Implementation of self-synchronizing Huijgens’ pendula

B.P. van den Elshout
DC 2010.035
Voor het licht wat nog steeds tweemaal zo sterk brandt...
Abstract

Synchronization is an essential fact of life and is a phenomena which appears in thousands of situations, starting at an atomic level and ending in the far reaches of the cosmos.

Synchronization is a complex dynamical mechanism and a better understanding of the mechanism is necessary to influence these processes. According to all literature Christiaan Huijgens was the first one who observed and described such a synchronizing phenomena and many people performed studies and research in this area after Huijgens. This thesis is also inspired by Huijgens’ observations and presents new experimental results in this area. These experiments are executed on an existing experimental setup at the Eindhoven University of Technology.

This research project has four main goals:

- The design and implementation of a robust, observer based, structure for system identification
- Implementation of Huijgens like dynamics on the experimental setup
- Implementation of energy based controllers for Huijgens’ synchronization
- Get more insight in the principles of mechanical, mutually coupled, synchronization

All goals are realized using the fully actuated experimental setup which consists of two oscillators connected to a beam by leaf springs and the beam itself is connected by leaf springs to the fixed world. Due to mechanical inaccuracies like damping and non linearities it is hardly possible to show self synchronization phenomena on this setup. Therefore it is necessary to robustly identify all system dynamics whereafter these dynamics can be modified in such a way that self synchronization takes place.

The thesis starts by presenting and analyzing, on simulation level, a robust identification strategy to fully identify and compensate the dynamics of the experimental setup whereafter new specified dynamics can be introduced on the experimental setup. This strategy makes it possible to implement several synchronizing scenarios on the setup like the “synchronizing rotating discs” and makes it possible to adjust the parameters of these dynamics without making physical adjustments to the experimental setup and even do this during an experiment.

The last part of this thesis implements and validates the cancelation strategy on the experimental setup to fully cancel all dynamics, to decouple the inputs of the setup and to plug in new dynamics. The presented experiments can be divided in two parts. The first part shows the results for a stand alone linear as well as nonlinear driven oscillator, with the beam fixed in the origin. These designed oscillators are used in the second part for two types of synchronization experiments. The first one shows linear translational synchronization, the second one shows non linear rotational synchronization. For both scenarios a proof is given for the stability of the anti-phase synchronization manifold under certain conditions.

All experiments were executed successful what creates the opportunity to start research projects to gain more insight in synchronizing phenomena and use the experimental setup for demonstration purposes.
Samenvatting

Synchronisatie is een eerste levensbehoeftte, het houdt ons namelijk in leven! Per seconde synchroniseren ongeveer 10000 hartcellen in de sinus knoop om zo voldoende spanningspuls te genereren om te hartspier samen te laten trekken. Synchronisatie is een fenomeen wat zich op duizenden vlakken voltrekt, van een atomair niveau tot diep in de kosmos.

Synchronisatie is een complex dynamisch proces en meer inzicht hierin is noodzakelijk om deze processen te beïnvloeden. De eerste observaties en beschrijvingen van synchronisatie zijn volgens de literatuur gedaan door Christiaan Huijgens waarna vele in zijn voetsporen zijn getreden en onderzoek op dit vlak gedaan hebben, zo ook dit onderzoek. Dit onderzoekt presenteert nieuwe experimentele resultaten van synchroniserende dynamische systemen, waaronder een reconstructie van Huijgens’ experimenten. De resultaten zijn behaald op een al bestaande experimentele opstelling op de Technisch Universiteit Eindhoven.

Dit onderzoek heeft vier hoofddoelen:

- Het ontwerpen en implementeren van een robuuste structuur voor systeem identificatie
- Implementeren van diverse synchroniserende dynamische systemen, waaronder Huijgens’ dynamica
- Implementeren van, op energie gebaseerde, controllers ter vervanging van escapement mechanismen
- Meer inzicht verkrijgen in de verschillende synchronisatie regimes

Alle resultaten zijn behaald op de experimentele opstelling welke uit twee oscillatoren en een balk bestaat. De oscillatoren zijn met bladveren aan de balk bevestigd en de balk met bladveren aan de vaste wereld. De opstelling is volledig geactueerd. Door mechanische onvolkomenheden bevat de opstelling veel demping en niet lineaire eigenschappen waardoor zelf-synchronisatie niet optreedt. Om dit toch mogelijk te maken is het noodzakelijk alle dynamica te identificeren en deze vervolgens zodanig aan te passen dat zelf-synchronisatie wel zal optreden.

Het eerste deel van dit rapport beschrijft en analyseert op simulatie niveau een robuuste identificatie methode om zo alle dynamica van de opstelling te kunnen compenseren en nieuwe dynamica te kunnen introduceren. Deze methode maakt het mogelijk verschillende typen synchroniserende systemen te implementeren zonder fysieke aanpassingen te doen en zelfs om tijdens een experiment de parameters van het systeem te wijzigen.

In het tweede deel van het rapport wordt de ontwikkelde methodiek op de opstelling toegepast en gevalideerd. De eerste experimenten tonen (niet) lineaire stand alone aangedreven oscillatoren waarbij de balk gefixeerd is. De tweede set toont deze zelfde oscillatoren maar dan gebruikt in twee synchroniserende systemen, een transiterend lineair systeem en een roterend niet lineair systeem. De stabiliteit van de optredende anti-phase synchronisatie is bewezen onder bepaalde omstandigheden.

Alle experimenten zijn succesvol uitgevoerd waarmee het mogelijk is geworden om bijvoorbeeld Huijgens’ synchronisatie in de praktijk te tonen en hier verder onderzoek naar te doen om meer en betere inzichten te verkrijgen in synchronisatie.
Contents

Abstract v
Samenvatting vii
Preface xi

1 Introduction 1
1.1 Synchronization, where Huijgens started 2
1.2 Problem formulation 3
1.3 Report outline 3

2 Huijgens’ synchronization 5
2.1 Synchronization concept 5
2.1.1 Synchronization requirements 5
2.1.2 Definitions and terminology 7
2.2 Previous research inspired by Huijgens 8

3 Experimental setup, model description and the design of a robust structure for identification purposes 11
3.1 Setup and model description 11
3.2 Structure to identify and modify 1DOF system dynamics 13
3.2.1 Sliding mode identifier design 16
3.2.2 Observer design 17
3.3 Identification and modifying the full system dynamics 23
3.3.1 Concept 23
3.3.2 Sliding mode observer 24
3.3.3 Luenberger observer design 25
3.3.4 Input decoupling 26
3.3.5 Simulation results 26
3.3.6 Robustness 27
3.4 Conclusions 27

4 Self sustained oscillators 29
4.1 Driving mechanism 29
4.2 Linear oscillator 29
4.2.1 Controller design 30
4.2.2 Robustness 30
4.2.3 Experimental results 31
4.3 Huijgens’ oscillator 32
4.3.1 Controller design 32
4.3.2 Parameter tuning and coordinate transformation 33
4.3.3 Experimental results and fine tuning of the cancelation mechanism 34
4.4 Conclusions 36
Preface

This thesis presents the results of my graduation project at the Dynamics and Control group at the Eindhoven University of Technology. My graduation project deals with the design and implementation of a system identification mechanism with the final goal to show a replication of Huijgens’ synchronization experiments. The first part of the report describes this identification methodology theoretically and the second part shows the implementation of this methodology and presents the results of the replication of Huijgens’ experiments.

I would like to thank all people who made my graduation project possible, especially the members of my graduation committee: prof. dr. H. Nijmeijer, dr. A.Yu. Pogromsky, dr. ir. R.H.B. Fey and dr. ir. P. Heuberger.

