Abstract: In this paper we study the well-posedness (existence and uniqueness of solutions) of linear relay systems with respect to two different solution concepts. We derive necessary and sufficient conditions for well-posedness in the sense of Filippov of linear systems of relative degree one and two in closed loop with relay feedback. For systems of relative degree one and two it is shown that uniqueness of Filippov solutions follows if the first nonzero Markov parameter is positive. By means of an example it is shown that this result is untrue for relative degree larger than two.

Keywords: Hybrid modes, relay control, discontinuous control, complementarity problems.

1. INTRODUCTION

Relay systems are important as they are used in various control problems like sliding mode control (Tsypkin, 1984; Utkin, 1978; Filippov, 1988; Johansson, 1997), and in idealized models of (Coulomb) friction phenomena. Still quite some fundamental issues of such systems (even if the underlying system is linear) are unclear and have received considerably attention in recent years. Analysis of simulation of such systems (Mattson, 1996; Heemels et al., 2000b), well-posedness, that is the existence and uniqueness of solutions (Filippov, 1988; Lootsma et al., 1999), existence of fast switches (Johansson, 1997) are largely open problems.

The study of relay systems can also be motivated from the perspective of hybrid systems, i.e. systems having mixed discrete (logic-switching) and continuous (analog) dynamics. Such systems are intensively studied recently (Pnueli and Sifakis, 1995; Antsaklis and Nerode, 1998; Morse et al., 1999). Also in this area the basic question of existence and uniqueness of trajectories is certainly non-trivial and of interest (Lygeros et al., 1999). Moreover, a problem that is typical for continuous-time hybrid dynamical systems is the occurrence of an infinite number of (relay) switches in a finite time interval, which is called the Zeno behaviour. This behaviour gives rise to many difficulties in analysis and simulation. Filippov (Filippov, 1988) already noticed the influence of this phenomenon on uniqueness of solutions to relay systems. One of the uniqueness theorems (Thm. 2.10.4 in (Filippov, 1988)) excludes the Zeno behaviour as one of the conditions to assure uniqueness. However, this condition is difficult to verify a priori as results guaranteeing the absence of Zenoess are extremely rare (Johansson et al., 1999). Filippov also came up with an example containing two relays in which the uniqueness of
solutions does not hold due to Zeno behavior (see page 116 of (Filippov, 1988)).

As such, the subject of this paper is the investigation of uniqueness (as existence follows easily, see the next section) and Zeno behaviour of solutions of a linear single input single output (SISO) system

\[ \dot{x}(t) = Ax(t) + Bu(t); \ y(t) = Cx(t), \]  

in a closed loop with the relay feedback

\[ u(t) = -\text{sgn}(y(t)). \]  

Here \( x \in \mathbb{R}^n, y \in \mathbb{R}^1 \) and the matrices \( A, B, C \) are of corresponding dimensions. Observe that for \( y = 0 \) the \text{sgn} function is set-valued, i.e. \( \text{sgn}(0) = [-1, 1] \).

To be precise, we will extend the results obtained in (Lootsma et al., 1999) for (1), (2) in the sense that we will prove uniqueness for the Filippov solution concept, which is more general than the solution concept used in (Lootsma et al., 1999). In (Lootsma et al., 1999) a particular kind of Zeno-ness has been excluded in the solution concept (left-accumulations of relay switching times are not allowed), such that their proof of existence of theFilippov solutions is not straightforward. Although the solution concept in (Lootsma et al., 1999) has this disadvantage, it applies to a broader class of hybrid systems, called linear complementarity systems (Van der Schaft and Schumacher, 1996; Van der Schaft and Schumacher, 1998; Heemels et al., 2000a; Heemels et al., 1999), which involve discontinuous dynamical systems like unilaterally constrained mechanical systems. Filippov’s notion of solutions does not allow such a generalization. However, for relay systems, the Filippov concept seems more natural and the associated uniqueness is of independent interest. It plays also an important role for proving consistency of a numerical simulation method based on time-stepping (Heemels et al., 2000b).

The conditions derived in (Lootsma et al., 1999) will be shown to be necessary and sufficient for Filippov uniqueness for systems of the form (1), (2) for which the underlying linear system (1) has relative degree one or two. However, by a counterexample of a triple integrator in closed loop with negative relay feedback, it is shown that the conditions are not sufficient in general.

