Control of mixing via entropy tracking

Maarten Hoeijmakers,1 Francisco Fontenele Araujo,2 GertJan van Heijst,2 Henk Nijmeijer,1 and Ruben Trieling2

1Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
2Department of Physics, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

(Received 6 January 2010; revised manuscript received 26 March 2010; published 4 June 2010)

We study mixing of isothermal fluids by controlling the global hydrodynamic entropy (s). In particular, based on the statistical coupling between the evolution of (s) and the global viscous dissipation (e), we analyze stirring protocols such that (s) ~ \tau^\alpha \langle e \rangle ~ \tau^{-1}, with 0 < \alpha \leq 1. For a model array of vortices [Fukuta and Murakami, Phys. Rev. E 57, 449 (1998)], we show that: (i) feedback control can be achieved via input-output linearization, (ii) mixing is monotonically enhanced for increasing entropy production, and (iii) the mixing time \tau_m scales as \tau_m ~ (\langle e \rangle)^{-1/2}.

DOI: 10.1103/PhysRevE.81.066302

PACS number(s): 47.51.+a

I. INTRODUCTION

Since the seminal paper by Maxwell [1], control theory has played an important role in science and technology. Besides the numerous applications in industrial motion systems—often using proportional/differential controllers—other problems range from the control of chaos in dynamical systems [2] to the design of engineering devices for ship maneuvering [3].

In fluid dynamics, control theory has stimulated many advances in thermal convection [4,5], drag reduction [6–8], mixing [9–11], turbulence [12,13], and flow over bluff bodies [14]. A common feature in such contexts is the feedback actuation, which usually involves blowing/suction through specific boundary points [7,11] or modulation of electromagnetic forces in the neighborhood of walls [6,9]. Whatever the mechanism, successful control crucially depends on the flow regime.

Stokes flows, for instance, can be controlled and even optimized via linear methods [9]. However, beyond such regime, hydrodynamic nonlinearities pose major challenges for control theory [10,12]. To circumvent part of the technical issues that arise, one often resorts to simplified nonlinear models with the aim of extracting insight on how to enhance (or suppress) a particular flow response [5]. In this spirit, we focus on mixing—an essential process in industrial applications and an inspiring subject for basic research. Inspiring but nontrivial, nontrivial in the sense of its quantification, issues that arise, one often resorts to simplified nonlinear feedback design, and experimental calibration. Despite many attempts to characterize mixing in terms of statistical properties of flow snapshots [9,15–17], there is still no consensus on which mixing measure m(\tau) best represents the process at time \tau. Moreover, from a control standpoint, feedback design requires substantial knowledge of m as explicit function of the velocity field U. But at present, such analytic relation is inaccessible from fundamental principles as well as realistic coordination of sensors with actuators remains a challenge for experiments.

Here, we do not address such experimental issues. Instead, we focus on mixing in a simplified flow model: an array of vortices [18] whose mathematical structure is susceptible to classical control theory [19,20]. In particular, we wish to reveal how mixing depends on hydrodynamic quantities such as the global viscous dissipation.

\[ \langle e \rangle = \frac{\nu}{2} \left( \sum_{\alpha} \sum_{\beta} \left( \frac{\partial U_{\alpha}}{\partial X_{\beta}} + \frac{\partial U_{\beta}}{\partial X_{\alpha}} \right) \right)^2, \]  

where \nu denotes the kinematic viscosity of the fluid, U_\alpha velocity components, X_\alpha Cartesian coordinates, and \langle \cdot \rangle space average over the flow domain.

To accomplish that, the present paper is organized as follows. Section II sketches the general lines of our approach. After pointing out the gap between the purely statistical and the purely kinematical descriptions of mixing, we adopt the hydrodynamic entropy (S) as intermediate between the two descriptions. The benefit of such approach is threefold: (i) it relates mixing with velocity gradients. (ii) It allows feedback control in terms of global flow quantities (\langle e \rangle and S) rather than local measurements [7,11]. (iii) It facilitates the control actuation via adjustments in the driving force of the flow instead of blowing and/or suction through ad hoc boundary points [7,11]. In this spirit, we choose (S) as a control target such that (e) is a statistically stationary/decaying function of time. The stirring protocol is then dynamically adjusted and the status of the mixture further characterized in terms of the mixing measure proposed by Stone and Stone [15].