Special thanks to my supervisor, prof. dr. H. Nijmeijer. I think you had a hard time with me sometimes and especially in the first phase of the graduation project. Thanks a lot for the commitment and the second change you gave me in september 2009!

Unas palabras de agradecimiento a Jonatan. Yo he apreciado nuestro colaboración y como me has supportado este período pasado. Tu estás un professor simpático. Además estimó mucho tu carácter y personalidad.

Thanks to all people who contributed to my graduation project by supporting me. In particular, and in random order, I would like to thank: My father, my girlfriend Sandra and Lourdess for their continuous support and love. Although I know you always believed that I was able to finish my project it will be a big relief for all of you!

Last but not least...... Mom. Thanks for everything you did for me to make it possible to finish my Masters. I know you would have been really proud on me. Although you are not here any more I’m convinced that you followed everything I did and you saw my renewed enthusiasm for my master thesis project!

Ben van den Elshout, June 2010
Chapter 1

Introduction

Synchronization: One of the most deeply rooted and pervasive drives in nature. Where the laws of entropy state a naturally increase of chaos, sync results in the opposite: spontaneous order \[25\] (Steven Strogatz, 2003)

Sync is everywhere, from an atomic scale to the far reaches of the cosmos. First of all (and quit important): Sync keeps you alive! The sinoatrial node, the pacemaker of your heart, consists of approximately 10000 synchronizing cells that generate the electric impulse to trigger your cardiac contraction! On the contrary, sync can cause serious trouble like epileptic syndromes, when millions of brain cells discharge in sync. Sync can protect individuals: swarms of birds and fishes sync to protect themselves for predators. The swarm creates the possibility to communicate, waves can propagate within the swarm to send signals over a large distance to an other swarm.

Sync means fun: dancing, singing and running together or running while listening to your iPod but again sync can cause trouble. For example: you are running while listening to music. Probably you noticed that you, at least try to, adjust your running rhythm to match with the beat of the music. A music beat close to your own running rhythm feels comfortable but when the difference starts to increase it feels more uncomfortable. At a certain beat, near twice your walking rhythm, running gets comfortable again. Your body is capable to, effortless, adjust its running rhythm within a certain margin and when exceeding this margin, your body needs more effort to match the rhythms. This is a typical example of uni-directional synchronization based on weak coupling. The multi-directional variant can be running together: when both individual running rhythms are close to each other they will sync at a sort of mean rhythm but when the rhythms differ to much it feels uncomfortable to sync, although you want to! The example can be extended to more syncing objects when considering making music in an orchestra. Everybody makes music in its own rhythm and all rhythms are adjusted marginally to meet each other at a common rhythm, they synchronize, possibly influenced by a conductor.

This kind of rhythm adjustments occurs in thousands of different situations.
1.1 Synchronization, where Huijgens started

Approximately 350 years ago, 1650, Christiaan Huijgens, a Dutch scientist, made a remarkable observation. According to all literature found on synchronization, Huijgens was the first one who observed and described synchronization between two, mechanically coupled, oscillating objects. In his situation the oscillating objects were pendulum clocks which are mechanically connected by their cases via a supporting beam. Figure 1.3 shows a drawing of the situation. Appendix A gives a detailed description of the pendulum clock principles. When the motion of one, or both, of the clocks was disturbed the anti-phase synchronous motion returned in a short time. It happened always, so it was not a coincidence. The synchronization took place despite the fact that the clocks were not exactly similar to each other and they did have a slightly different period due to mechanical differences. Apart of anti-phase synchronization Huijgens observed in-phase motion when he changed the distances between the coupled clocks.

Synchronization is a phenomenon which occurs in many situations where objects are coupled (interacting with each other). When this coupling is weak, both systems influence each other, one can think about synchronization in electrical systems (Chua circuit), mechanical systems (pendula clocks) and biological systems (flashing of fireflies and neural networks). When the coupling is strong, resulting in a master-slave configuration, one can find this type of synchronization in observer designs, communication channel synchronization (handshaking in a network) and for example running on music. Lots of research is done in the last category of synchronization but the first one is more delicate and less research is done over here. Due to the frequent appearance of mutually coupled synchronization it is important to get more insight in why, and under which circumstances, this type of synchronization takes place.
1.2 Problem formulation

To gain more insight in Huijgens’ synchronization an experimental setup was designed by Tillaart [26] and tested by Rijlaarsdam [22] to expand the gained synchronization knowledge of Oud [15] on a more sophisticated setup. Several synchronization experiments were carried out on the experimental setup by Rijlaarsdam but not all of them were successful, mainly because of model identification problems. The experimental setup, detailed described in chapter 3, consists of two translational oscillators coupled by a rigid beam. Both oscillators and the beam can be actuated by external forces. This research project has four main goals:

- The design and implementation of a robust, observer based, structure for system identification
- Implementation of Huijgens like dynamics on the experimental setup
- Implementation of energy based controllers for Huijgens’ synchronization
- Get more insight in the principles of mechanical, mutually coupled, synchronization

1.3 Report outline

The first part of this report, chapter 2 and 3, gives a brief explanation of synchronization concepts and provides a state of the art overview of all research inspired by Huijgens’ observation. Chapter 3 presents the experimental setup used for the experiments in this thesis and this chapter also presents the concept to modify the dynamics of the experimental setup to make it possible to introduce new specified dynamics for a linear translational oscillator and for nonlinear rotational pendula.

The idea behind the approach is to cancel all dynamics whereafter specified dynamics are introduced. The second part, chapter 4 to 6, uses the specified dynamics concept to implement different types of oscillators and different types of synchronization regimes on the experimental setup. These chapters present a combination of simulations and experimental results to validate the theory formulated in the first part. The report ends up with conclusions and recommendations.

The research project is done on the Eindhoven University of Technology in the Dynamics and Control group of the Mechanical Engineering department under supervision of prof. dr. H. Nijmeijer.
Chapter 2

Huijgens’ synchronization

The synchronization phenomena observed and described by Huijgens in the "Horoloqium Oscilatortium" [11] was in and anti-phase synchronization between two, mutually coupled, self sustained oscillators. Firstly this chapter gives a conceptual idea of synchronization whereafter these general concepts are mapped against literature which is inspired by Huijgens’ observations.

2.1 Synchronization concept

Synchronous comes from: συν: syn, meaning “the same” or “common” and χρνoς: chronous, meaning “time”. So synchronizing objects show the same behavior in time. Pikovsky [18] gives a definition for synchronization which clearly matches Huijgens’ descriptions:

“synchronization is the adjustment of rhythms of oscillating objects due to interaction”

In this context oscillating objects can be anything represented by the following examples:

• The menstruation cycles of women, living together in one house, synchronizes.
• The flashing of male fireflies in nature synchronizes under certain circumstances.
• People who are clapping in a room synchronize.
• Synchronization of pendulum clocks.

The interaction between the objects in the examples is totally different: mechanical, biological or chemical, but all examples have two things in common:

• All objects are acting autonomous with a certain periodicity.
• All objects are weakly coupled with each other.

These aspects are described in detail for pendulum like mechanical systems in the following sections, as this fully fits within the framework of this report.