The following notations and definitions are used in the paper. A point \( \tau \in \mathcal{E} \subset \mathbb{R} \) is called a right-accumulation point of \( \mathcal{E} \), if there exists a sequence \( \{ \tau_i \}_{i \in \mathbb{N}} \) such that \( \tau_i \in \mathcal{E} \) and \( \tau_i < \tau \) for all \( i \) and furthermore, \( \lim_{i \to \infty} \tau_i = \tau \). A left-accumulation point is defined similarly by replacing \( < \) by \( > \).

An accumulation point of \( \mathcal{E} \) is either a left- or a right-accumulation point of \( \mathcal{E} \).

The closure of a set \( S \subset \mathbb{R}^n \) is denoted by \( \text{cl}S \).

2. FILIPPOV SOLUTIONS

Consider the following system of differential equations:

\[ \dot{x} = f(x), \]  

where \( f : G \to \mathbb{R}^n, G \subset \mathbb{R}^n \), is a piece-wise smooth vector field undergoing (possible) jumps on a set \( M \subset G \) of zero measure. In the simplest “convex” definition of Filippov (Filippov, 1988) (which is actually equivalent to Utkin’s equivalent control definition (Utkin, 1978) for the system (1), (2)) the equation (3) is transformed into a differential inclusion \( \dot{x} \in F(x) \), where for each \( x \in G \) the set \( F(x) \) is defined to be the smallest convex closed set containing all the limit values of the function \( f \).

**Definition 1.** A Filippov solution of (3) is a solution of the differential inclusion

\[ \dot{x} \in F(x), \]  

that is, an absolutely continuous function \( x(t) \) defined on an interval \( I \), for which \( \dot{x}(t) \in F(x(t)) \) almost everywhere on \( I \).

Note that for the system (1), (2) \( M \) is the set \( \{ x \in \mathbb{R}^n \mid Cx = 0 \} \) and \( F(x) = \{ Ax + B \} \) for \( Cx < 0 \), \( F(x) = \{ Ax - B \} \) for \( Cx > 0 \) and \( F(x) = Ax + B[-1, 1] \) when \( Cx = 0 \).

Suppose that \( F(x) \) is bounded on \( G \). The obtained set valued function \( F(x) \) is upper semicontinuous (see Lemma 2.6.3 in (Filippov, 1988)) and hence, according to theorem 2.7.1 in (Filippov, 1988) for arbitrary initial conditions from \( G \) the solution of the differential inclusion (4) locally exists.

We say that a solution (in the sense of Filippov) \( x(t,x_0) \) with initial conditions \( x_0 \) is (locally) right unique if there is a \( t_1 > 0 \) such that each two solutions \( x_1(t,x_0) \) and \( x_2(t,x_0) \) on the interval \([0,t_1]\), satisfying \( x_1(0,x_0) = x_0, x_2(0,x_0) = x_0 \) coincide on the interval \( 0 \leq t \leq t_1 \). A similar definition for left uniqueness can be given. If a solution \( x(t,x_0) \) is both (locally) right and left unique, we will say that it is (locally) unique (Filippov, 1988).

Here we mention some important properties of Filippov solutions:

1. **Compactness of the set of solutions.**

   The set of solutions defined on \( \alpha \leq t \leq \beta \)
with initial conditions from a given compact set is compact with respect to the $C[\alpha, \beta]$ topology, (see theorem 2.7.3 (Filippov, 1988)).

(2) **Continuous dependence on initial conditions.** Uniqueness of solutions implies continuous dependence on the initial data, (see theorem 2.8.2 (Filippov, 1988)).

### 3. FORWARD SOLUTIONS

The solution concept used in (Lootsma et al., 1999) stems from hybrid dynamical systems (Pnueli and Sifakis, 1995; Antsaklis and Nerode, 1998; Morse et al., 1999) and more precisely from (linear) complementarity systems (LCS), (Van der Schaft and Schumacher, 1996; Van der Schaft and Schumacher, 1998; Heemels et al., 2000a; Heemels et al., 1999). The discrete part of the behaviour is related to the idea that an ideal relay element is given by three modes of operations (“discrete states”) given by the three branches of the characteristic:

\[
\begin{align*}
A_1 & \quad y(t) > 0, u(t) = -1 \\
A_2 & \quad y(t) < 0, u(t) = 1 \\
A_3 & \quad y(t) = 0, -1 \leq u(t) \leq 1
\end{align*}
\]

During the evolution of the system, the relay switches between these three modes, which have their own characteristic laws of motion. In such hybrid systems the solutions are usually considered in the following “forward sense.”