Sections III and IV are devoted to the application of the above strategy to a simplified flow model. We begin by introducing the array of vortices derived by Fukuta and Murakami [18], which is inspired by experiments on a shallow layer of fluid driven by electromagnetic forcing [21,22]. The governing equations for the stream function are given in Sec. III. In terms of them, we compute the global viscous dissipation of the flow, discussing its geometrical representation in state space and its relation to entropy production. Then in Sec. IV, we show that the vortex model, although nonlinear, is amenable to feedback control via input-output linearization [19,20]. In this framework, we prescribe stirring protocols such that the dimensionless viscous dissipation (e) evolves as (e) ~ \tau^{1+\alpha}, with 0 < \alpha \leq 1. In particular, for statistically stationary (e), we define a mixing time \tau_m and show that \tau_m ~ (e)^{-1/2}. For the statistically decaying case, we show that mixing is monotonically enhanced for increasing entropy production.

Finally, Sec. V provides a summary of results, conclusions, and open questions.
II. MIXING, ENTROPY, AND VISCOUS DISSIPATION

Mixing is traditionally described at two levels: (i) kinematically, in terms of stretching/folding of fluid elements [16,17] and (ii) statistically, by taking snapshots of the mixture and computing average properties of each image [15–17]. The former is related to velocity gradients; the latter to entropy [23–25]. But how to connect these two pictures in a consistent way? Answers to this question could be formulated in terms of vorticity, persistence of strain [26,27], or viscous dissipation, to cite just a few possibilities. Among those, the global viscous dissipation (1) provides a convenient alternative since it is related to the hydrodynamic entropy via the differential equation (see Landau and Lifshitz [28], p. 195, Eq. (49.6)):

\[
\frac{d}{dt} \langle \rho S \rangle = \kappa \left( \nabla T \right)^2 + \left( \frac{\rho}{T} \right) + \left( \frac{\xi}{T} \left( \nabla \cdot U \right)^2 \right),
\]

where \( \rho \) denotes the density of the fluid, \( T \) the temperature, \( \kappa \) the thermal conductivity, and \( \xi \) the second viscosity. In particular, for incompressible and isothermal flow, the evolution of the global entropy \( \langle S \rangle \) is simplified to

\[
\frac{d}{dt} \langle S \rangle = \frac{1}{T} \langle \epsilon \rangle.
\]

Here, it is convenient to introduce dimensionless variables such that \( X_t = \frac{X}{L}, \tau = \frac{T}{t}, U = \frac{U}{U_0}, \epsilon = \frac{\epsilon}{\epsilon_0}, \) and \( S = \frac{S}{2T}, \)

where \( L \) is a typical length scale of the flow. Thus

\[
\frac{d}{dt} \langle s \rangle = \langle \epsilon \rangle. \tag{2}
\]

Physically, Eq. (2) establishes an interesting connection between average velocity gradients and entropy. For instance, if \( \langle s \rangle \) evolves as a power law, so does \( \langle \epsilon \rangle \). From the control standpoint, this suggests the specification of \( \langle s \rangle \) as a control target. Thus, the study of stirring protocols based on statistical properties of \( \langle s \rangle \) and \( \langle \epsilon \rangle \) may contribute to a better understanding of the mixing dynamics. That is the main point of the present paper.

