Stokes flow in a rectangular cavity with a cylinder

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Abstract

The Stokes flow in a rectangular cavity with two opposite fixed and two opposite moving walls and a (rotating) cylinder in the centre is investigated. This flow is used as a prototype flow for studying distributive mixing. A general analytical method of superposition is implemented. The method is illustrated with several examples of steady flow and the topology of the streamlines is discussed. Some typical examples of laminar mixing under periodic wall motion are examined.

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1. Introduction

The aim of this paper is to study steady and periodic two-dimensional creeping flow of an incompressible viscous fluid inside a rectangular cavity with a circular cylinder placed in its centre. The flow is induced by tangential velocities at the cavity top and bottom walls as well as at the cylinder surface.

Hellou and Coutanceau (1992) addressed numerically and experimentally the problem for the cavity with fixed walls and rotating cylinder. They used the solution of the governing biharmonic equation for the stream function in polar coordinates together with matching of the arbitrary coefficients by a collocation method at the cavity boundary. These authors obtained accurate results, even for quite long cavities, with good agreement between experimental and calculated streamline patterns for corner and cellular eddies. Such eddies were previously studied by Moffatt (1964) for a wedge cavity and by Bourot (1984) for flow between parallel flat walls, inducted by a rotating cylinder in the centre. These results coincide qualitatively with numerical analysis of Lewis (1979),
where the problem of a square cavity with a cylinder was studied for non-zero Reynolds numbers using a finite difference method.

Robertson and Acrivos (1970) carried out an experimental study of a cylinder rotating with a prescribed angular velocity in a Stokes shear flow created by parallel walls moving in opposite directions with equal velocities. For the theoretical analysis a model by Bretherton (1962) of a cylinder in an infinite shear flow was applied. The authors classified five possible regimes of streamlines near the cylinder depending on the ratio of angular velocity of the cylinder and strength of the shear flow (analogous results were also described by Jeffrey and Sherwood, 1980). The influence of the fixed side walls on the structure of the flow was not discussed. Besides, the more general case of a cylinder placed arbitrary out of the centre of shear (or, equivalently, the velocities of the moving opposite walls, non-equal in the absolute value) was not considered.

On the other hand, experimental and numerical studies of periodic Stokes flows in a rectangular cavity (Leong and Ottino, 1989; Carey and Shen, 1995) or in an annulus between eccentric cylinders (Aref and Balachandar, 1986; Ottino, 1990) addressed mixing processes induced by a chaotic advection. A passive dyed blob was considered as a collection of a large number of individual points (markers). Because of exponential divergence of neighbouring points in chaotic regions, it was hardly possible to accurately reconstruct the shape of the blob even after only a few periods.

In the present paper we develop an accurate analytical method for determining the velocity field inside the cavity and an algorithm of line tracking which provides the determination of the position and the shape of a passive dyed blob at any moment of time. Some typical streamline patterns created in the cavity due to the presence of the obstacle are analysed and examples of mixing are discussed.

The paper is organized as follows: the formulation of the problem, the analytical method for its solution and the algorithm for contour line tracking are described in Section 2. The test results of the computational accuracy, the analysis of typical steady streamline patterns as well as a comparison with the flow visualization, and results of mixing analysis are presented in Section 3. The conclusions are given in Section 4. Appendix A provides a short description of the experimental setup.

2. Problem formulation and analytical solution

2.1. Statement of the problem

Consider a two-dimensional creeping flow of an incompressible viscous fluid in a rectangular cavity \(|x| \leq a, \ |y| \leq b\) with a circular cylinder of radius \(R\) located in its centre (Fig. 1). The geometry of the cavity can be characterized by two dimensionless parameters \(h = a/b\) and \(R_0 = R/b\). The fluid motion is produced by the uniform tangential velocities \(V_{\text{top}}\) and \(V_{\text{bot}}\) applied at the top \((y = b)\) and the bottom \((y = -b)\) walls, respectively, and the uniform rotation of the cylinder with the circumferential velocity \(U\). The side walls \(x = \pm a\) are unmovable.

For an incompressible fluid the components \(u_x, u_y\) of the velocity field in Cartesian coordinates \((x, y)\) and the components \(u_r, u_\theta\) in the polar coordinates \((r, \theta)\) can be expressed by means of stream function \(\Psi\) as

\[
  u_x = \frac{\partial \Psi}{\partial y}, \quad u_y = -\frac{\partial \Psi}{\partial x} \tag{1}
\]
Fig. 1. A schematic presentation of the flow region under study. The cavity length is 2a, its width is 2b and the radius of the cylinder is R.

and

\[ u_r = \frac{1}{r} \frac{\partial \Psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \Psi}{\partial r}, \]  

respectively.

For slow motion (Stokes approximation) the inertial terms may be neglected and the stream function \( \Psi \) satisfies the biharmonic equation

\[ \nabla^2 \nabla^2 \Psi = 0, \]  

where \( \nabla^2 \) stands for the Laplace operator.

The no-slip conditions

\begin{align*}
    u_x &= V_{\text{top}}, \quad u_y = 0, \quad \text{at } y = b, \quad |x| \leq a, \\
    u_x &= V_{\text{bot}}, \quad u_y = 0, \quad \text{at } y = -b, \quad |x| \leq a, \\
    u_x &= 0, \quad u_y = 0, \quad \text{at } x = \pm a, \quad |y| \leq b, \\
    u_r &= 0, \quad u_\theta = U, \quad \text{at } r = R, \quad 0 \leq \theta \leq 2\pi
\end{align*}  

are prescribed at the whole boundary.

The linearity of the boundary problem (3), (4) permits to seek for the solution as a sum

\[ \Psi = V_A \Psi_A + V_B \Psi_B + U \Psi_C, \]  

where

\[ V_A = \frac{V_{\text{top}} - V_{\text{bot}}}{2}, \quad V_B = \frac{V_{\text{top}} + V_{\text{bot}}}{2}, \]  

and the stream functions \( \Psi_A, \Psi_B \) and \( \Psi_C \) are the solutions of three basic boundary problems: \( A \) with \( V_{\text{top}} = 1, \ V_{\text{bot}} = -1, \ U = 0 \), \( B \) with \( V_{\text{top}} = 1, \ V_{\text{bot}} = 1, \ U = 0 \) and \( C \) with \( V_{\text{top}} = 0, \ V_{\text{bot}} = 0, \ U = 1 \).
Fig. 2. Representation of the solution of boundary problem with arbitrary velocities $V_{\text{top}}$, $V_{\text{bot}}$ and $U$ as linear combination of the solutions of three stationary boundary problems $A$, $B$ and $C$. To show the symmetry properties of the velocity field, described by problems $A$, $B$ and $C$, velocity components are depicted at some points: $(x, y)$, $(x, -y)$ and $(-x, y)$.