2.1.1 Synchronization requirements

Self sustained oscillator

It is important to notice the difference between synchronization and resonance. Resonance can look like synchronization but it is not. Synchronization is a complex dynamical process which requires an oscillating object to be self sustainable and autonomous. The object needs an internal source of energy to keep its movement going and needs to have a robust limit cycle. According to these requirements a free moving pendulum is not a self sustained oscillator, but a driven pendulum is. The driven pendulum can be
approximated by an other example of a self sustained oscillator: the "Van der Pol" oscillator [28] described by the following equation of motion:

$$\ddot{x} = \mu (1 - x^2)\dot{x} - x,$$

(2.1)

where $x$ denotes the position, $\dot{x}$ the velocity and $\mu$ denotes a nonlinear damping parameter. Choosing $\mu = 0.01$ results in a stable limit cycle for the oscillator, independent of the initial conditions, which approximates the sinusoidal behavior of a pendulum clock. This limit cycle is depicted in the phase plane in the center and left plot of Figure 2.1 for two different initial conditions. In contrary to the Van der Pol oscillator, which starts from any initial condition except for the origin, a typical property of the conventional pendulum clock is that it needs a certain amount of startup energy to reach the described limit cycle. This area is depicted in Figure 2.1 by the dotted circle. Starting within this dashed circle the pendulum damps out and model (2.1.1) is replaced by:

$$\ddot{x} = -\dot{x} - x.$$

(2.2)

In the steady state situation the oscillator moves over the limit cycle with a certain frequency and the position on the limit cycle is defined as the “phase” of the oscillator and has a period of $(2\pi)$. The phase of the oscillator needs to be able to move freely along the limit cycle to make synchronization possible. This results in restrictions on the design of the self sustained oscillator, described in chapter 4.

### Coupling type

A categorization is needed when analyzing synchronizing systems. The categorization does not say anything about the type of objects in the system or their behavior, it only describes the way objects in the system are interacting. Four different configuration types can occur:

**Not coupled**  The obvious case where the objects do not influence each other

**Mutually coupled**  Bidirectional coupling between objects. A typical example is Huijgens’ synchronizing pendulum clocks. Both objects influence each other but non of them is master. A lot of variants can be found on mutually coupled objects where the balance and strength of the coupling varies from weakly coupled to almost master slave coupled.

**Master slave coupled**  A unidirectional coupling between objects. A typical example is given by Pecora Carrolls [17] observer technique.

**Statically coupled**  Objects can be seen as dynamically one object. Figure 2.2 shows an example of statically coupled pendula.

For the synchronization phenomena we are discussing in this thesis only the weak mutually coupled category of systems is of interest.

---

Figure 2.1: Oscillator limit cycles using Van der Pol oscillator according to (2.1.1) using $\mu = 0.01$ for the left and center plot. The right plot shows the behavior when no driving mechanism is active within the dashed circle.
2.1. SYNCHRONIZATION CONCEPT

2.1.2 Definitions and terminology

In [6] (Blekhman et al., 1997) the following more rigorous definition for synchronization is provided:

**Definition 2.1.** (Asymptotic Synchronization [6] (Blekhman et al., 1997))

Given $k$ systems with state $\xi_i \in X_i$ and output $y_i \in Y_i$, $i = 1, \ldots, k$ and given $l$ functionals $g_j : Y_1 \times \cdots \times Y_k \times T \rightarrow \mathbb{R}^l$, where $T$ is a set of common time instances for all $k$ systems and $Y_i$ are the sets of all functions from $T$ into the outputs $Y_i$. Furthermore, defining a shift operator $\sigma_\tau$ such that $\left((\sigma_\tau y_1)(,.), \ldots, (\sigma_\tau y_k)(,.), y\right) \equiv 0 \ \forall j = 1, \ldots, l$ (2.3) is valid for $t \rightarrow \infty$ and some $\sigma_\tau \in T$.

The phase difference between both pendulum-like systems needs to reach a constant value within a margin due to measurement noise, disturbances and system inequalities. Because exact synchronization is not expected to be observed in real life systems Blekhman et al. 1997 provides a definition for approximate asymptotical synchronization:

**Definition 2.2.** (Approximate Asymptotic Synchronization (Blekhman et al., 1997))

Using the notations introduced in Definition 2.1, the solutions $\xi_1(,.), \ldots, \xi_k(,.)$ of systems $\Sigma_1, \ldots, \Sigma_k$ with initial conditions $\xi_{i0}, \ldots, \xi_{ko}$ are called approximately asymptotically synchronized with respect to the functionals $g_1, \ldots, g_l$ if:

$$|g_j(\sigma_\tau y_1(,.), \ldots, \sigma_\tau y_k(,.), y)| \leq \epsilon \ \forall j = 1, \ldots, l$$

(2.4) is valid for $t \rightarrow \infty$ and some $\sigma_\tau \in T$.

Apart of (approximate asymptotic) synchronization a special type of (approximate asymptotic) synchronization so called “anti-phase” (approximate asymptotic) synchronization is of special interest in this report. The following definition is provided for (exact) anti-phase synchronization:

**Definition 2.3.** (Approximate Anti-phase Synchronization [22] (Rijlaarsdam, 2008))

Consider two systems $\Sigma_1$ and $\Sigma_2$ with initial conditions $\xi_{10}$ and $\xi_{20}$ and corresponding solutions $\xi_1(\xi_{10}, t)$ and $\xi_2(\xi_{20}, t)$. Furthermore, assume that both $\xi_1(\xi_{10}, t)$ and $\xi_2(\xi_{20}, t)$ are periodic in time with period $T$. The solutions $\xi_1(\xi_{10}, t)$ and $\xi_2(\xi_{20}, t)$ are called (approximately) asymptotically synchronized in anti-phase if they are (approximately) asymptotically synchronized according to Definition 2.1 or 2.2, with:

$$g(.) = \xi_1(.) - \alpha \sigma_\tau(\xi) \xi_2(.)$$

(2.5)

with $\alpha \in \mathbb{R}_{>0}$ a scale factor and $\sigma_\tau(\xi)$ a shift operator over half an oscillation period.
2.2 Previous research inspired by Huijgens

The last century a lot of people were, and still are, interested in synchronization phenomena. Table 2.1 summarizes all research done in the Huijgens synchronization area and some variants on this. Some variants ([29], [21], [2], [10]), clustered at the bottom part of the table, use a model with spring-coupled pendula, in stead of beam-coupled pendula.

Half a century after Huijgens’ observations, John Ellicott [9] made more or less the same observation but he called this observation “communication of motion”, in stead of synchronization. He did all kind of experiments mainly resulting in the fact that the “communication of motion” depends on the way the clocks were coupled. Korteweg [13] was the first one in the beginning of the 20th century who was mathematically describing the synchronization phenomena (without any driving and damping in the model) whereafter Blekhman [5] in 1988 examined experimentally and theoretically a synchronizing system and comes to the conclusion that two synchronization states can occur, in and anti-phase.

Recently five experiments are done by Bennett et al. [4], Pantaleone [16], Senator [24], Oud [15] and Rijlaarsdam [22] to re-examine and/or reproduce Huijgens’ original results. Bennett, Pantaleone and Oud used 3DOF systems containing two, by their cases, coupled oscillators. Senator used a more complicated model. Blekhman, Pantaleone, Senator as well as Oud observed “in” and “anti” phase synchronization while Bennett only observed anti-phase synchronization. Main difference between the developed theories is how the authors handle energy dissipation and energy supply in their clock models and especially the escapement mechanism models.

The overview in Table 2.1 shows that a lot of effort is done on controlling Huijgens’ synchronization and its spring coupled variant but also hardly shows any real life experiments done with a control focus and hardly any analysis is done based on a linear Huijgens-like system. Using the fully actuated experimental setup described in the next chapter, and in detail by Rijlaarsdam [22], gives to possibility to implement and analyse any type of observer and controller. An other advantage of the fully actuated system is that the dynamical properties of the system can be cancelled and specified dynamics can be introduced. Using these specified dynamics the experimental setup can be used to simulate more Huijgens-like synchronization phenomena like the “synchronizing rotating discs” or “synchronizing Duffing oscillators”.

