A - KC)e
\] (30)
where \(e = \xi - \tilde{\xi}\). Thus if all solutions of the system
(28) are well defined on the infinite time interval
then the system (29) is an observer with globally
asymptotically stable linear time-invariant dynam-
ics. Moreover, if the pair \((A, C)\) is observable, that
means the matrix \((C, CA, CA^2, \ldots, CA^{n-1})\) has a
full rank, then the gain \(K\) can be chosen such that
the matrix \(A - KC\) has desired eigenvalues and
therefore convergence of the error to zero can be
made arbitrary fast.

 Necessary and sufficient conditions under which
the original system can be transformed via coordi-
nate change and output rescaling to a form which
has an observer with linear error dynamics can be
found in [Tchon & Nijmeijer, 1993], see also
[Krener & Isidori, 1983; Krener & Respondek, 1985].

Now suppose that when implementing the output
injection in (29) the term \(K(\eta - \tilde{\eta})\) cannot be
realized exactly but with some uncertainty that gives
rise to the following error system
\[
\begin{align*}
\dot{e} &= A_e e + B q(z) \\
z &= C e
\end{align*}
\] (31)
where \(e(t) \in \mathbb{R}^n, z(t) \in \mathbb{R}, A_e = A - KC, B\) is an
\(n \times m\) matrix and the function \(q : \mathbb{R} \to \mathbb{R}^m\) ex-
presses an uncertainty in the output injection. We
will study a more general situation when the un-
certainty can be characterized by a local constraint
of quadratic type. To this end rewrite the system
equations in the following form
\[
\begin{align*}
\dot{e} &= A_e e + B u \\
z &= C e
\end{align*}
\] (32)
where \(u(t) \in \mathbb{R}^m\). Assume that the uncertainty can
be specified by a local constraint of the following form
\[
w(e, u) \leq 0
\] (33)
where the function \(w\) is a quadratic form of its ar-
gments:
\[
w(e, u) = e^T Ge + 2e^T Du + u^T \Gamma u
\] (34)
with matrices \(G = G^T, \Gamma = \Gamma^T\) and \(D\) of dimen-
sions \(n \times n, m \times m\) and \(n \times m\), respectively. For
example for scalar \(z\) and \(u\) the constraint (34) may
characterize a sectorial constraint of the following form
\[
w(e, u) = (\mu_1 z - u)(\mu_2 z - u) \leq 0.
\] (35)
This constraint arises when the relation between
input \(u\) and output \(z\) is given by some function
\(u = q(z)\) which satisfies the following inequalities
\[
\mu_1 \leq \frac{q(z)}{z} \leq \mu_2
\] (36)
which are in fact equivalent to the constraint (35).
Hence the problem of robust synchronization is to
find conditions which guarantee asymptotic stabil-
ity of the Lur’e system (31) for all possible Lipschitz
continuous functions satisfying (36).

Once the class of admissible constraints is de-
defined one can seek for the observer with absolutely
stable dynamics, i.e. for the class of observers which
can be defined by the quadratic form \(w\). To solve
this problem we have to find a Lyapunov function
which proves stability of the error dynamics for all
\(z\) and \(u\) which satisfy (33). We will seek this Lyap-
unov function as a quadratic form
\[
V(e) = \frac{1}{2} e^T P e
\]
where \(P = P^T\) is some symmetric matrix. Taking
the time derivative of \(V\) with respect to the sys-
tem (32) we come to the following problem: Find a
positive definite matrix \(P\) such that
\[
e^T P(A_e e + B u) \leq 0 \quad \text{with} \quad w(e, u) \leq 0.
\] (37)
As it was proved by Yakubovich [1971] this problem is equivalent to the following problem: Find a positive definite matrix $P$ such that

$$e^T P(A_c e + B u) \leq w(e, u) \quad (38)$$

The fact that (38) implies (37) is obvious. The converse statement can be proved applying the $S$-procedure which is always lossless (it means that (37) implies (38)) in case of two quadratic functions.