It can be seen (Fig. 2) that the stream functions $\Psi_A$ and $\Psi_C$ are even on both coordinates $x$ and $y$, while the streamfunction $\Psi_B$ is even on $x$ and odd on $y$. Generally speaking, problems $A$ and $C$ can be combined into one, providing the traditional splitting of the initial problem into symmetric and antisymmetric parts. However, separate consideration of these three problems gives the possibility to consider any (including time dependent) boundary condition (4) when solving problems $A$, $B$ and $C$ only once for given values of the geometrical parameters $h$ and $R_0$.

2.2. Solutions of the basic biharmonic problems

The solutions of problems $A$, $B$ and $C$ can be constructed analytically by the method of superposition, see Meleshko (1996) for details. The main idea of this method consists in the representation of the stream function as a sum of three ordinary Fourier series on the complete systems of trigonometric functions on the coordinates $x$, $y$ and $\theta$. All these series satisfy identically the biharmonic equation (3) inside the cavity and have sufficient functional arbitrariness to fulfill the two boundary conditions at some part of the boundary. Because of the interdependency, the expression for the Fourier coefficient in one series will depend on all coefficients of the other two series. Thus, the final solution involves the solution of an infinite system of linear algebraic equations giving the relations between all the coefficients. We will consider in detail only the construction of the solution for problem $A$. The stream function $\Psi_B$ and $\Psi_C$ can be obtained in a similar manner.

Let us represent the biharmonic stream function $\Psi_A$ in the form of a sum:

$$\Psi_A = \psi^{(\text{rec})} + \psi^{(\text{cyl})},$$

(7)
where

\[
\psi^{(sec)} = \frac{b}{a} \sum_{m=1}^{\infty} (-1)^m X_m \left( \frac{\sinh m b}{\cosh m b} + \frac{y \sinh m y}{b \cosh m b} \right) \cos m x - a \sum_{j=1}^{\infty} \frac{Y_j}{\beta_j} \left( \frac{\sinh \beta_j x}{\cosh \beta_j a} - \frac{x \sinh \beta_j y}{a \cosh \beta_j a} \right) \sin \beta_j y + \sum_{m=1}^{\infty} f_m \frac{\sin m y}{m} \cos m x + \sum_{j=1}^{\infty} g_j \frac{\sin \beta_j x}{\beta_j \cosh \beta_j a} \sin \beta_j y
\]

with

\[
x_m = \frac{2m - 1}{2a} \pi, \quad \beta_j = \frac{2j - 1}{2b} \pi
\]

and

\[
\psi^{(cyl)} = E_0 R \ln \left( \frac{r}{R} \right) + R \sum_{j=1}^{\infty} E_j \left[ \left( \frac{R}{r} \right)^{2j-2} - \left( \frac{R}{r} \right)^{2j} \right] \cos 2j \theta + R \sum_{j=1}^{\infty} \frac{w_j}{2j} \left( \frac{R}{r} \right)^{2j} \cos 2j \theta
\]

with yet to be specified coefficients \( X_m, Y_j, f_m, g_j, E_j, w_j \). Here we chose in the general representation of the biharmonic function in the polar coordinates (Michell, 1899) terms which are decreasing or minimal increasing with \( r \).

Using the relations

\[
\frac{\cos 2j \theta}{r^{2j}} = xC_j(x, y) + yS_j(x, y), \quad \frac{\cos 2j \theta}{r^{2j-2}} = (x^2 + y^2)[xC_j(x, y) + yS_j(x, y)]
\]

with

\[
C_j(x, y) = \frac{\cos(2j + 1) \theta}{r^{2j+1}} = \frac{1}{(2j)!} \int_0^\infty \lambda^{2j} e^{-\lambda x} \cos \lambda y \, d\lambda = \frac{(-1)^j}{(2j)!} \int_0^\infty \lambda^{2j} e^{-\lambda y} \sin \lambda x \, d\lambda,
\]

\[
S_j(x, y) = \frac{\sin(2j + 1) \theta}{r^{2j+1}} = \frac{1}{(2j)!} \int_0^\infty \lambda^{2j} e^{-\lambda x} \sin \lambda y \, d\lambda = \frac{(-1)^j}{(2j)!} \int_0^\infty \lambda^{2j} e^{-\lambda y} \cos \lambda x \, d\lambda,
\]

the stream function \( \psi^{(cyl)} \) can be readily expressed in Cartesian coordinates. By means of Eq. (1) we can calculate the components of the velocity \( u^{(cyl)}_x(x, y) \) and \( u^{(cyl)}_y(x, y) \) of the “cylindrical” part in the general solution (7). At the rectangle’s sides these velocities define the following smooth functions \( f(x) = -u^{(cyl)}_x(x, b) \) and \( g(y) = u^{(cyl)}_y(a, y) \). If we choose now the arbitrary sets of coefficients \( f_m \) and \( g_j \) in Eq. (8) as coefficients of Fourier expansions of the functions \( f(x) \) and \( g(y) \)

\[
f(x) = \sum_{m=1}^{\infty} f_m \sin m x, \quad g(y) = \sum_{j=1}^{\infty} g_j \sin \beta_j y,
\]