**Definition 2.** Suppose that there is an $\varepsilon > 0$ and a triple $(u, x, y) : [0, \varepsilon) \rightarrow \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ satisfying for $x_0 \in \mathbb{R}^n$

1. $(u, x, y)$ is real-analytic on $[0, \varepsilon)$;
2. $x(0) = x_0$;
3. (1) is satisfied on $[0, \varepsilon)$; and
4. there exists an $i \in \{1, 2, 3\}$ such that for all $t \in [0, \varepsilon)$ $Ai$ holds.

Then $x$ is called a local solution on $[0, \varepsilon)$ to (1), (2) in the sense of (Lootsma et al., 1999) (called a “forward solution” from now on) with initial condition $x_0$.

This means that a local forward solution satisfies the dynamics of one mode only on an interval of the form $[0, \varepsilon)$ and does not account for the possibility of left-accumulation of the relay switching times. As a consequence, solutions starting with an infinite number of time instances of leaving and reaching the switching surface $y = 0$ in (1), (2) (as e.g. in the example in (Filippov, 1988, p. 116)) are excluded from the definition of forward solutions.

In (Lootsma et al., 1999) one proves existence and uniqueness of forward solutions under suitable conditions for the multiple relay case. This result can be formulated for the single relay case in terms of the Markov parameters of the system (1), which are defined by $H^i := CA^{i-1}B$ for $i = 1, 2, \ldots$. The leading Markov parameter is defined as the first Markov parameter that is non-vanishing, i.e. it is given by $H^\rho$ with

\[
\rho := \min\{i = 1, 2, \ldots \mid H^i \neq 0\}
\]

provided that not all Markov parameters are zero. We call $\rho$ the relative degree of the system (1).

The following result gives a sufficient condition for uniqueness of forward solutions.

**Theorem 1.** (Lootsma et al., 1999) Let the relay system $(1,2)$ be given. If the leading Markov parameter $H^\rho$ is positive, then from any initial condition $x_0$ there exists a unique local forward solution.

Hence, uniqueness of forward solutions can be checked by means of the simple calculation of the sign of the leading Markov parameter. In (Van der Schaft and Schumacher, 1999) it is observed on page 110, that the conditions of Filippov (see section 2.10 in (Filippov, 1988)) have to be checked on a point-by-point basis, while here the determination of the leading Markov parameter suffices. However, the results of Filippov are applicable to general nonlinear systems coupled to relays, while Theorem 1 is valid for linear relay systems only.

In (Lootsma et al., 1999) (see also (Heemels et al., 2000a)) it is discussed how the local solutions can be extended to obtain a global forward solution (i.e. on $[0, \infty)$) by concatenation of local forward solutions. Recall that forward solution might have right-accumulations of relay switching times, but no left-accumulations.

### 4. FILIPPOV VERSUS FORWARD SOLUTION

However, there is an open question: *how are the conditions of Theorem 1 related to the uniqueness (left or right) of the solutions of the closed loop system when the solutions are understood in the sense of Filippov?* As mentioned in the introduction, a first indication of possible problems stems from an example constructed by Filippov (Filippov, 1988, p. 116) for which besides the zero solution (which is both a Filippov and a forward solution) there exists an infinite number of other
trajectories (being Filippov, but not forward solutions) starting from the origin. The nonzero solutions leave the origin due to left-accumulations of the relay switching times and as such are included in the Filippov solution concept, but not as forward solutions. However, Filippov’s example does not satisfy the conditions for uniqueness given in (Lootsma et al., 1999) for the multiple relay case. Hence, there is no counterexample known so far in the literature showing that the conditions of (Lootsma et al., 1999) are not sufficient for Filippov uniqueness as well.

At this point we make a simple observation. Note that the sign of the even Markov parameter is invariant with respect to time inversion. Thus, if a system has an even relative degree with a positive Markov parameter, then together with the right uniqueness we have immediately the left uniqueness (for the forward solution). By analogy, for an odd relative degree the above mentioned result gives a uniqueness condition only for right uniqueness.

4.1 Relative degree one

First we consider the case of relative degree one. In this case the right uniqueness of Filippov solutions indeed follows from the positivity of the first Markov parameter.

Theorem 2. Consider a linear SISO system (1) of relative degree one in closed loop with the feedback (2). Then for all initial conditions the Filippov solution is right-unique if and only if the first Markov parameter $CB$ is positive.

4.2 Relative degree two

Now let us consider the case of relative degree two.

Theorem 3. Consider a linear SISO system (1) of relative degree two in closed loop with the feedback (2). Then for all initial conditions the Filippov solution of (1), (2) is unique if and only if the second Markov parameter $CAB$ is positive.