But how to assess the plausibility of such argument? To answer this question, we should somehow quantify mixing. We do so by adopting the mixing number \( M \) proposed by Stone and Stone [15], since its sensitivity on image resolution is considerably weaker than in other methods. For details, we refer the reader to Ref. [15]. Here, we just present an informal definition of \( M \) for mixing between two species in a rectangular domain. The basic idea is as follows. Let a snapshot image \( I(t) \) discretized in \( N \) cells, each of which colored as black or white. Given a cell \( C_a \), consider the set \( O_a(t) \) whose color is opposite to that of \( C_a \). Then, introduce the distance between \( C_a \) and \( O_a \) as \( D(C_a, C_b) = \min_{O_a(t)} d(C_a, C_b) \), where \( d(C_a, C_b) \) denotes the Cartesian distance between cells. In this way, the mixing number \( M(t) \) is defined as [15]

\[
m(t) = \frac{\sum_{a=1}^{N} \Delta^2(C_a, O_a(t))}{N} \tag{3}
\]

Qualitatively, Eq. (3) measures the average distance between black and white species at time \( t \).

In summary: to bridge the gap between the statistical and kinematic descriptions of mixing, we adopt the hydrodynamic entropy \( \langle s \rangle \) as intermediate between them. In particular, Eq. (2) suggests the choice of \( \langle s \rangle \) as a control target, which is presumably related to mixing [via Eq. (3)]. Next, we apply this idea to the case of mixing in an array of vortices.

III. MIXING IN AN ARRAY OF VORTICES

Vortices are pervasive structures in nature. In geophysical flows, for instance, they emerge in a variety of length scales, encompassing extreme events such as tornadoes and snow avalanches [29]. In condensed matter physics, they play a less threatful but important role in superconductivity [30], Bose-Einstein condensates [31], and fluid dynamics as a whole.

Under laboratory conditions, vortex distributions in regular lattices offer a convenient platform for research on flow structures. Among the many examples are vortex arrays in soap films [32] and in shallow layers of electrolytes [21,33,34]. The latter, in particular, is experimentally realized by setting a row of evenly spaced magnets underneath the flow container and then passing an electric current through the fluid. Such simple setup has inspired theoretical studies on drag reduction [6], three-dimensional resonant mixing [34,35], and mixing control in two-dimensional Stokes flow [9].

In the present paper, we address neither three-dimensionality issues nor the Stokes regime. Instead, our focus is on feedback control in a two-dimensional nonlinear model [18].

A. Model

Consider a two-dimensional, viscous, and incompressible flow modeled by the streamfunction [18],

\[
\psi(x,y,t) = \psi_0(t) \sin(kx) \sin(y) + \psi_1(t) \sin(y) + \psi_2(t) \cos(kx) \sin(2y), \tag{4}
\]

where the time amplitudes \( \psi_{0,1,2}(t) \) are governed by

\[
\frac{d\psi_0}{dt} = \frac{k(2k^2 + 3)}{2(k^2 + 1)} \psi_1 \psi_2 - \frac{k^2 + 1}{R} \psi_0 + \frac{(k^2 + 1)F}{R}, \tag{5}
\]

\[
\frac{d\psi_1}{dt} = -\frac{3k^2}{4} \psi_0 \psi_2 - \frac{1}{R} \psi_1, \tag{6}
\]

\[
\frac{d\psi_2}{dt} = -\frac{k^3}{2(k^2 + 4)} \psi_0 \psi_1 - \frac{k^2 + 4}{R} \psi_2. \tag{7}
\]

Equations (4)–(7) mimic the spatiotemporal dynamics of a shallow layer of fluid driven by a Lorentz force [18,22].
such a context, $F$ is related to the amplitude of the electric current through the fluid, $k$ to the distance between magnets, and $R$ to the viscosity of the electrolyte.

Insight into the basic structure of the model is provided by plotting streamlines of Eq. (4). As shown in Fig. 1, the $\psi_0$-mode induces an array of counter-rotating vortices [plate 1(a)]; $\psi_1$ a pure shear flow [plate 1(b)], and $\psi_2$ a vortex lattice [plate 1(c)]. From this perspective, proper combinations of $\psi_{0,1,2}$ may be sought in order to achieve a desired flow response. Since our focus here is on mixing, we shall specify an appropriate hydrodynamic output as follows. First, we relate viscous dissipation and entropy in the vortex system [Eqs. (4)-(7)]. Then, we adopt a control strategy based on input-output linearization and systematically quantify mixing for different stirring protocols.