then the normal component of the velocity at the rectangle’s sides is identically equal to zero.
In the same way, by using the expansions
\[
\cosh \alpha y \cos \alpha x = 1 + \sum_{j=1}^{\infty} (-1)^j \frac{(\alpha r)^{2j}}{(2j)!} \cos 2j\theta,
\]
\[
2\alpha y \sinh \alpha y \cos \alpha x = (\alpha r)^2 + \sum_{j=1}^{\infty} (-1)^j \left[ \frac{(\alpha r)^{2j+2}}{(2j+1)!} + \frac{(\alpha r)^{2j}}{(2j-1)!} \right] \cos 2j\theta,
\]
\[
\cosh \beta x \cos \beta y = 1 + \sum_{j=1}^{\infty} \frac{(\beta r)^{2j}}{(2j)!} \cos 2j\theta,
\]
\[
2\beta x \sinh \beta x \cos \beta y = (\beta r)^2 + \sum_{j=1}^{\infty} \left[ \frac{(\beta r)^{2j+2}}{(2j+1)!} + \frac{(\beta r)^{2j}}{(2j-1)!} \right] \cos 2j\theta,
\]
we can express the function \( \Psi^{(\text{rec})} \) in polar coordinates and calculate by Eq. (2) the components \( u^{(\text{rec})}_r(r, \theta) \) and \( u^{(\text{rec})}_\theta(r, \theta) \) of the velocity of the “rectangular” part of the general solution (7). At the boundary \( r = R \) the component \( u^{(\text{rec})}_r \) of the velocity define the smooth function \( w(\theta) = u^{(\text{rec})}_r(R, \theta) \). Again, if we choose the set of coefficients \( w_j \) in (9) as Fourier coefficients of the function \( w(\theta) \),
\[
w(\theta) = \sum_{j=1}^{\infty} w_j \sin 2j\theta, \quad w_j = \frac{1}{2\pi} \int_0^{2\pi} w(\theta) \sin 2j\theta \, d\theta,
\]
then the normal component of the velocity at the surface of the cylinder is identically equal to zero.

The boundary conditions for the tangential components of the velocity at the surface of both, the rectangle and the cylinder, lead to the infinite systems
\[
X_m A(x_m b) - \sum_{l=1}^{\infty} Y_l \frac{4\beta l x_m^2}{b(b^2 + x_m^2)^2} = P_m, \quad 1 \leq m \leq \infty,
\]
\[
Y_l A(\beta l a) - \sum_{m=1}^{\infty} X_m \frac{4\alpha m \beta l^2}{a(a^2 + \beta l^2)^2} = Q_l, \quad 1 \leq l \leq \infty
\]
and
\[
E_0 = \frac{1}{2} s_0, \quad E_j = s_j, \quad 1 \leq j \leq \infty,
\]
respectively, for defining the coefficients \( X_m, Y_l, E_j \). The following notations:
\[
A(\xi) = \tanh \xi + \frac{\xi}{\cosh^2 \xi},
\]
\[
P_m = 2 - \alpha m a(u_m - f_m \tanh x_m b)(-1)^m - 2x_m^2 \sum_{l=1}^{\infty} (-1)^l \frac{g_l}{\beta l^2 + x_m^2},
\]
\[
Q_l = \beta l b(v_l - g_l \tanh \beta l a)(-1)^l - 2\beta l^2 \sum_{m=1}^{\infty} (-1)^m f_m \frac{a}{x_m^2 + \beta l^2}
\]
were introduced. The sets of coefficients $u_m, v_l$ and $s_0, s_j$ represent the Fourier coefficients of the smooth functions $u(x) = -u_{x}^{(cyl)}(x, b), v(y) = u_{y}^{(cyl)}(a, y)$ and $s(\theta) = u_{0}^{(rec)}(R, \theta)$:

$$
u(x) = \sum_{m=1}^{\infty} u_m \cos \alpha_m x, \quad v(y) = \sum_{l=1}^{\infty} v_l \cos \beta_l y, \quad s(\theta) = \frac{s_0}{2} + \sum_{j=1}^{\infty} s_j \cos 2j\theta. \quad (14)$$

The explicit expressions for the sets of coefficients $u_m, v_l, f_m, g_l$ through the unknowns $E_0, E_j$, as well as the set of coefficients $w_j, s_j$ through the unknowns $X_m, Y_l$ can be obtained by straightforward algebra.

The structure of the infinite system (12), (13) is clear. The first part, Eqs. (12), is (mainly) responsible for the flow field in the “pure” rectangular cavity for nonuniform velocities of the walls. This nonuniform distribution is induced by the presence of the cylinder inside the cavity. The second part, Eqs. (13), (mainly) defines the flow field outside the cylinder created by the nonuniform velocity distribution on its boundary. This nonuniformity reflects the presence of the rectangular boundary of the cavity.

Using the equalities

$2\alpha_m \sum_{l=1}^{\infty} (-1)^l \frac{g_l}{\beta_l^2 + \alpha_m^2} = -\int_{-b}^{b} g(y) \frac{\sinh \alpha_m y}{\cosh \alpha_m b} dy$,

$2\beta_l \sum_{m=1}^{\infty} (-1)^m \frac{f_m}{\alpha_m^2 + \beta_l^2} = -\int_{-a}^{a} f(x) \frac{\sinh \beta_l x}{\cosh \beta_l a} dx$.

and by standard procedure of integration by parts, it can be shown that

$$P_m = 2 - 2 \frac{Z}{\alpha_m} \tanh \alpha_m b + O(\alpha_m^{-2}), \quad m \to \infty,$$

$$Q_l = 2 \frac{Z}{\beta_l} \tanh \beta_l a + O(\beta_l^{-2}), \quad l \to \infty$$

with $Z = f'(a) + g'(b)$. Here the relations $u(a) + g(b) = 0, v(b) + f(a) = 0$ were used. Therefore, following the general analysis of an infinite system like Eq. (12), developed by Meleshko and Gomilko (1997), we may write

$$X_m = \frac{2\pi^2}{\pi^2 - 4} - \frac{Z}{\alpha_m} + x_m, \quad Y_l = \frac{4\pi}{\pi^2 - 4} + \frac{Z}{\beta_l} + y_l$$

with

$$x_m = \text{Re} \left( \frac{D_\lambda}{\alpha_m^\lambda} \right) + o(\alpha_m^{-2\lambda}), \quad m \to \infty, \quad y_l = -\text{Re} \left( \frac{D_\lambda}{\beta_l^\lambda} \right) + o(\beta_l^{-2\lambda}), \quad l \to \infty,$$

where $D_\lambda$ is a complex constant, and $\lambda \approx 2.739593 + i1.119025$ is the root of the transcendental equation $\sin(\pi \lambda/2) + \lambda = 0$ with the lowest positive real part.