Note that in case $CAB$ is positive, Zeno behaviour can be excluded for the relay system (1), (2) (in the class of Filippov solutions) as we know that forward solutions exists which have the special property that left-accumulations of relay switching times cannot occur. By considering these solutions for both the original and the time-reversed system it follows that the unique Filippov solution is a forward solution and cannot have accumulations of switching times at all. Conditions excluding Zeno-behaviour are rare, cf. (Johansson et al., 1999) and as such this result is quite interesting. Note that for the situation, where $CB$ is positive, left-accumulations of relay switching times cannot occur based on a similar reasoning, but right-accumulations cannot be excluded in this case. This discussion is summarized in the following corollary.

Corollary 4. Consider the relay system given by (1), (2). In case $CB > 0$ Filippov solutions are right unique and do not have left-accumulations of relay switching times. In case $CB = 0$ and $CAB > 0$ Filippov solutions are unique and accumulations of relay switching times do not occur.

4.3 Relative degree three

Up to this point one may conjecture that a statement similar to Theorem 1 is true for the Filippov solutions as well. However, the following counterexample which is inspired by (Johansson, 1997) proves that this is not true in general.

Consider the following system

$$\begin{align*}
\dot{x} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x \\
u &= -\text{sgn} y
\end{align*}$$

with $x = (x_1, x_2, x_3)^T$.

The system (6) has relative degree three and its third Markov parameter is 1, so the system (6), (7) has right unique forward solutions for arbitrary initial conditions. Our goal is to show that if one accepts Filippov’s concept of solution, then there are at least two solutions of (6), (7) starting at the origin.

Theorem 5. There are at least two Filippov solutions of the system (6), (7) starting at the origin.

The proof of this result is based on the following lemma that is obtained from (Johansson, 1997).

Lemma 6. Any solution of th system (6), (7) starting at $t = 0$ in the set

$$\Omega = \{ x \in \mathbb{R}^3 : x_1 = 0, x_2 > 0 \}$$

is right unique and undergoes an infinite number of switches with intervals between them of length $t_1, t_2, \ldots$ for which the following estimate is true

$$t_k > (\sqrt{2} + 1)^{k-1} t_1.$$
Observe that if (9) is true, then \( x(1, x_0) \) starting at the origin.
Given an arbitrary solution \( x(t, x_0) \) with \( x_0 \in S_\varepsilon^0 \) and let \( T(x_0) \geq 0 \) be a moment of the first intersection of the trajectory of \( x(t, x_0) \) with the switching plane \( x_1 = 0 \) for a time instant larger or equal than \( t = 1 \), i.e., \( x_1(T(x_0), x_0) = 0 \). Denote
\[
 t_{\max} = \sup_{z \in S_\varepsilon^0} T(z)
\]
By definition, \( t_{\max} \geq 1 \). Moreover, \( t_{\max} > 1 \) (otherwise we have a contradiction with the previous lemma). Additionally, we claim that \( t_{\max} \) is bounded. This fact follows from lemma 6.

Consider an arbitrary converging sequence \( \{x_0^i\} \), \( i = 1, \ldots, \infty \), with \( x_0^i \in S_\varepsilon^0 \), \( x_0^i \to 0 \) as \( i \to \infty \). Since the set of Filippov solutions defined on the finite interval \( [0, t_{\max}] \) for the initial conditions from the compact set \( \text{cls}S_\varepsilon^0 \) is compact, a subsequence of \( x(t, x_0^i) \) converges to a solution \( \xi_0(t) = x(t, x_0^i) \), where \( x(t, x_0) \) is some Filippov solution of the closed loop system and \( x_0 \in \text{cls}S_\varepsilon^0 \) (see Section 2). Without loss of generality, we may assume that the sequence itself is convergent in the \( C[0, t_{\max}] \)-topology. This implies that
\[
 \forall t \in [0, t_{\max}] \ x(t, x_0^i) \to \xi_0(t) \text{ as } i \to \infty
\]
Since \( 0 \in \text{cls}S_\varepsilon^0 \) and \( x(0, x_0^i) = x_0^i \to 0 \) as \( i \to \infty \), it follows that \( \xi_0(0) = x_0 = 0 \). Thus \( \xi_0(t) \) is a Filippov solution of the closed loop system starting at the origin.

Our goal is to show that for the sequence \( \{x_0^i\} \) of points in \( \mathbb{R}^3 \) there is a sequence of times \( \{q^i\} \), \( 1 \leq q^i \leq t_{\max} \) such that
\[
 \lim_{i \to \infty} \inf ||e(q^i, x_0^i)|| > 0. \quad (9)
\]
Observe that if (9) is true, then \( \xi_0 \neq 0 \) and hence, we constructed a non-zero solution starting from the origin.