### II. Viscous dissipation and entropy

To determine the global viscous dissipation [Eq. (1)] in the vortex system [Eqs. (4)-(7)], we write the velocity components in terms of the streamfunction as $u_x=\partial \psi / \partial y$ and $u_y=-\partial \psi / \partial x$. Then, we compute the corresponding velocity gradients and take the space average, 

$$\langle \cdots \rangle = \frac{1}{L_x L_y} \int_0^{L_y} \int_0^{L_y} (\cdots) dx dy,$$

with $L_x = 2\pi/k$ and $L_y = \pi$. In this way, the dimensionless viscous dissipation $\langle \epsilon \rangle$ is given by

$$\langle \epsilon \rangle = A_0 \psi_0^2 + A_1 \psi_1^2 + A_2 \psi_2^2,$$

where $A_0 = \frac{1}{4}(1+k^2+k^4)$, $A_1 = \frac{1}{2}$, and $A_2 = 1+2k^2+k^4$. Equation (8) has geometrical counterparts in state space. For instance, $\langle \epsilon \rangle = r_0 = \text{constant}$ corresponds to an ellipsoid, as shown in Fig. 2.

### IV. CONTROL

Once a physically relevant output $\xi$ is identified, it may be desirable to drive the system to a target response $r$, so that the difference $\xi-r$ asymptotically converges to zero. Examples of such outputs include measurements of wall-shear stresses in channel flows [7,11] and local temperatures [4,5] in thermal convection. In the case of Eqs. (5)-(7), we choose $\xi=\langle \epsilon \rangle$ and implement feedback control via input-output linearization [19,20].

FIG. 1. Streamlines of the model (4) for $k=1$. (a) Counter-rotating vortices generated by the main mode $\psi_0$. (b) Pure shear flow induced by $\psi_1$. (c) Vortex lattice due to $\psi_2$.

FIG. 2. Geometric structures in state space for $k=1$ and $R=10$. Ellipsoid: surface for which $\langle \epsilon \rangle = 0.4$. Straight lines: stable equilibria for $0 \leq \langle \epsilon \rangle \leq 1$.

FIG. 3. Geometric structures in state space for $k=1$ and $R=10$. Ellipsoid: surface for which $\langle \epsilon \rangle = 2$. Straight lines (black): stable equilibria for $2 \leq \langle \epsilon \rangle \leq 2.5$. Vectors: internal dynamics. U-shaped line (white): typical trajectory of the closed loop system.
A. Input-output linearization

Consider a dynamical system of the form

$$\frac{d\psi}{dt} = a(\psi) + Fb(\psi),$$  \hspace{1cm} (10)

where $\psi=(\psi_0,\ldots,\psi_{m-1})$ denotes the state vector, $a$ and $b$ vector fields, and $F$ a scalar input.

In terms of Eq. (10), the time derivative of the output $\dot{\xi}$ can be written as

$$\frac{d\xi}{dt} = \nabla \dot{\xi} \cdot a + F \nabla \xi \cdot b,$$  \hspace{1cm} (11)

where $\nabla \xi= (\partial \xi_1/\partial \psi_0, \ldots, \partial \xi_m/\partial \psi_{m-1})$. Since $d\xi/dt$ explicitly depends on the input $F$, system [Eq. (10)] has relative degree $\gamma=1$ [19,20]. Under such property, a tracking feedback controller may be achieved by choosing $F$ as

$$F = \frac{-\nabla \xi \cdot a - \Gamma(\xi - r) + \frac{dr}{dt}}{\nabla \xi \cdot b},$$  \hspace{1cm} (12)

where $\Gamma>0$ is the control gain. This so-called input-output linearization is valid for $\nabla \xi \cdot b \neq 0$, cf. [19,20].