The infinite system (12), (13) expressed in terms of the unknowns $x_m$, $y_l$ can be treated by the simple reduction method, i.e., by putting

$$x_m = 0, \quad m > M, \quad y_l = 0, \quad l > L, \quad E_j = 0, \quad j > J.$$
and solving the finite system of $M + L + J + 1$ equations for the unknowns $x_m$ ($1 \leq m \leq M$), $y_l$ ($1 \leq l \leq L$), $E_j$ ($1 \leq j \leq J$) and $E_0$. The approximate value of the complex constant $D_\lambda$ can be found from the equations

$$\Re \left( \frac{D_\lambda}{\zeta^\lambda} \right) = x_M, \quad \Re \left( \frac{D_\lambda}{\beta_\lambda^\lambda} \right) = -y_L. \quad (18)$$

The presence of the nonzero constants in the asymptotics of $X_m$ and $Y_l$ for problem $A$ explains the difficulties of the simple reduction method when solving the original infinite system (12), (13).

Subsequent transformations of $\Psi^{\text{rec}}$, in order to obtain rapidly convergent Fourier series for the velocity field everywhere including the rectangle’s boundary, are similar to those presented in Meleshko (1996).

2.3. Local behaviour of the stream function

Expanding the stream functions $\Psi^{\text{cyl}}$ and $\Psi^{\text{rec}}$ into Taylor series on $a - x$ and $b - y$ at the corner point $(a, b)$ and putting $x = a - \rho \cos \chi$, $y = b - \rho \sin \chi$, $0 \leq \chi \leq \frac{\pi}{2}$, after some transformations, we finally obtain

$$\Psi_A(\rho, \chi) = \rho \frac{4}{\pi^3 - 4} \left[ \chi \cos \chi + \left( \frac{\pi}{2} - \frac{\chi}{2} \right) \sin \chi \right]$$

$$+ \Re \left( \rho^{1+1/2} \beta_\lambda \left[ \sin \left( \frac{\pi}{2} - \frac{\chi}{2} \right) \right] \sin \left( \frac{\chi}{2} \right) \right) + O(\rho^4). \quad (19)$$

The term linear in $\rho$ in Eq. (19) represents the well-known solution by Goodier (1934) and Taylor (1962) for the Stokes flow in a wedge under the unit tangential velocity along the side $\chi = 0$. The second term in Eq. (19), proportional to $\rho^{1+1/2}$, represents an infinite sequence of the Moffatt eddies near the wedge apex.

2.4. Separation and stagnation points in the flow

The positions of separation points at the unmovable boundaries, where a streamline splits into two parts, and stagnation points in the interior of the cavity, where the velocity is equal to zero, are important for studying the topology of the flow.

Expanding, for example, the stream function $\Psi$ near the separation point $(R, \theta_c)$ at the cylinder we can obtain the following equations:

$$\frac{\partial \tilde{u}_0}{\partial r} \bigg|_{(R, \theta_c)} = 0, \quad \tan \chi_c = \frac{\partial^2 \tilde{u}_0}{\partial r^2} \bigg|_{(R, \theta_c)} / \left( \frac{\partial^2 u_c}{\partial r^2} \bigg|_{(R, \theta_c)} \right), \quad (20)$$

which determine both the positions $\theta_c$ of the separation point and the angle of inclination $\chi_c$ of the dividing streamline to the cylinder.

A full search of stagnation points in the cavity for any value of the top, bottom, and cylinder velocities can be done by analysing the contour levels of the stream function and by looking for its local extremum. Besides, by symmetry conditions, the position of a stagnation point $(0, y_n)$ on the
y-axis can be found by solving the equation \( u_x(0, y_{st}) = 0 \), because the component \( u_x \) is equal to zero on this line. For the problems \( A \) and \( C \) (or, for their arbitrary linear combinations) there can also exist stagnation points \((\pm x_{st}, 0)\) on the x-axis, which are defined from the equation \( u_y(x_{st}, 0) = 0 \).

The stagnation points can be divided (see Perry and Chong, 1987, for details) into elliptic stagnation points, where \( \Psi \) has an extremum value (maximum or minimum) and nearby streamlines surrounding it have the form of ellipses, or hyperbolic (saddle) stagnation points, where \( \Psi \) has a minimax value and nearby streamlines form two sets of hyperbolae which are locally not closed.

2.5. Adaptive contour line tracking

Although the velocity field is calculated for all situations, we restrict our consideration of a distributive mixing of a passive dye in the cavity to the cases of a steady motion of the top wall \( (V_{\text{top}} = V, \ V_{\text{bot}} = 0, \ U = 0) \) and a periodic discontinuous co-rotational wall motion with period \( T \) and a fixed cylinder:

\[
V_{\text{bot}}(t) = V, \ V_{\text{top}}(t) = 0, \quad \text{if} \quad kT < t \leq (k + \frac{1}{2})T, \quad U = 0
\]

\[
V_{\text{bot}}(t) = 0, \ V_{\text{top}}(t) = -V, \quad \text{if} \quad (k + \frac{1}{2})T < t \leq (k + 1)T,
\]

with \( k = 0, 1, 2, \ldots \). In the latter case the dimensionless parameter

\[
D = \frac{VT}{2a}
\]

characterises the mixing protocol. It is equal to the total dimensionless displacements of the walls for one period and may serve as a measure of the work done during mixing. It should be stressed that an infinite amount of work should be applied to produce discontinuous velocities at the corner points. Smoothing of the velocity near the corner leads to a finite value of the work done; the velocity field inside the cavity does not considerably change.

The system of ordinary differential equations

\[
\frac{dx}{dt} = \frac{\partial \Psi}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial \Psi}{\partial x},
\]

with the initial conditions \( x = x_{in}, y = y_{in} \) at \( t = 0 \) describes the motion of an individual (Lagrangian) particle which occupies the position \((x, y)\) at time \( t \) in the known Eulerian velocity field given by the stream function \( \Psi \). Numerical integration of the system (23) was performed by using the Runge–Kutta scheme with adaptive stepsize (Press et al., 1992).