As Huijgens observed in his experiments that the distance between the pendula plays a key role in the type of synchronization occurring, the beam could not be considered as a rigid body. To allow to include the distance between the pendula the beam should be considered more realistic as an infinite dimensional system described by partial differential equations. Dilão [8] introduces this concept where the pendula are coupled by a beam which is divided in two parts coupled by a spring-damper combination.

For now, this thesis uses the rigid beam approach according to most of the models presented in Table 2.1.
### Table 2.1: Overview of all literature inspired by Huijgens’ synchronization.

<table>
<thead>
<tr>
<th>Author</th>
<th>Year</th>
<th>Focus</th>
<th>Coupling</th>
<th>Type</th>
<th>Drive</th>
<th>Oscil type</th>
<th>Drive</th>
<th>Sync type</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellicott [9]</td>
<td>1739</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pikovsky [18]</td>
<td>2001</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pantaleone [16]</td>
<td>2002</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rijaardam [22]</td>
<td>2008</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓ d</td>
<td></td>
</tr>
<tr>
<td>Dilao [8]</td>
<td>2009</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zheng [29]</td>
<td>2000</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>Distributed frequency analysis</td>
</tr>
<tr>
<td>Quinn [21]</td>
<td>2000</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>Distributed frequency analysis</td>
</tr>
</tbody>
</table>

- Internal stable typ of oscillator
- Escapement replica
- Hamiltonian and energy like functions
- Using virtual dynamics to implement several types of synchronizing systems
- Beam is divided in two parts and not treated as a fully rigid body
- Using virtual dynamics to implement several types of synchronizing systems
Chapter 3

Experimental setup, model description and the design of a robust structure for identification purposes

This chapter gives a brief explanation of the experimental setup used for the implementation of several synchronizing systems, presents the model of the experimental setup used for all experiments presented in this thesis and presents a concept to use desired dynamics on the experimental setup.

3.1 Setup and model description

Figure 3.1: Photograph of the experimental setup in the DCT lab at the Eindhoven University of Technology.

The setup as depicted in Figure 3.1 is a fully translational version of the original Huijgens’ clocks. The “blue” and “red” triangles are the translational oscillators connected to the supporting, translational, beam by leaf springs. The beam is connected to the fixed world by leaf springs as well. The construction restricts all degrees of freedom and only allows one degree of freedom in the translational direction for each
oscillator. The stroke along this degree of freedom for both oscillators and beam is approximately 1 [cm].
A disadvantage of the leaf springs is the non linear behavior for large displacements, presented in Figure 3.2. The experimental setup is actuated by amplifier-voice-coil combinations which have a limited input

![Stiffness characteristics](image)

**Figure 3.2:** Stiffness characteristics of both oscillators and beam

of \( u_i(\text{max}) = \pm 0.42 \ [V] \) corresponding to a maximal force of \( F_i(\text{max}) = \pm 9 \ [N] \) for both oscillators and \( F_3(\text{max}) = \pm 0.5 \ [N] \) for the beam. The positions of both oscillators and the supporting beam are measured, the velocity needs to be derived out of the position signals. A TUeDACS system is used as interface to the system. A detailed description of all used separate hardware components can be found in Rijlaarsdam [22].

Figure 3.3 presents the experimental setup schematically. The masses are given by \( m_i \in \mathbb{R}_{>0}, i = 1, 2, 3 \), the displacements by \( x_i \in \mathbb{R} \), \( \beta_i : \mathbb{R} \rightarrow \mathbb{R} \) and \( \kappa_i : \mathbb{R} \rightarrow \mathbb{R} \) represent the non-linear damping and stiffness respectively. The equations of motion are derived by the Euler-Lagrange method where the displacement

![Schematic representation](image)

**Figure 3.3:** Schematic representation of the experimental setup.

vector is defined with respect to the fixed world. Appendix B.1 shows the derivation in detail for the linear approximation of the system resulting in the following equations of motion:

\[
\ddot{x}_1 = -\omega_1^2 \Delta x_1 - 2\zeta_1 \omega_1 \dot{x}_1 + c_1 u_1(t)
\]

\[
\ddot{x}_2 = -\omega_2^2 \Delta x_2 - 2\zeta_2 \omega_2 \dot{x}_2 + c_2 u_2(t)
\]

\[
\ddot{x}_3 = \sum_{i=1}^{2} \mu_i \left[ \omega_i^2 \Delta x_i + 2\zeta_i \omega_i \dot{x}_i - c_i u_i(t) \right] + \omega_3^2 x_3 - 2\zeta_3 \omega_3 \dot{x}_3 + c_3 u_3(t),
\]

where \( \Delta x_i = x_i - x_3 \ [m] \), \( \omega_i = \sqrt{\frac{k_i}{m_i}} \ [rads^{-1}] \), \( \zeta_i = \frac{d_i}{2\omega_i m_i} \ [-] \) and \( \mu_i = \frac{m_i}{m_3} \ [-] \) for the subsystems \( i = 1, 2 \) are the displacement with respect to the beam, the undamped eigenfrequency, the dimensionless...
3.2 STRUCTURE TO IDENTIFY AND MODIFY 1DOF SYSTEM DYNAMICS

The states $x_1$ and $x_2$ correspond to the positions of the “blue” and “red” oscillator respectively, $x_3$ corresponds to the position of the “beam” and $\omega_3 = \sqrt{\frac{k_3}{m_3}} \text{ [rads}^{-1}] \text{ and } \zeta_3 = \frac{d_3}{2\omega_3 m_3} [-]$ represent the undamped eigenfrequency and dimensionless damping of the beam respectively. Table 3.1 presents the numerical values of all parameters. The system inputs are represented by $u_i \text{ [V]}$ and the motor constants by $c_i \text{ [ms}^{-2}\text{V}^{-1}]$. The damping and stiffness are linear approximated by the parameters $d_i \text{ [Nsm}^{-1}] \text{ and } k_i \text{ [Nm}^{-1}]$ respectively what causes that the model does not capture all system dynamics any more.

Table 3.1: Parameter values for the experimental setup according to model (3.1).

<table>
<thead>
<tr>
<th>Blue oscillator (i=1)</th>
<th>Red oscillator (i=2)</th>
<th>Supporting beam (i=3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_i$</td>
<td>12.5521</td>
<td>14.0337</td>
</tr>
<tr>
<td>$\zeta_i$</td>
<td>0.3362</td>
<td>0.4296</td>
</tr>
<tr>
<td>$\mu_i$</td>
<td>0.0411</td>
<td>0.0578</td>
</tr>
<tr>
<td>$c_i$</td>
<td>20.9218</td>
<td>23.2465</td>
</tr>
</tbody>
</table>

Although the system description is quite similar to the description in Rijlaarsdam [22] an essential difference is made in the treatment of the input signals and their conversion to force via the motor constants and coupling strength. The following equations present this difference where (3.2) presents the resulting input force on the beam according to the model of Rijlaarsdam and (3.3) shows this force according to the model presented in this report.

$$F_{beam \ input \ Rij} = c_3(u_3 - u_1 - u_2) \quad (3.2)$$

$$F_{beam \ input \ Els} = c_3u_3 - c_1\mu_1u_1 - c_2\mu_2u_2 \quad (3.3)$$

The goal of the experimental setup is to reconstruct Huijgens’ synchronizing experiments and to implement other synchronizing variants. Under ideal circumstances and without mechanical inaccuracies the experimental setup is able to show self synchronization. Due to mechanical inaccuracies, both oscillators are not identical and a lot of damping is present in the experimental setup, no synchronization appears. The systems’ dynamics need to be modified.