Thus, we apply feedback control [Eq. (12)] to the system,

$$\psi = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{bmatrix},$$  \hspace{1cm} (13)

$$a = \begin{bmatrix} \frac{k(k^2+3)}{2(k^2+1)} \psi_1 \psi_2 - \frac{k^2+1}{R} \psi_0 \\ -\frac{3k}{4} \psi_0 \psi_2 - \frac{1}{R} \psi_1 \\ -\frac{k^3}{2(k^2+4)} \psi_0 \psi_1 - \frac{k^2+4}{R} \psi_2 \end{bmatrix},$$  \hspace{1cm} (14)

$$b = \begin{bmatrix} \frac{k^2+1}{R} \\ 0 \\ 0 \end{bmatrix},$$  \hspace{1cm} (15)

$$\xi = \langle \epsilon \rangle = A_0 \phi_0^2 + A_1 \psi_1^2 + A_2 \psi_2^2,$$  \hspace{1cm} (16)

and

$$r = r(t),$$  \hspace{1cm} (17)

where the target $r$ remains to be specified (we do so in the next subsection). In addition, since $\nabla \xi \cdot b = \frac{2k(k^2+1)}{R} \psi_0$, input-output linearization [Eq. (12)] holds on $L=\{\psi \in \mathbb{R}^3; \psi_0 \neq 0\}$. As shown in Fig. 3, the internal dynamics [19] of system [Eqs. (10)–(16)] for $r=r_0=\text{constant}$ evolves on a surface $T = \{\psi \in L; \xi = r_0\}$.

B. Mixing

Now we consider mixing in the array of vortices [Eq. (10)–(16)]. In particular, we study stirring protocols such that the viscous dissipation $\langle \epsilon \rangle$ is targeted at

$$r(t) = D(t + t_0)^{\alpha - 1},$$  \hspace{1cm} (18)

where the coefficient $D$ is related to the stirring strength, $\alpha$ is the scaling exponent ($0 < \alpha \approx 1$), and $t_0$ a time offset ($t_0=1$). Physically, target [Eq. (18)] corresponds to time-decaying dissipation. But in contrast to the several studies on freely decaying flows [36,37], here the driving force $F(t)$ is dynamically adjusted by the controller [Eq. (12)] so that $\langle \epsilon \rangle$ converges to the power law [Eq. (18)]. In this sense, $\langle \epsilon \rangle = D(t + t_0)^{\alpha - 1}$ is equivalent to

$$\langle \epsilon \rangle \approx D \alpha (t + t_0)^{\alpha},$$  \hspace{1cm} (19)

since Eq. (2) allows control of mixing via entropy tracking.

In order to characterize the mixing dynamics in terms of $D$ and $\alpha$, we perform numerical simulations as follows. First, we fix the model parameters at representative values, namely: $k=1$ and $R=10$. Then, we discretize the physical space $(x,y)$ in an $200 \times 100$ grid. At initial configuration, we consider a horizontal layer of black fluid occupying the central third of the grid while the remaining area (bottom and top) is filled with white fluid. In this setting, imaging of the mixture is performed via forward advection and the mixing number [Eq. (3)] is computed as function of $D$ and $\alpha$. 

1. Role of the coefficient $D$

To reveal the dependence of the mixing dynamics on $\langle \epsilon \rangle$, we fix the scaling exponent at $\alpha=1$ and compute the mixing number [Eq. (3)] for increasing values of $D$. As shown in Fig. 4, increasing $D$ leads to faster mixing. This result supports the notion that entropy and mixing are closely related. Furthermore, note that the curves tend to collapse on a minimum mixing number $m=6 \times 10^{-3}$. In this ultimate regime, the time series become indistinguishable from each other due to the spatial resolution of the mixing number.

Clearly, mixing tends to evolve faster for increasing $\langle \epsilon \rangle = D$. But what is the connection between the viscous dissipation rate and the mixing rate? To answer this question, we introduce a mixing time $t_m$ defined as the instant at which the
average distance between two species of the mixture has halved. In terms of the mixing number this corresponds to $m(t_m)/m(0)=0.25$ (see Fig. 4).