Any algorithm of contour line tracking comes down to tracking of points distributed along the initial blob boundary and, after this point tracking, connecting neighbouring points. Being obviously easy for the initial boundary, the general problem how to connect those points by smooth curves, without intersections of these curves, to get the boundary at any instant, provides some problems. Because of nonuniform stretching and folding of the line, two neighbouring points may appear far away from each other at some future time, and to maintain smoothness during the flow evolution can be difficult. The obvious way to overcome this problem is to increase the number of points. In
order to avoid much computational effort, this should not be done uniformly — but only at those parts of the initial line where considerable stretching or folding occurs.

The essence of the algorithm employed is

1. Divide the time interval into small time steps \( \Delta t \). Start at \( t_0 = 0 \) with small number of points \( N_0 \) uniformly distributed along the initial contour line. Solve system (23) for each point and trace the positions of all these points up to the moment \( t_1 = \Delta t \).

2. Calculate the distance \( \Delta l_n \) between points of numbers \( n \) and \( n + 1 \). If it appears that some distance \( \Delta l_k \) becomes larger than some initially prescribed value \( l_{dis} \), insert an additional point on the initial contour in the middle between points \( k \) and \( k + 1 \), solve the system (23) for that one point, and renumber correspondingly the initial and final arrays of points. After completing this operation we have \( N_d \geq N_0 \) points with distances between each of two neighbors less than \( l_{dis} \). Connect these points by straight lines and form a \( N_d \)-polygon.

3. Take in any turn three points \( m - 1, m, m + 1 \) \((m = 2, 3, \ldots, N_d + 1)\) and find the angle \( \gamma_m \) at the vertex \( m \) (computationally, it is preferable to calculate only the cosine of this angle from known distances between these points). If the angle \( \gamma_m \) appears to be smaller than some prescribed value \( \gamma \), insert (if necessary) additional points at the initial contour line between points \( m - 1, m, m + 1 \) in such a way that, finally, the distances between all “old” and “new” points not exceed the value \( l_{cur} \) or the angles in the polygon are larger than \( \gamma \).

The next step is now obvious: proceeding to the instant \( t_2 = 2 \Delta t \) we move firstly the \( N_1 \)-vertex polygon and then apply the same algorithm again. In presented computations values of the parameters were: \( l_{dis} = 0.02b \), \( l_{cur} = 0.005b \), \( \gamma = 120^\circ \).

A check on the proposed algorithm is the accuracy of fulfilling the area conservation condition. The area enclosed by the deformed curve was calculated using the Stokes theorem as a line integral, which for a \( N \)-polygonal form of the closed curve provides the simple expression

\[
s = \frac{1}{2} \sum_{i=1}^{N} (x_i y_{i+1} - y_i x_{i+1}), \quad \text{where } x_{N+1} = x_1, y_{N+1} = y_1.
\]  

(24)

2.6. Periodic points and Poincaré mapping

Special points of interest in the periodic flow induced by the wall’s motion (21) are periodic points \((x^*, y^*), (x'^*, y'^*), \ldots \) of period-1, -2, …, that return to their initial positions after time \( T, 2T, \ldots \), respectively. The algorithm of the searching and classification of the periodic points is similar to that outlined by Meleshko and Peters (1996). It is based upon one-dimensional search of points which, for example, being located at the moment \( T/4 \) on \( y \)-axis at the moment \( 3T/4 \) cross again the same axis. It should be noted that in this case, because of the obstacle inside the cavity, not all periodic points (even of the period-1) can be found by this algorithm. A more general approach of dividing the whole area into small squares, tracking for time \( T, 2T, \ldots \) of the boundary of each of them and subsequent mapping onto the initial one indicates those squares which can contain periodic points (details to be addressed in Anderson et al., 1998). Subsequent determination of the periodic points inside the square can be done by an ordinary Newton method. The symmetry algebra technique developed by Ottino (1991) can provide several additional periodic points if one manages to find one periodic point in the flow.
The Poincaré map for the periodic mixing protocol (21) was constructed by choosing one arbitrary point (which, supposedly, lies in the chaotic region) inside the cavity and plotting its position after each period, and continuing this for a large number of periods. Such a map reveals both elliptic islands surrounding the elliptic periodic points and the chaotic “sea” related to the hyperbolic periodic points in the flow.

3. Results and discussion

3.1. Accuracy of calculations

The advantage of the method of superposition, in spite of the rather tedious algebra involved, is the accuracy of calculations with small number of the terms in Fourier series. Test results for the velocities for problem A (Tables 1–3) show both the excellent satisfaction of all boundary conditions (also at the corner points) as well as stable values of the velocity components in some inner points inside the cavity.

The similar accuracy of calculations is obtained for the problems B and C. For the problem C when the walls of the rectangle are fixed and the flow is induced by the steady rotating cylinder Hellou and Coutanceau (1992) showed the accuracy up to $10^{-7}$ of satisfying of zero velocities at the rectangle’s walls by using only the general solution in polar coordinates like Eq. (9), whereas the boundary conditions on the cylinder are fulfilled identically. The authors used 30 terms in the

\begin{table}[h]
\centering
\begin{tabular}{ccc|cccc}
\hline
M & L & J & \multicolumn{4}{c}{x coordinate of point on the top wall $y = b$} \\
\hline
& & & 0.0 & 0.3a & 0.6a & 0.9a & a \\
20 & 12 & 10 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 0.9999 \\
10 & 6 & 6 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 0.9998 \\
5 & 3 & 3 & 1.0001 & 1.0000 & 0.9999 & 0.9998 & 0.9996 \\
\hline
\end{tabular}
\caption{Dependence of the value $u_x$ on the top wall on $x$ coordinate and number of unknowns $M$, $L$, $J$ for the problem $A$, $h = 1.67$, $R_0 = 0.3$}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{ccc|cccc}
\hline
M & L & J & Angular coordinate $\theta$ of the point $(R_0, \theta)$ \\
\hline
& & & 0 & $\pi/4$ & $\pi/2$ \\
& & & $u_\theta$ & $u_\rho$ & $u_\theta$ & $u_\rho$ & $u_\theta$ & $u_\rho$ \\
20 & 12 & 10 & $-9.0 \times 10^{-9}$ & $-9.8 \times 10^{-9}$ & $-5.4 \times 10^{-9}$ & $8.4 \times 10^{-16}$ & $-1.5 \times 10^{-8}$ \\
10 & 6 & 6 & $-1.1 \times 10^{-8}$ & $-9.8 \times 10^{-9}$ & $-5.5 \times 10^{-9}$ & $6.1 \times 10^{-15}$ & $-1.5 \times 10^{-8}$ \\
5 & 3 & 3 & $3.2 \times 10^{-5}$ & $1.0 \times 10^{-4}$ & $3.8 \times 10^{-4}$ & $3.5 \times 10^{-8}$ & $-6.5 \times 10^{-4}$ \\
\hline
\end{tabular}
\caption{Dependence of the values $u_\rho$ and $u_\theta$ on surface of a fixed cylinder on the number of unknowns. The value of $u_\rho$ for $\theta = 0$ is omitted because it is exactly zero in all cases. Problem $A$, $h = 1.67$, $R_0 = 0.3$}
\end{table}
Table 3
Dependence of the calculated values $u_x$ and $u_y$ at some test points inside the cavity on the number of unknowns. Problem $A$, $h = 1.67$, $R_0 = 0.3$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$L$</th>
<th>$J$</th>
<th>Coordinates of point $(x, y)$ inside the cavity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(0.6a, 0)$</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>10</td>
<td>$0.0$</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>6</td>
<td>$0.0$</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td>$0.0$</td>
</tr>
</tbody>
</table>