3.2 Structure to identify and modify 1DOF system dynamics

The approach chosen in this report to modify the system dynamics is to firstly cancel all internal dynamics of the experimental setup whereafter new dynamics are introduced on the system, the specified dynamics. These dynamics can be obtained from any oscillating system for example: identical linear oscillators without damping (described in chapter 5), or more complicated like Huijgens’ dynamics in combination with translational to rotational coordinate transformations (described in chapter 6). This section describes, in particular for a 1DOF system, the method used to identify almost all internal dynamics, how to cancel these dynamics and how to introduce specified dynamics. The presented results are generated on a single stand alone oscillator on the experimental setup using a fixed beam situation. The next section handles the three degree of freedom identifying and specified dynamics.

As stated, this section describes the identification process for a single degree of freedom. The beam is fixed in the experimental setup causing the internal dynamics (3.1) to reduce to:

$$\ddot{x}_i = f_i(x_i, \dot{x}_i) + c_iu_i, \quad (3.4)$$

where $f_i$ represents the internal system dynamics, ie. stiffness and damping, $c_i$ the motor constant and $u_i$ the input voltage on the system where $i = 1$ represents the “blue” oscillator and $i = 2$ the “red” one. Consider the structure in Figure 3.4 where we consider only a single degree of freedom under
ideal circumstances without unknown dynamics. As stated in the previous chapter, only the position state variable \( x_i \) is measured and since measurement noise \( n_i \) is present this results in the measured position signal \( x_{im} \). Since we do not have access to the complete state a full state observer is needed to reconstruct the system states resulting in \( \hat{x}_i \) and \( \dot{\hat{x}}_i \) as the observed position and velocity respectively. The added value of the observer on the observed position signal is to reduce the noise \( n_i \) in the measured signal \( x_{im} \). The design of the “state observer” is discussed later but assume for the moment that the “state observer” is able to reconstruct the system states. From now on the observed states are used in stead of the measured position signal.

The “cancelation” block is assumed to have the inverse dynamics of the original setup and can be defined as:

\[
M_i = -f_i(\hat{x}_i, \dot{\hat{x}}_i),
\]

and the “specified dynamics” block contains the specified dynamics \( D_i \) for the system resulting in the following closed loop dynamics:

\[
\ddot{x}_i = D_i(\hat{x}_i, \dot{\hat{x}}_i) + f_i(e_i, \dot{e}_i) + c_is_i,
\]

where \( f_i(e_i, \dot{e}_i) = f_i(x_i, \dot{x}_i) - f_i(\hat{x}_i, \dot{\hat{x}}_i) \) represents the remaining cancelation error and \( s_i \) represents an extra input to the system which is used in a later stage to add controller functionalities to the system, for example an initialization controller. The remaining cancelation error is assumed to converge to zero for the moment resulting in the specified closed loop behavior of system:

\[
\ddot{x}_i = D_i(\hat{x}_i, \dot{\hat{x}}_i) + c_is_i.
\]

Not all internal dynamics are known in real life situations, represented in the following extended system dynamics:

\[
\ddot{x} = f_i(x, \dot{x}) + \gamma_i(x, \dot{x}) + c_iu_i,
\]

where \( f_i \) represents the known part of the dynamics and \( \gamma_i \) the non identified part. Applying the cancelation as defined in (3.2) not results in a full cancelation of all dynamics. Apart of this, in the design of the state
observer the influence of $\gamma_i$ is not considered ending up in incorrect observed states. This situation is depicted in Figure 3.5. To identify these unknown dynamics a sliding mode identifier is introduced and assumed is that this identifier fully determines the unknown dynamics $\gamma_i$. The design of the sliding mode identifier is discussed later. The output of this identifier $\hat{\gamma}_i$ can be used to enrich the state observer or to cancel these dynamics in the original system. Both situations are schematically presented in Figure 3.6. The dark-blue area represents the identification error made by the sliding mode identifier defined as: $\epsilon_i = \gamma_i - \hat{\gamma}_i$. Not only the unknown dynamics are identified by the sliding mode observer but it also identifies the system states, denoted by $x_{is}$ and $\dot{x}_{is}$. These states almost perfectly match the states of the experimental setup but contain a lot of high frequent components and therefore are not used in this concept. The right scenario in Figure 3.6 shows the effect when subtracting $\gamma_i$ from the original system resulting in equal dynamics of the system and the state observer. Applying cancelation (3.2), called “a priori cancelation” from now on, on this scenario results in the intended effect of full dynamical cancelation, apart of the remaining error $\epsilon$. Figure 3.7 presents the concept in a blockscheme where the dotted output line of the sliding mode identifier represents both scenarios depicted in Figure 3.6. The left scenario uses the line denoted by “1” and the right scenario uses line “2” in Figure 3.7. The left scenario depicted in Figure 3.6 is used for testing the sliding mode identifier without actuating the system. The right scenario, including a priori masking, is used for all experiments presented in this thesis.
Mathematically the idea is represented by the following equations:

\[
\dot{x}_i = f_i(x_i, \dot{x}_i) + \gamma_i(x_i, \dot{x}_i) + c_i u_i
\]

\[
= f_i(x_i, \dot{x}_i) + \gamma_i(x_i, \dot{x}_i) + c_i \left[ -\frac{1}{c_i} f_i(\hat{x}_i, \dot{\hat{x}}_i) - \frac{1}{c_i} \dot{g}_i(\hat{x}_{is}, \dot{\hat{x}}_{is}) + \frac{1}{c_i} D_i(\hat{x}_i, \dot{\hat{x}}_i) + s_i \right],
\]

what results in the closed loop dynamics:

\[
\ddot{x}_i = D_i(\hat{x}_i, \dot{\hat{x}}_i) + c_i s_i + f_i(e_i, \dot{e}_i) + e_i(x_i, \dot{x}_i),
\]

where \( e_i = x_i - \hat{x}_i \) and \( \dot{e}_i = \dot{x}_i - \dot{\hat{x}}_i \) represent the state observer error. The remaining error from the sliding mode identifier is contained in the term \( e_i(x_i, \dot{x}_i) \) under the assumption that \( x_i = \hat{x}_{is} \) and \( \hat{x}_i = \dot{\hat{x}}_{is} \).

Under ideal circumstances (3.10) reduces to the ideal specified dynamics:

\[
\ddot{x}_i = D_i(\hat{x}_i, \dot{\hat{x}}_i) + c_i s_i.
\]

### 3.2.1 Sliding mode identifier design

The sliding mode identifier presented in [23] is used here to determine the non-identified part of the system dynamics. Starting with the extended model dynamics (3.8) written in state space form results in:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_i(x_1, x_2) + \gamma_i(x_1, x_2) + c_i u_i
\end{align*}
\]

where \( x_1 \) and \( x_2 \) represent \( x_i \) and \( \dot{x}_i \) as position and velocity respectively for a single degree of freedom \( i \).