A solution to the problem (38) can be obtained by a seminal result due to Kalman and Yakubovich [Yakubovich, 1962; Kalman, 1963] which is usually referred to as the Kalman–Yakubovich lemma. Here we present this result in a form convenient to study asymptotic absolute stability.

Denote $\chi(s) = C(sI_n - A_c)^{-1}B$ as the transfer function of the system (32) and $w_H$ as the Hermitian extension of the quadratic form $w$:

$$w_H(z_r + iz_i, u_r + iu_i) := w(z_r, u_r) + w(z_i, u_i)$$

for real $z_r, z_i, u_r, u_i, i^2 = -1$.

**Theorem 4.1.** Let the pair $(A_c, B)$ be controllable, that is the matrix $(B, A_c B, A_c^2 B, \ldots, A_c^{n-1} B)$ has full rank. Then the following two statements are equivalent:

(i) There exists a matrix $P = P^T$ satisfying the inequality

$$e^T P(A_c e + B u) \leq w(e, u). \quad (39)$$

(ii) The frequency domain inequality

$$w_H(\chi(i\omega) u, u) \geq 0, \quad i^2 = -1 \quad (40)$$

holds for all $\omega \in \mathbb{R}$ such that det$(i\omega I_n - A_c) \neq 0$ and all complex-valued vectors $u \in \mathbb{C}^m$. If the matrix $A_c$ is Hurwitz, $G \geq 0$ and the pair $(A_c, D)$ is observable then any matrix $P$ satisfying (39) is positive definite: $P = P^T > 0$.

This theorem gives a solution for an uncertainty described by a unique local quadratic constraint. The case of absolute stability of linear systems under integral quadratic constraint is considered in [Yakubovich, 1973], the case of multiple integral constraints is considered in [Megretsky & Treil, 1993].

As an example we design an observer with linear time-invariant absolutely stable dynamics for the Rössler system:

$$\begin{cases}
\dot{x}_1 = -x_2 - x_3 \\
\dot{x}_2 = x_1 + ax_2 \\
\dot{x}_3 = c + x_3(x_1 - b)
\end{cases} \quad (41)$$

with output $y = x_3 = (0 \ 0 \ 1)x$. In the above (41) the coefficients $a, b, c > 0$. First one can notice that $x_3 = 0$ implies that $\dot{x}_3 = c > 0$. It means that whenever $x_3(0) \geq 0$ then $x_3(t) = y(t) > 0$ for all $t > 0$ (the graph of $x_3(t)$ cannot intersect the line $x_3 = 0$ since otherwise $\dot{x}_3$ should be nonpositive for $x_3 = 0$). Keeping this fact in mind we may use the comparison function $V = (x_1^2 + x_2^2 + x_3)/2 > 0$. Taking the time derivative of this function along the solutions of the Rössler system yields $\dot{V} = ax_2^2 + c - bx_3 \leq 2aV + c$ which implies that the solutions (with $x_3(0) \geq 0$) are well defined on the infinite time interval $\mathbb{R}_+$. The observer problem is hence well posed at least for $x_3(0) \geq 0$.

Suppose that $x_3(0) > 0$. We introduce the following change of coordinates

$$(\xi_1 \ \xi_2 \ \xi_3) = (x_1 \ x_2 \ ln \ x_3), \quad \eta = ln \ y$$

which is well defined since as we have proved $x_3(t) > 0$ for all $t > 0$.

In the new coordinates the system equations are given by:

$$\begin{pmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{\xi}_3
\end{pmatrix} = \begin{pmatrix}
0 & -1 & 0 \\
1 & a & 0 \\
1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix} + \begin{pmatrix}
en\eta \\
0 \\
-b + ce^{-\eta}
\end{pmatrix}
$$

which is a particular case of the system (28). In this case the linear part of (42) is observable and therefore one can design an observer for (42) in the form