Fourier series and defined 61 coefficients by means of least square procedure at 181 minimization points at the rectangle’s boundary. Details of their error estimation remain, however, unclear.

In our case a satisfactory accuracy for the analysis of the flow structure (although higher accuracy is preferable for mixing simulations) was achieved with $M = 5$, $L = 3$, $J = 3$ in Eq. (17), that is with only 12 unknown coefficients to find (see Tables 1–3).

3.2. Steady streamline patterns

We examined the flow regimes for some typical values of the parameters $h$ and $R_0$ and combinations of the boundary velocities $V_{\text{top}}$, $V_{\text{bot}}$ and $U$.

Let us start with problem $A$ when the flow in the cavity is generated by moving the top and bottom wall in opposite directions with the same velocity and a fixed cylinder. The structure of the flow strongly depends on the aspect ratio $h$. In a square cavity, $h = 1$, two eddy zones, attached to the cylinder, are observed (Fig. 3a). The total vertical size of these eddies and the cylinder, which can be treated as the “effective size” of an obstacle, only slightly depends on the radius of the cylinder (Fig. 3b). As the aspect ratio of the cavity increases, the eddy zones rapidly become more “flat” and disappear when $h = 1.12$. Then, within a wide range of the $h$, the flow has a simple structure without eddies. The flow near the corner points exhibits remarkable similarities with the above mentioned Goodier–Taylor solution: the difference between exact and approximate values of the stream function is less than ten percents even for $\rho = 0.75b$ for the linear term on $\rho$ in Eq. (19). If we continue to elongate the cavity, a new pair of eddies is present on the left and right hand side of the cylinder when $h > 1.95$ (see Fig. 5b below).

For problem $B$ when the flow is generated by moving the top and bottom walls of the cavity in the same direction with equal velocities, the streamline patterns form two symmetrical domains (Fig. 4). Inside each of the subdomains, depending on the values $h$ and $R_0$, two eddies enclosed by a separatrix can exist (Fig. 4a–4c). The separatrix passes through a saddle type stagnation point. For $h = 1.67$ the size of the obstacle does not change essentially the flow structure (compare Fig. 4a and 4c). Even for $R_0 \to 0$ the flow near the center of cavity would differ from one for the “pure” rectangular cavity because of presence of the singular stagnation point (where otherwise the velocity would be non-zero). Therefore, we have an important influence of the obstacle on the flow structure. For $h = 1$ a significant change occurs (Fig. 4d) with decreasing of the cylinder radius (compare
Fig. 3. Streamline patterns for the cavity flow, generated by the motion of top and bottom walls in opposite directions, $V_{\text{top}} = V$ and $V_{\text{bot}} = -V$, and a fixed cylinder, $U = 0$ – problem A. The different sets of geometrical parameters $a$ and $h$ are: (a) $h = 1$, $R_0 = 0.3$; (b) $h = 1$, $R_0 = 0.1$.

Fig. 4b and 4d): both pairs of eddies merge, forming one eddy occupying, respectively, the whole upper and bottom parts of the cavity.

The flow generated by a rotating cylinder and fixed cavity walls (the problem C) exhibits the corner eddies as well as the primary one. The evolution of the corner eddies with increasing aspect ratio $h$ of the cavity (merging of eddies and formation of a typical cellular flow structure), was studied in detail by Hellou and Coutanceau (1992). Our results completely coincide with the results of these authors.

The next flow regime considered is the combination of oppositely moving walls ($V_{\text{top}} = V$ and $V_{\text{bot}} = -V$) and a rotating cylinder. The analogy of this flow with an unbounded shear flow studied by Robertson and Acrivos (1970) and Jeffrey and Sherwood (1980) is obvious. These authors investigated the case of a cylinder, rotating at angular velocity $\Omega$, immersed in an unbounded shear flow of strength $\kappa$ and centered on the stagnation line of undisturbed flow. They used for that analysis the exact Stokes solution for an infinite domain

$$
\Psi_{\text{shear}} = \frac{1}{4} \kappa \left[ 2r^2 \sin^2 \theta + \left( R^2 - \frac{R^4}{r^2} \right) \cos 2\theta - 2\omega \log \left( \frac{r}{R} \right) - 1 \right],
$$

(25)

where $\omega = 1 + 2\Omega/\kappa$, originally proposed by Bretherton (1962). They distinguished five possible flow regimes depending on the value $\omega$.

With a relatively long cavity ($h = 2.5$ and $R_0 = 0.3$) and top and bottom walls moving with opposite velocities (the intensity of shear flow equals $\kappa = V/b$), we can mimic this flow in the central zone
Fig. 4. Streamline patterns for the cavity flow, generated by the motion of top and bottom walls in the same direction, \( V_{\text{top}} = V \) and \( V_{\text{bot}} = V \), and a fixed cylinder, \( U = 0 \) – problem B. The different sets of geometrical parameters \( a \) and \( h \) are: (a) \( h = 1.67, R_0 = 0.3 \); (b) \( h = 1, R_0 = 0.3 \); (c) \( h = 1.67, R_0 = 0.1 \); (d) \( h = 1, R_0 = 0.1 \).

of the cavity and estimate the influence of the side walls. The velocity profile in the vicinity of the cylinder suggests, that for the given parameters the cavity flow characteristics near the cylinder are similar to those of the unbounded shear flow. Five qualitatively different flow regimes, analogous to those described by Jeffrey and Sherwood (1980), may be generated varying the cylinder rotation speed \( \Omega = U/R \).