Defining the sliding mode identifier according to:

\[
\begin{align*}
\hat{x}_{s1} &= \hat{x}_{s2} + k_{i1} e_{s1} \\
\hat{x}_{s2} &= f_i(\hat{x}_{s1}, \hat{x}_{s2}) + k_{i2} e_{s1} + k_{i3} \text{sign}(e_{s1}) + c_i u_i,
\end{align*}
\]

where \( k_{i1}, k_{i2} \) and \( k_{i3} \) are tuning parameters of the sliding mode identifier and \( \hat{x}_{s1} \) and \( \hat{x}_{s2} \) represent the identified position and velocity respectively. The error dynamics become:

\[
\begin{align*}
\dot{e}_{s1} &= e_{s2} - k_{i1} e_{s1} \\
\dot{e}_{s2} &= f_i(e_{s1}, e_{s2}) + \gamma_i(x_1, x_2) - k_{i2} e_{s1} - k_{i3} \text{sign}(e_{s1}),
\end{align*}
\]
where \( e_{s1} = x_1 - \dot{x}_1 \) and \( e_{s2} = x_2 - \dot{x}_2 \). The stability analysis is based on the theory on “equivalent control” [23] where a coordinate transformation is used according to:

\[
\begin{align*}
v_{s1} &= e_{s1} \\
v_{s2} &= e_{s2} - k_1 e_{s1}.
\end{align*}
\]  
(3.15)

The sliding surface \( s \) can now be defined according to:

\[
s = v_{s1} = v_{s2} = 0.
\]  
(3.16)

Writing (3.14) in the new coordinates results in:

\[
\begin{align*}
\dot{v}_{s1} &= v_{s2} \\
\dot{v}_{s2} &= f_i(v_{s1}, v_{s2}) - k_2 v_{s1} - k_1 v_{s2} + \gamma_i(x_1, x_2) - k_i sign(v_{s1}),
\end{align*}
\]  
(3.17)

where \( k_1 \) and \( k_2 \) are constants used to guarantee stability and are chosen as indicated in [23]. The identifier (3.17) has a discontinuity surface in \( v_{s1} = 0 \) and the discontinuous term presents a second order sliding mode, moreover, the discontinuous term gives robustness against uncertainties and unmodeled dynamics. By using the equivalent control it is possible to estimate the non identified dynamics and external disturbances. In order to determine the behavior of system (3.17) at the sliding surface \( v_1 = v_2 = 0 \) the equivalent control method is used, hence:

\[
\begin{align*}
\dot{v}_{s1} &= f_i(v_{s1}, v_{s2}) - k_2 v_{s1} - k_1 v_{s2} + \gamma_i(x_1, x_2) - U_{eq-i} = 0, \\
\end{align*}
\]  
(3.18)

implying \( e_{s1} = e_{s2} = 0 \) and \( \dot{x}_s = \dot{x} \). Therefore the equivalent control input is given by:

\[
U_{eq-i} = \gamma_i(x_1, x_2).
\]  
(3.19)

In [27] it is established that the equivalent control coincides with the low frequency content of the discontinuous control. Hence

\[
\lim_{\omega_i \to \infty} U_{eq-i} \approx \bar{k}_i sign(e_{s1}),
\]  
(3.20)

where the upper bar represents a lowpass filtered version of \( k_i sign(e_{s1}) \) by cutoff frequency \( \omega_i \). Due to this filtering information gets lost and to cope with this problem we define:

\[
\bar{U}_{eq-i} = U_{eq-i} + k_i e_{s1},
\]  
(3.21)

where \( k_i e_{s1} \) represents the extra compensation to compensate for the phase lag and information loss appearing by filtering the discontinuous term.

### 3.2.2 Observer design

To reconstruct the state vector a classical Luenberger observer is introduced. The dynamics of the observer are equal to the dynamics of the original system and an extra correction term is introduced to guarantee that the error between the real output and the observed one is asymptotically stable. Starting with system (3.8) in state space form results in:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_i(x_1, x_2) + \gamma_i(x_1, x_2) + c_i u_i,
\end{align*}
\]  
(3.22)

where \( x_1 \) and \( x_2 \) represent \( x_1 \) and \( \dot{x}_i \) for a single degree of freedom \( i \). Defining the observer according to the following state space model:

\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 + k_i e_1 \\
\dot{\hat{x}}_2 &= f_i(\hat{x}_1, \hat{x}_2) + \bar{U}_{eq-i} + c_i u_i,
\end{align*}
\]  
(3.23)
results in error dynamics according to:

\[
\begin{align*}
\dot{e}_1 &= -k_{15}e_1 + e_2 \\
\dot{e}_2 &= f_i(e_1, e_2) + \gamma_i(x_1, x_2) - \tilde{U}_{eq-i} + \text{hfc} \tag{3.24}
\end{align*}
\]

where \( e_1 = x_1 - \hat{x}_1 \) and \( e_2 = x_2 - \hat{x}_2 \). Due to the lowpass filtering of the discontinuous term high frequent components get lost, presented by “hfc”. Assuming \( \gamma_i(x_1, x_2) = \tilde{U}_{eq-i} \) and as long as \( k_{i5} \in \mathbb{R}_{>0} \) the eigenvalues of the error dynamics all have a negative real part hence, the error dynamics are asymptotically stable. The larger \( k_{i5} \) is chosen, the faster the error dynamics converge to zero and the more noise is captured by the observer, due to the higher bandwidth of the observer. During the experiments in chapter 4 the exact tuning is determined for the observer gain for each oscillator to obtain the best results in reality. The parameters used for the state observer and sliding mode identifier for all simulations in this chapter are given in Table 3.2.

### Identifying unknown dynamics

Figure 3.8 depicts simulation results based on the isolated “blue” oscillator. The parameters used for the model are given in chapter 3 where the linear stiffness is replaced by a non-linear function, similar to the linear approximation. The presented simulations use a PID controller to bring the “blue” oscillator to its initial position \( x_{0\text{ blue}} = -5 \text{ [mm]} \) during the first 10 seconds, whereafter the cancelation and specified dynamics are switched on at \( t = 10 \text{ [s]} \). The cancelation contains all linear approximated terms of the model dynamics and the specified dynamics should force an undamped oscillation at \( \omega_s = 2\pi \text{ [rads}^{-1}] \). The results are presented in Figure 3.8 without sliding mode compensation and in Figure 3.9 with sliding mode compensation. Two remarkable things can be seen comparing both figures. During the initialization the model not stabilizes at the correct initial value, where the observer does in Figure 3.8. This is caused by the fact that the PID controller uses the observed signals as reference and the observer gives an incorrect result due to the lack of robustness. For to the same reason the specified dynamics does not exactly behave as it should. Figure 3.9 presents the expected, specified, result. The tracking error is less than 1 percent and can be neglected.

---

Table 3.2: Parameters for sliding mode identifier and state observer tuned in simulation mode.

<table>
<thead>
<tr>
<th>Blue oscillator (i=1)</th>
<th>Red oscillator (i=2)</th>
<th>Beam (i=3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_{11} = 1 )</td>
<td>( k_{21} = 1 )</td>
<td>( k_{31} = 20 )</td>
</tr>
<tr>
<td>( k_{12} = 25 )</td>
<td>( k_{22} = 25 )</td>
<td>( k_{32} = 10 )</td>
</tr>
<tr>
<td>( k_{13} = 0.7 )</td>
<td>( k_{23} = 1 )</td>
<td>( k_{33} = 0.05 )</td>
</tr>
<tr>
<td>( \omega_{c1} = 50\pi \text{ [rad/s]} )</td>
<td>( \omega_{c2} = 50\pi \text{ [rad/s]} )</td>
<td>( \omega_{c3} = 50\pi \text{ [rad/s]} )</td>
</tr>
<tr>
<td>( k_{14} = 1000 )</td>
<td>( k_{24} = 1000 )</td>
<td>( k_{34} = 500 )</td>
</tr>
<tr>
<td>( k_{15} = 1 )</td>
<td>( k_{25} = 1 )</td>
<td>( k_{35} = 10 )</td>
</tr>
</tbody>
</table>

\(^1\text{hfc: High Frequent Components. Appearing due to the low pass filtering of discontinuous term in 3.20}\)
Figure 3.8: Simulation results for isolated “blue” oscillator only using a priori cancelation. The specified dynamics should result in an undamped oscillation of 1 [Hz]. Due to the mismatch between the dynamics of the observer and the model the results are not fully meeting the requirements.