$$\begin{pmatrix}
\tilde{\xi}_1 \\
\tilde{\xi}_2 \\
\tilde{\xi}_3
\end{pmatrix} = \begin{pmatrix}
0 & -1 & 0 \\
1 & a & 0 \\
1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\tilde{\xi}_1 \\
\tilde{\xi}_2 \\
\tilde{\xi}_3
\end{pmatrix} + \begin{pmatrix}
en\eta \\
0 \\
-b + ce^{-\eta}
\end{pmatrix} + \begin{pmatrix}
K_1 \\
K_2 \\
K_3
\end{pmatrix} (\eta - \tilde{\eta}) \quad (43)$$
where \( \tilde{\eta} = \tilde{\xi}_3 = (0 \ 0 \ 1) \tilde{\xi} \) and \( K_1, K_2, K_3 \) are the gain coefficients which can be chosen to provide the error system with arbitrarily desirable dynamics. Simple calculations show that if the factors \( K_1, K_2, K_3 \) are chosen to satisfy the inequalities

\[
-K_3^2 a + K_3(a^2 + K_1) + K_2 - a > 0
\]

\[
-K_1 a - K_2 + K_3 > 0
\]

then the error system is asymptotically stable and therefore the observer problem is solved.

Now suppose that when implementing two synchronizing Rössler systems the term \( K_3(\eta - \tilde{\eta}) \) can be realized only with some degree of uncertainty and the equations of the observer are given by

\[
\begin{pmatrix}
\dot{\tilde{\xi}}_1 \\
\dot{\tilde{\xi}}_2 \\
\dot{\tilde{\xi}}_3
\end{pmatrix} = \begin{pmatrix}
0 & -1 & 0 \\
1 & a & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{\xi}_1 \\
\tilde{\xi}_2 \\
\tilde{\xi}_3
\end{pmatrix} + \begin{pmatrix}
e^{-\eta} \\
0 \\
-b + ce^{-\eta}
\end{pmatrix}
\]

\[
+ \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} (\eta - \tilde{\eta}) + \begin{pmatrix} 0 \\ 0 \\ q(\eta - \tilde{\eta}) \end{pmatrix} \tag{44}
\]

where the nonlinear function \( q \) satisfies the sector constraint

\[
\mu_1 \leq \frac{q(e_3)}{e_3} \leq \mu_2 \tag{45}
\]

for some numbers \( \mu_1 < \mu_2 \). It is clear that the error system in this case is given by the equations

\[
\begin{pmatrix}
\dot{\tilde{\xi}}_1 \\
\dot{\tilde{\xi}}_2 \\
\dot{\tilde{\xi}}_3
\end{pmatrix} = \begin{pmatrix}
0 & -1 & -K_1 \\
1 & a & -K_2 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{\xi}_1 \\
\tilde{\xi}_2 \\
\tilde{\xi}_3
\end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ q(e_3) \end{pmatrix}
\]

\[
= Fe + [0 \ 0 \ 1] \tilde{q}(e_3) \tag{46}
\]

where the asterisk "\(^*\)" stands for complex conjugation. Then

\[
w_H(\chi(i\omega)u, u) = \text{Re}\{(\mu_1 \chi(i\omega) - 1)^*(\mu_2 \chi(i\omega) - 1)\}|u|^2
\]

where

\[
\chi(i\omega) = [0 \ 0 \ 1](i\omega I - F)^{-1}[0 \ 0 \ 1]^T
\]

\[
\frac{\det \begin{pmatrix} i\omega & 1 \\ -1 & i\omega - a \end{pmatrix}}{\det(i\omega I - F)^{-1}}
\]

\[
= \frac{(1 - \omega^2) - i \omega}{(a\omega^2 - K_2 - K_1 a) + i((K_1 + 1)\omega - \omega^3)}.
\]

The frequency-domain inequality hence is given by the following relation

\[
\text{Re}\{(\mu_1 \chi(i\omega) - 1)^*(\mu_2 \chi(i\omega) - 1)\} \geq 0. \tag{47}
\]

In the literature on automatic control this relation is usually referred to as the circle criterion because of its geometric interpretation. Indeed in the complex plane the boundary of the domain (47) is a circle which passes through the points \((\mu_1^{-1}, 0), (\mu_2^{-1}, 0)\) with the center on the real axis. For \( \mu_1 > 0, \mu_2 > 0 \) the inequality means that the graph of \( \chi(i\omega) \), \( \omega \in \mathbb{R} \), lies outside this circle.