Given \( i \), denote by \( t_{j}^i \) the length of the intervals between successive switches for the solution \( x(t, x_0^i) \). Denote by \( n^i \) the minimal number of switches for the solution \( x(t, x_0^i), t \in [0, t_{\max}] \) necessary to satisfy the following inequality
\[
 q^i := \sum_{j=1}^{n^i} t_{j}^i \geq 1 \quad (10)
\]
By definition, \( q^i \leq t_{\max} \) for all \( i \).

Let \( r = 1 + \sqrt{2} \). By virtue of lemma 6, we have
\[
 t_{j}^i < t_{n^i} \left( \frac{1}{r} \right)^{n^i-j}, \quad j < n^i
\]
and hence from (10)
\[
 1 \leq \sum_{j=1}^{n^i} t_{j}^i < t_{n^i} \sum_{j=0}^{n^i-1} r^{-j}.
\]
Thus
\[
 t_{n^i} > \left[ \frac{1}{\sum_{j=0}^{n^i-1} r^{-j}} \right] > \left[ \frac{1}{\sum_{j=0}^{\infty} r^{-j}} \right] = 2 - \sqrt{2}
\]
This value is separated from zero for any \( i \). From the third equation of the closed loop system it follows that
\[
 \lim_{i \to \infty} |x_3(q^i, x_0^i)| > 0.
\]

The constructed non-zero Filippov solution with zero initial state starts with a left-accumulation point of relay switching times and hence is not a forward solution as used in (Lootsma et al., 1999). Note that any small perturbation of the zero initial condition immediately gives rise to a solution moving away from the origin (see Lemma 1) in which the switching times increase exponentially according to (8).

5. DISCUSSION

In this paper we compared two different solution concepts for a linear time-invariant system coupled to an ideal relay, namely Filippov solutions and forward solutions. One advantage of the forward solution concept is that is extendable to classes of hybrid dynamical systems (e.g. linear complementarity systems), while the Filippov solution is not. Another major advantage of the forward solution is that the (available) well-posedness conditions for linear relay systems are relatively easy to check. This is partly because the possibility of left-accumulations of relay switching times (a particular kind of Zeno behaviour) is excluded.

In this paper we actually showed that if the underlying linear system has relative degree one or two the conditions are still sufficient and even necessary for uniqueness of Filippov solutions. In particular, the existence of left accumulation points of switching periods can be excluded a priori on the basis of these conditions which facilitate simulation and analysis of these systems. However, for higher relative degrees this is unfortunately not the case. The system (6), (7) (with one relay and of relative degree three) demonstrates the following drawbacks of the forward solutions:

1. **Reversing time.** If \( x(t) \) is a forward solution to \( \dot{x} = f(x) \), then the reversed-time
trajectory $x(-t)$ is not necessarily a forward solution to $\dot{x} = -f(x)$.

(2) **Compactness of the set of solutions.** The set of forward solutions defined on $\alpha \leq t \leq \beta$ with initial conditions from a given compact set is not compact with respect to $C[\alpha, \beta]$ topology and hence, with respect to any weaker topology. This property immediately follows from the example (6), (7) for which we proved the existence of multiple Filippov solutions originated from the origin. At the same time, theorem 1 predicts the uniqueness of the forward solutions. Then using compactness property of the Filippov solutions (see Section 2) it is possible to build a fundamental sequence of forward solutions converging in $C^0$-topology (and, therefore, in any weaker topology) to a Fillipov solution, which in turn is not a forward solution since it starts from the origin and is not identically zero.

(3) **Continuous dependence on the initial conditions.** Uniqueness of a forward solution does not necessarily imply continuous dependence on the initial data. This property again follows from the example (6), (7): the unique forward solution (see theorem 1) starting at the origin does not continuously depend on the initial conditions.

The price paid for using a concept that is applicable to broader classes of hybrid dynamical systems is that certain properties which do not hold in these classes cannot be expected to hold for the specific subclasses as indicated here. For instance, the first and third property do typically not hold for hybrid dynamical systems. The second property requires a space in which the solutions live and a topology on this space. As the global solution concept is given by a construction based on concatenation of local solutions, this seems like a non-trivial task.

It is worth mentioning that in the Introduction to monograph (Filippov, 1988) discussing possible approaches to define a solution for discontinuous systems Filippov claimed that compactness of the solutions is a mandatory property for any possible solution concept. Unfortunately, the forward solution being very convenient for applications does not possess this property due to possible left-accumulation points.

6. REFERENCES