Figure 3.9: Simulation results for isolated “blue” oscillator using a priori cancelation and the cancelation of the identified part of the dynamics. The specified dynamics force an undamped oscillation of 1 [Hz].
CHAPTER 3. EXPERIMENTAL SETUP, MODEL DESCRIPTION AND THE DESIGN OF A ROBUST STRUCTURE FOR IDENTIFICATION PURPOSES

Noise reduction

Another advantage of the observer is that the noise level of the reconstructed states is reduced. Figure 3.10 shows the results of the observer when noise is added to the measured output signal. The figure clearly shows that the signals are still tracked and the noise levels are reduced with approximately a factor 5.

Figure 3.10: Simulation results for isolated “blue” oscillator when adding measurement noise and using the full state observer to reconstruct the states.

Robustness

Considering Figure 3.7 three blocks use the estimated system parameters to determine the control signal to cancel the full dynamics. The results of the used structure are presented in Figure 3.11 when changing the estimated frequency and the estimated damping of the “blue” oscillator with 50%. The figure clearly shows that the control structure is capable to cope with the introduced model inaccuracy. To achieve this result the sliding mode identifier needs more effort resulting in a larger contribution of the sliding mode identifier in the total system input. On the contrary the contribution of the a priori cancelation decreases, finally resulting in approximately the same system input. The effort of the sliding mode identifier and the a priori cancelation can be exchanged as long as the sliding mode identifier is able to capture all unknown dynamics. This phenomena is depicted in Figure 3.13 where the left column shows the control signals for the case that 100 percent of the a priori cancelation is used and the right column shows the case where 50 percent a priori cancelation is used. The last row clearly shows that the final controller signals are behaving the same and the rows above show the difference in the several controller components.

Important for a correct full dynamics cancelation is that the internal dynamics of the sliding mode identifier are equal to the applied a priori cancelation, depicted in Figure 3.12. The experimental setup is represented by a known dynamical part $f_i$ and an non identified part $\gamma_i$. The internal dynamics of the sliding mode identifier are represented by $\tilde{f}_i$. According to the theory on equivalent control the output of the identifier is:

$$U_{eq-i} = \dot{\gamma}_i(\cdot) = f_i(\cdot) + \gamma_i(\cdot) - \tilde{f}_i(\cdot) - \epsilon_i(\cdot),$$  \hspace{1cm} (3.25)

where $\epsilon_i(\cdot)$ represents the identification error. Subtracting $U_{eq-i}$ without a priori cancelation from the original system dynamics results in $\tilde{f}_i(\cdot) + \epsilon(\cdot)$, presented in the center bar of Figure 3.12. To result in full dynamics cancelation, presented in the right bar, the a priori cancelation term needs to be equal to the internal dynamics of the sliding mode identifier. Using for example the correct known dynamics $f(\cdot)$ as a priori cancelation results in the following closed loop dynamics:

$$\ddot{x}_i = f_i(\cdot) + \gamma_i(\cdot) - \dot{\gamma}_i(\cdot) + \epsilon_i(\cdot) - f_i(\cdot) = \tilde{f}_i(\cdot) + \epsilon_i(\cdot),$$  \hspace{1cm} (3.26)

presented in the fourth bar which clearly not meets the cancelation requirement and $\tilde{f}_i(\cdot)$ presents the difference between $f_i(\cdot)$ and $\tilde{f}_i(\cdot)$.
3.2. STRUCTURE TO IDENTIFY AND MODIFY 1DOF SYSTEM DYNAMICS

Figure 3.11: Simulation results “blue” oscillator using sliding mode identifier to capture the error made in the a priori dynamics cancelation. The introduced error is 50% on natural frequency and 50% on the dimensionless damping.

Figure 3.12: Schematically representation of the effect of incorrect a priori cancelation. As long as the a priori compensation equals the internal dynamics of the sliding mode identifier the best compensation result is obtained.
Figure 3.13: Simulation results showing the differences for several controller signal components, depending on the amount of a priori cancelation used. The final composed controller signals are behaving equally.
3.3 Identification and modifying the full system dynamics

The previous chapter describes the design and simulation results of the masking dynamics for a single oscillator. This chapter extends the concept to a full state masking system.

3.3.1 Concept

The concept used for the 3 DOF observer is equal to the concept shown in the previous section. A sliding mode identifier is used to determine three unknown dynamical parts, one for each degree of freedom whereafter these dynamics are subtracted from the experimental setup, according to the right scenario in Figure 3.6. All a priori knowledge is canceled from the experimental setup and the specified dynamics are introduced. The specified dynamics and a priori cancelation use the reconstructed observed states in stead of the measured ones because of the high noise in the measured signals.

Figure 3.14 shows the concept in more detail, where all signals and states are vectors containing the signals for both oscillators and the beam. The design of the “sliding mode identifier” and “observer” blocks is described in sections 3.3.2 and 3.3.3 respectively. Assume for the moment that all unknown dynamics are correctly identified, so: $\hat{\gamma}(.) = \gamma(.)$. The outputs of the sliding mode identifier $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\gamma}_3$ can be (partially) added to the state observer to enrich the state observer with the missing dynamics or can be (partially) subtracted from the experimental setup to cancel these dynamics. All combinations result in equal dynamics for the experimental setup and the state observer, according to the described scenarios in Figure 3.6 for the one degree of freedom situation.

The experimental setup is described in a generalized form by the following set of equations, derived in appendix B.1.1:

$$
\Delta \ddot{x}_1 = \hat{f}_1 + \hat{\gamma}_1 + \hat{\eta}_1 \\
\Delta \ddot{x}_2 = \hat{f}_2 + \hat{\gamma}_2 + \hat{\eta}_2 \\
\Delta \ddot{x}_3 = \hat{f}_3 + \hat{\gamma}_3 + \hat{\eta}_3,
$$

(3.27)

where $\hat{f}_i$, $\hat{\gamma}_i$ and $\hat{\eta}_i$ represent the known dynamics, the unknown dynamics and the inputs for state $i = 1, 2, 3$ respectively.
3.3.2 Sliding mode observer