When the cylinder is counter-rotating \( (U > 0) \), the flow in the domain is divided into four zones (Fig. 5a). In the vicinity of the cylinder the fluid is rotating in the same direction as the cylinder does, whereas the fluid in the outer zone is forced to move in opposite direction. This leads to saddle type stagnation points on the vertical axis of symmetry and to large eddies on the left- and right-hand sides of the cylinder. For unbounded shear flow, these eddies corresponding to the zones of blocked flow extend to infinity.

If the cylinder rotation speed is slowed down, no qualitative changes are observed until the cylinder is stopped. Then, the zone where fluid is rotating against the shear flow disappears. The flow structure becomes more simple with only two eddy zones attached to the right and left side of the cylinder (Fig. 5b).
Fig. 5. Streamline patterns for different flow regimes caused by the cylinder rotation in the quasi-shear flow \((V_{top} = V, V_{bot} = -V)\), generated in a long cavity \((h = 2.5, R_0 = 0.3)\). The cylinder is respectively: (a) counterrotating, \(U = 0.25V\); (b) fixed, \(U = 0\); (c) slowly corotating, \(U = -0.075V\); (d) rotating with the critical velocity \(U = -0.12V\); (e) fast corotating, \(U = -0.25V\).

When the cylinder is slowly co-rotating \((-0.12V < U < 0)\), the flow domain again becomes subdivided into four zones. There are saddle points on the horizontal axis of symmetry. The separatrix, corresponding to these stagnation points, separates two eddies and a zone around the cylinder from the main flow (Fig. 5c).

In an unbounded shear flow, according to Jeffrey and Sherwood (1980), the next qualitative change (the disappearance of the stagnation points) occurs when the angular velocity of the cylinder \(\Omega\) reaches a critical value when the cylinder becomes freely rotating and the total torque on the cylinder applied by the fluid is zero. The calculation of the torque acting on the cylinder provides the relation \(U = -0.15V\) for chosen values of \(h\) and \(R_0\). This value coincides with the results of Robertson and Acrivos (1970) for the shear flow in the infinite domain. For other geometrical parameters \(h\) and \(R_0\), the values of \(\Omega_{free} = U/R\) are different (Table 4).
Table 4
Dependence of the angular velocity $\Omega_{\text{free}}$ of the freely rotating cylinder on the aspect ratio $h$ of the cavity and the radius of the cylinder $R$. In all cases $V_{\text{top}} = V$, $V_{\text{bot}} = -V$

<table>
<thead>
<tr>
<th>Cylinder radius $R_0$</th>
<th>Aspect ratio of the cavity $h$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>0.1</td>
<td>−0.3905</td>
</tr>
<tr>
<td>0.3</td>
<td>−0.3903</td>
</tr>
</tbody>
</table>

Fig. 6. Streamline patterns in the cavity with geometrical parameters $h = 1.67$ and $R_0 = 0.3$: (a) flow generated by the moving top wall ($V_{\text{top}} = V$), while the bottom wall and the cylinder are fixed, $V_{\text{bot}} = 0$, $U = 0$. (b) flow generated by the simultaneous motion of the top wall ($V_{\text{top}} = V$) and the cylinder ($U = 0.9V$). The bottom wall is fixed.

The presence of the side walls stimulates the collapse of the vortex structures and the disappearance of stagnation saddle points at a lower speed of cylinder rotation. For the given geometry of the cavity the critical value is $U_c = -0.12V$. The streamline pattern for this critical regime is presented in Fig. 5d. And, finally, when the cylinder is co-rotating fast ($U_c < -0.12V$) the flow structure becomes very simple with no stagnation points (Fig. 5e).

From these results we can conclude that all typical regimes for the shear flow around a cylinder in the infinite domain are also observed in a finite domain.

The presence of a cylinder can strongly influence the structure of the flow generated by one moving wall. Two typical cases, namely, a fixed and a counter-rotating cylinder, respectively, are considered. For the fixed cylinder (Fig. 6a), a zone with two eddies attached to the cylinder arises on the side of the moving wall. It is separated from the rest of the cavity by the separatrix starting and ending at the cylinder. This zone contains two eddies enclosed by another separatrix associated with the saddle stagnation point located on the vertical axis of symmetry.

For the counterrotating cylinder the eddy zone, consisting of the pair of eddies, is separated from the cylinder (Fig. 6b). An additional stagnation point arises in the bottom half of the cavity and a “dog snout” streamline pattern is formed. The presence of the isolated eddy zone in the main flow can be of significant importance for the usage of this kind of flow for studying mixing phenomena.
Fig. 7. The structure of the steady mixing pattern generated by the motion of the top wall \( (V_{\text{top}} = V) \). The bottom wall and the cylinder are fixed. The geometrical parameters of the cavity are \( h = 1.67 \) and \( R_0 = 0.3 \). (a) Flow visualization. (b) Numerical tracking of the four initially horizontal lines after time \( t = 22.12a/V \).

It should be noted that for all closed streamlines in Figs. 5 and 6 the Poincaré–Hopf theorem, stating that the sum of the indices of the stagnation points is equal to the Euler characteristic of the region, holds well. The index of an interior elliptic point is \(+1\) and the index of an interior hyperbolic point is \(-1\). A (parabolic) separation point at the boundary has the index \(-\frac{1}{2}\). The Euler characteristic of the \( n \)-connected region is equal to \( 2 - n \).