Figure 5 shows that $t_m\sim\langle \epsilon \rangle^{-0.51}$. Such scaling is surprisingly robust. Further numerical simulations indicate that the exponent $-0.51(1)$ remains basically unchanged for ratios $0.1\leq m(t_m)/m(0)\leq 0.8$.

On dimensional grounds, the result of Fig. 5 may be written as

$$t_m = \alpha \sqrt{\frac{v}{\langle \epsilon \rangle}}, \quad (20)$$

where $\alpha$ is a dimensionless coefficient. This simple analysis supports our numerical result and evidences that the mixing dynamics is related to velocity gradients.

2. Role of the exponent $\alpha$

Now we characterize the mixing dynamics in terms of $\alpha$. In particular, we fix the stirring strength at $D=1$ and compute the mixing number as function of: (i) time, (ii) entropy, (iii) viscous dissipation, and (iv) control effort.

To begin, note that the larger the exponent $\alpha$ in Eq. (19), the larger the entropy production $d\langle s \rangle/dt$ in the flow. This suggests that mixing should be enhanced for increasing $\alpha$.

Indeed, this is the case. Figure 6 shows that the time series $m(t)$ decays faster as $\alpha$ is increased from 0.2 to 0.8.

To further quantify the decaying dynamics, we plot $m$ as function of $\langle \epsilon \rangle$. As shown in Fig. 7, the larger the entropy the better the mixing. But here the $\alpha$-dependence is more subtle: for a given value of $\langle \epsilon \rangle$, the mixing number is smaller for decreasing $\alpha$ (compare, for instance, the curves for $\alpha=0.2$ and $\alpha=0.8$). Such trend reveals that just the magnitude of the entropy is insufficient to assure the mixing quality; the temporal dynamics of $\langle \epsilon \rangle$ is also crucial for achieving low values of $m$.

The findings above may be complemented by plotting $m$ as function of the viscous dissipation $\langle \epsilon \rangle$. As shown in Fig. 8, at $\langle \epsilon \rangle=1$ [i.e., at $t=0$, according to Eq. (18)], the normalized mixing number is equal to 1 for all values of $\alpha$. Then, as $\langle \epsilon \rangle$ decays [again, cf. Eq. (18)], $m$ experiences a transient and eventually a monotonic decrease. Comparing the curves at $\langle \epsilon \rangle=0.5$, for instance, one infers that the mixing performance is improved for increasing $\alpha$. This trend is physically reasonable in agreement with the limiting case $\alpha=1$ (cf. Sec. IV B 1). In addition, Fig. 8 shares a common feature with Figs. 6 and 7, namely: a glitch around $m(t)/m(0)=0.05$. Such lethargic interval reflects suppression of the higher modes ($\psi_1$ and $\psi_2$) and dominance of the main flow ($\psi_0$) in the final stage of the mixing process (see Fig. 1).
hydrodynamic [Eq. (2)] and statistical [Eq. (3)] aspects of the phenomenon.

For the array of vortices modeled by [Eqs. (4)–(7)], we succeeded to design a feedback controller via input-output linearization [Eqs. (10)–(16)]. Moreover, we found that the mixing time for statistically stationary viscous dissipation scales as \( t_m \sim \langle \epsilon \rangle^{1/2} \). Nevertheless, it remains to be seen whether this relation is flow specific or somehow more general. Analysis of mixing time series from other models could clarify this issue.

In any scenario, the main conclusion is that the study of stirring protocols based on statistical properties of \( \langle s \rangle \) and \( \langle \epsilon \rangle \) may indeed contribute to a better understanding of the mixing dynamics. From this perspective, the present work could be extended in several ways. For instance, instead of specifying \( \langle \epsilon \rangle \) as statistically stationary or as a statistically decaying power law, one may consider other functions of time that enhance or suppress mixing.

Finally, notwithstanding the intrinsic aesthetics of fluid mixtures in nature and technology, nonlinear control of mixing remains a tremendous challenge from the experimental standpoint.

ACKNOWLEDGMENTS

We thank the referees for the constructive suggestions. This work was supported by the TU/e stimulation program on “Fluid and Solid Mechanics.”