To demonstrate the design procedure of observers with absolutely stable error dynamics for the Rössler system we carried out a computer simulation for the following parameters: \( a = 0.2, b = 5.7, c = 0.2 \). A typical trajectory of the Rössler system for such parameters is depicted in Fig. 1. Parameters of the observer (43) were chosen as \( K_1 = 0, K_2 = 0.8, K_3 = 1 \). A typical transient behavior of the error is shown in Fig. 2. Now suppose that the feedback \( q(e_3) \) is realized with uncertainty. If it were just a proportional feedback \( q(e_3) = K_3 e_3 \) then the Hurwitz criterion gives the following stability margins: \( 0.8 < K_3 < 1.8349 \). If the function \( q \) is assumed to lie in the sector (45) then the stability margins are significantly narrowed. Figure 3 shows the graph of \( \chi(i\omega) \) and the circle which corresponds to the bounds \( 1.3 - 0.305 \leq q(e_3)/e_3 \leq 1.3 + 0.305 \). The circle criterion asserts that any locally Lipschitz continuous functions satisfying these sector constraints ensure asymptotic stability of the error dynamics for arbitrary initial conditions for which
the solutions of the Rössler system are well defined on the infinite time interval. It should be noticed that this criterion gives sufficient but not necessary conditions. However if the feedback $q$ satisfies a more general integral constraint

$$
\int_0^{t_j} (\mu_1 e_3(t) - q(e_3(t)))(\mu_2 e_3(t) - q(e_3(t)))dt \leq \gamma
$$

for $\gamma > 0, t_j \to \infty$ as $j \to \infty$ then the bounds obtained by the frequency domain inequality are both necessary and sufficient [Yakubovich, 1973].

5. Conclusion

In this paper we have demonstrated an observer-based approach to the problem of robust synchronization of dynamical systems. We posed and solved the following problem: How to describe a set of admissible observers, given the transmitter dynamics. In the general situation the set of observers is described by a set of integral constraints, however in practical analysis one can use stronger local

It is seen that the sector of absolute stability lies strictly inside of the sector given by Hurwitz criterion and in case of local quadratic constraints stability margins derived from the circle criterion may be conservative. To obtain a less conservative margin one can use several results devoted to absolute stability. Here we will mention only one of them. In 1957, Kalman conjectured that if a nonlinearity is differentiable and lies in the Hurwitz sector together with its derivative then the system is globally asymptotically stable. This statement is known as the Kalman conjecture [Kalman, 1957]. It turned out that in general case this conjecture is not true [Barabanov, 1988]. However it is true for systems of the order not greater than 3: $n \leq 3$ [Barabanov, 1988]. In our case the error dynamics is described by autonomous system of the third order. Thus if for some differentiable nonlinearity we have

$$
0.8 < \frac{q(e_3)}{e_3} < 1.8349, \quad 0.8 < \frac{1}{e_3} \frac{dq(e_3)}{de_3} < 1.8349,
$$

then the error dynamics is globally asymptotically stable, that is, the goal of synchronization is achieved for all $x_1(0), x_2(0), \tilde{x}_1(0), \tilde{x}_2(0), \tilde{x}_3(0)$ and all $x_3(0) > 0$. 

Fig. 1. A typical trajectory of the Rössler system for $a = 0.2, b = 5.7, c = 0.2$.

Fig. 2. A typical transient behavior of $\eta_3(t) - \tilde{\eta}_3(t)$ for $K_1 = 0, K_2 = 0.8, K_3 = 1$.

Fig. 3. The graph of $\chi(i\omega)$ for $K_1 = 0, K_2 = 0.8$. 

Fig. 2. A typical trajectory of the Rössler system for $a = 0.2, b = 5.7, c = 0.2$. 

Fig. 3. The graph of $\chi(i\omega)$ for $K_1 = 0, K_2 = 0.8$. 

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constraints. It was shown that some known synchronization schemes can be considered within the proposed framework. We hope that the presented approach allows to develop synchronizing circuits for the problems of communications using chaotic signals.

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