3.3. Mixing in a steady flow

Two-dimensional steady flows do not exhibit high mixing efficiency, because the material stretching there is at best linear in time (Ottino, 1990). Nevertheless, mixing in a steady flow in confined domain provides a simple tool to visualize experimentally the structure of the flow, by placing some dyed lines in the flow. They become stretched (due to velocity gradients) and aligned predominantly (after sufficiently long time) along the streamlines. In Fig. 7a the result of such a steady mixing, when only the top wall was moving, is presented. Not pretending to reproduce exactly the experimental picture (initial positioning of many dyed lines and duration of the motion were not precise), the overall visible structure and some of its peculiarities can be found in both the experimental and the numerical results (details on the experimental setup are given in Appendix A). Fig. 7b shows a few initially straight lines (plotted dashed) advected in the flow under study for the time \( t = 22.12a/V_{\text{top}} \). Kink-like and loop patterns clearly seen in this figure are similar to the experimental observation. Kinks are created in the flow when the dyed line originally crosses the same (curved) streamline twice.

3.4. Mixing in a periodic flow

All simulations for the periodic mixing protocol were performed for the cavity with \( h = 1.67 \), \( R_0 = 0.3 \) and \( D = 5 \). Two circular blobs of the radius \( 0.1b \) were considered, centered at the points
Fig. 8. Poincaré map of the periodic flow in the cavity with $h = 1.67$, $R_0 = 0.3$ and $D = 5$.

(0.6384$b$, 0) and (0.9586$b$, 0). For the chosen value of $D$ these points are the hyperbolic and elliptic periodic points, respectively, of period-1.

The Poincaré map constructed after 50 000 periods in Fig. 8 clearly reveals a relatively large island ($e$) associated with the elliptic point (0.9586$b$, 0) of period-1 (the whole island returns to its original position after one period) as well as the island $g$, surrounding the obstacle. In the chaotic “sea” there also exists a variety of smaller islands ($a, b, c, d, f$) and ($a', b', c', d', f'$) corresponding to the elliptic periodic points of period-2.

Fig. 9 shows the deformation of the blob surrounding the hyperbolic periodic point of period-1. After 3 periods the length of the circumference of the blob is already $l = 179.2 l_0$, the enclosed area becomes $s = 1.011 s_0$ (here $l_0$ and $s_0$ correspond to the initial circular blob). The increase of the blob area serves as the estimation of growing errors of the blob shape description. For this mixing regime the circumference of the blob increases exponentially as $\exp(\pi t/T)$ with $\pi \approx 1.47$. Such a stretching is typical for chaotic flows. Practically, all dyed material is stretched along the unstable manifold associated with this hyperbolic point.

The mixing pattern of a blob surrounding the elliptic point of period-1 is much simpler (Fig. 10). Practically all dyed material is “locked” inside the island, with only a minor portion of material escaping into the chaotic region. Only this filament is responsible for some increase of the blob’s circumference (30.2 $l_0$ after 4 periods).

Therefore, in spite of the close initial locations of the blobs (compare Fig. 9a and 10a), mixing patterns caused by the periodic flow are very different, once more demonstrating that knowledge of periodic points plays a crucial role in the evaluation of laminar mixing processes.

In order to show the effect of the presence of the obstacle, like the cylinder, on the efficiency of mixing we calculated the deformation of the the same initial blob for two flows, i.e. with and without the cylinder (Fig. 11). It appeared that for the given mixing protocol the cylinder considerably affects
Fig. 9. Deformation of the blob, centered originally around hyperbolic periodic point \((0.6383b, 0)\). Plots correspond to (a) 0, (b) 1.0, (c) 2.0, (d) 3.0 periods of motion.

Fig. 10. Deformation of the blob, centered originally around elliptic periodic point \((0.9586b, 0)\). Plots correspond to (a) 0, (b) 1.0, (c) 2.0, (d) 3.0 periods of motion.
the structure of the mixing pattern, “attracting” the folds of the dyed blob. In Fig. 11a the elliptic island is clearly visible, being closely wrapped by thin filaments. In contrast, the mixing pattern (Fig. 11b) in the “pure” cavity with the same initial position of the blob is relatively simple.

4. Conclusions

The primary goal of the present paper was to develop the analytical method of superposition for solving the two-dimensional Stokes flow problem for the rectangular cavity with a cylinder in order to study distributive mixing in this cavity. An accurate velocity field is one of the main advantageous results of the proposed analytical method. The algorithm permits one to obtain accurate results even near corner points by using only a few terms in the Fourier series. The computation time, necessary to achieve an accuracy that is good enough for most applications (like the analysis of the flow structure) is relatively small.

A variety of flow regimes with complicated structures, containing eddies, stagnation points, separation lines and cellular structures, were revealed in this relatively simple system and a strong dependence of the flow structure on the geometrical parameters was found. The calculated results also provide the possibility to compare them with some well-known, important studies on the shear flow around a cylinder.

An algorithm of contour line tracking was presented, which provides a reliable tool to describe blob deformation in any two-dimensional flow with known velocity field. Accordingly, we compared one of the typical steady mixing patterns with experimental observations.

The flow in a rectangular cavity was used as a prototype flow for studying mixing phenomena in periodic Stokes flows. The cavity with a cylinder is useful to investigate if the presence of an obstacle can stimulate mixing (which is repeatedly stretching and folding of material) due to existence of the eddy zones, created by the obstacle. Adding of the cylinder rotation might be a simple way to remove islands (dead zones where no mixing takes place) in mixing patterns. Just as for the cavity without a cylinder, the system is analytically and experimentally tractable with relative ease.
Acknowledgements

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Appendix A

A.1. Experimental setup

Flow visualisation experiments were performed in a rectangular cavity (46.5 × 77.5 × 150 mm³) in accordance with the one used by Leong and Ottino (1989). A schematical drawing of the experimental setup is given in Fig. 12. Two opposite walls are fixed while the other two, made of rubber belts, can move. The cavity is placed in a glass container (265 × 215 × 230 mm³) filled with glycerine (Lamers & Plenger, 2LZ250003), having a kinematic viscosity 18.5 cm² s⁻¹. A cylinder (diameter 14 mm), which can rotate independently from the moving walls, is placed in the centre of the cavity.

To visualize flow patterns, a passive fluorescent dye made by dissolving fluorescent powder (Aldrich, fluorescein salt) in glycerine, is used. The tracer is injected about 5–10 mm beneath the free surface of the fluid at a preselected location. Within typical observation times (1 h), the flow is two dimensional at this depth, and no diffusion of the tracer is observed. The dye is excited by two ultraviolet lamps (UVP, model B-100 AP). The experiments are recorded on film or on video tape.
References