The design of the sliding mode observer is based on system (3.27) written in state space form according to:

\[\begin{align*}
x_1 &= x_2 \\
x_2 &= f_1 + \bar{\theta}_1 + \bar{\gamma}_1 \\
x_3 &= x_4 \\
x_4 &= f_2 + \bar{\theta}_2 + \bar{\gamma}_2 \\
x_5 &= x_6 \\
x_6 &= f_3 + \bar{\theta}_3 + \bar{\gamma}_3,
\end{align*}\]  
(3.28)

where \(x_1 = \Delta x_1, x_2 = \Delta x_2, x_3 = \Delta x_2, x_4 = \Delta x_2, x_5 = \Delta x_3\) and \(x_6 = \Delta x_3\) according to the definitions in section 3.1. The sliding mode observer equations in state space are given by:

\[\begin{align*}
\dot{x}_{s1} &= \dot{x}_{s2} + k_{11}e_{s1} \\
\dot{x}_{s2} &= f_1(e_{s1}, e_{s2}) + \bar{\gamma}_1 - k_{12}e_{s1} - k_{13}\text{sign}(e_{s1}) \\
\dot{x}_{s3} &= \dot{x}_{s4} + k_{21}e_{s3} \\
\dot{x}_{s4} &= f_2(e_{s3}, e_{s4}) + \bar{\gamma}_2 - k_{22}e_{s3} - k_{23}\text{sign}(e_{s3}) \\
\dot{x}_{s5} &= \dot{x}_{s6} + k_{31}e_{s5} \\
\dot{x}_{s6} &= f_3(e_{s5}, e_{s6}) + \bar{\gamma}_3 - k_{32}e_{s5} - k_{33}\text{sign}(e_{s5}),
\end{align*}\]  
(3.29)

where \(e_{si} = x_i - \hat{x}_i\). The error dynamics of the observer are determined by:

\[\begin{align*}
\dot{e}_{s1} &= e_{s2} - k_{11}e_{s1} \\
\dot{e}_{s2} &= f_1(e_{s1}, e_{s2}) + \bar{\gamma}_1 - k_{12}e_{s1} - k_{13}\text{sign}(e_{s1}) \\
\dot{e}_{s3} &= e_{s4} - k_{21}e_{s3} \\
\dot{e}_{s4} &= f_2(e_{s3}, e_{s4}) + \bar{\gamma}_2 - k_{22}e_{s3} - k_{23}\text{sign}(e_{s3}) \\
\dot{e}_{s5} &= e_{s6} - k_{31}e_{s5} \\
\dot{e}_{s6} &= f_3(e_{s5}, e_{s6}) + \bar{\gamma}_3 - k_{32}e_{s5} - k_{33}\text{sign}(e_{s5}),
\end{align*}\]  
(3.30)

and can be treated as three separate subsystems in pairs of two. The analysis for the first subsystem is presented here and only the results of the second and third subsystem.

Consider the first two equations of (3.30). According to [23] a coordinate transformation is used:

\[\begin{align*}
v_1 &= e_{s1} \\
v_2 &= e_{s2} - k_{11}e_{s1},
\end{align*}\]  
(3.31)

what results in:

\[\begin{align*}
\dot{v}_1 &= v_2 \\
\dot{v}_2 &= f_1(v_1, v_2) - k_{12}v_1 - k_{11}v_2 + \bar{\gamma}_1 - k_{13}\text{sign}(v_1).
\end{align*}\]  
(3.32)

The constants \(k_{11}\) and \(k_{12}\) are used to guarantee stability. The term \(k_{13}\text{sign}(v_1)\) results in a second order sliding surface:

\[\dot{v}_1 = f_1(v_1, v_2) - k_{12}v_1 - k_{11}v_2 + \bar{\gamma}_1 - k_{13}\text{sign}(v_1) = 0.\]  
(3.33)

By using the “equivalent control” method this results in:

\[\dot{v}_1 = f_1(v_1, v_2) - k_{12}v_1 - k_{11}v_2 + \bar{\gamma}_1 - U_{eq} = 0.\]  
(3.34)
which results on the sliding surface in:

\[ U_{eq} = \dot{\gamma}_1, \tag{3.35} \]

and according to [27] this results in:

\[ \lim_{\omega_c \to \infty} U_{eq1} \approx k_{13} \text{sign}(v_1), \tag{3.36} \]

where the upper beam means the lowpass filtered \( k_{13} \text{sign}(v_1) \) signal by a second order filter with cutoff frequency \( \omega_c \). Due to this filtering a phase lag appears in the resulting signal and some information gets lost. To compensate for this an extra signal, \( k_{14}v_1 \), is added to the signal \( U_{eq1} \) what results in to following expression:

\[ \ddot{U}_{eq-1} = U_{eq-1} + k_{14}v_1 \tag{3.37} \]

The same procedure is followed for the other subsystems resulting in:

\[ \lim_{\omega_2 \to \infty} \ddot{U}_{eq2} \approx \ddot{\gamma}_2 \tag{3.38} \]

\[ \lim_{\omega_3 \to \infty} \ddot{U}_{eq3} \approx \ddot{\gamma}_3 \tag{3.39} \]

Transforming these results back to the absolute coordinate system results in the following expressions for the non identified dynamics:

\[ \gamma_1 = \ddot{U}_{eq-1} + \ddot{U}_{eq-3} \]

\[ \gamma_2 = \ddot{U}_{eq-2} + \ddot{U}_{eq-3} \tag{3.41} \]

\[ \gamma_3 = (1 + \mu_1 + \mu_2)\ddot{U}_{eq-3} + \mu_1 \ddot{U}_{eq-1} + \mu_2 \ddot{U}_{eq-2}, \]

which are scaled by the corresponding motor constants to retrieve the signals \( \dot{\gamma}_i = \frac{2}{c_i} \) in Figure 3.14

### 3.3.3 Luenberger observer design

Apart of the identified dynamics the sliding mode identifier results in all reconstructed system states. Before these states can be used they need to be filtered to remove all high frequent components due to the switching behavior of the identifier. The filtering causes an undesirable phase lag and therefore these signals are not suitable to use for control purposes. To reconstruct all system states without phase lag and with a reduced noise level an extra observer is introduced, classical Luenberger, which contains all known dynamics of the original system, enriched with correction terms (\( k_{i5} \)) on the velocity according to the following model in state space:

\[
\begin{align*}
\dot{x}_1 &= \dot{x}_2 + k_{15}e_1 \\
\dot{x}_2 &= f_1 + \dot{n}_1 + \dot{U}_{eq-1} \\
\dot{x}_3 &= \dot{x}_4 + k_{25}e_3 \\
\dot{x}_4 &= \dot{f}_2 + \dot{n}_2 + \dot{U}_{eq-2} \\
\dot{x}_5 &= \dot{x}_6 + k_{35}e_5 \\
\dot{x}_6 &= \dot{f}_3 + \dot{n}_3 + \dot{U}_{eq-3},
\end{align*}
\tag{3.42}
\]

where the equivalent control signals (\( \dot{U}_{eq-i} \)) from the sliding mode observer are added to the known linear dynamics of the state observer to ensure equal dynamics of the observer and the observed system.
Calculating the error dynamics results in:
\[
\begin{align*}
\dot{e}_1 &= e_2 - k_{15}e_1 \\
\dot{e}_2 &= \bar{f}_1(e_1, e_2) + \bar{\gamma}_1 - \bar{U}_{eq-1} \\
\dot{e}_3 &= e_4 - k_{25}e_3 \\
\dot{e}_4 &= \bar{f}_2(e_3, e_4) + \bar{\gamma}_2 - \bar{U}_{eq-2} \\
\dot{e}_5 &= e_6 - k_{35}e_5 \\
\dot{e}_6 &= \bar{f}_3(e_5, e_6) + \bar{\gamma}_3 - \bar{U}_{eq-3}.
\end{align*}
\]

Assuming the sliding mode observer correctly determines the signals \(\bar{U}_{eq-i}\), the terms \(\bar{\gamma}_1 - \bar{U}_{eq-1}\) in the error dynamics vanishes. The stability proof is in accordance with the stability proof described in the previous section using the coordinate transformation and the splitting of the dynamics in three pairs of two.

### 3.3.4 Input decoupling

The specified dynamics for a translational system can be used in two ways. The first one is to fully cancel the dynamics of both oscillators and only cancel the unknown part of the beam dynamics. This results in the following equations of motion for the experimental setup under ideal circumstances:
\[
\begin{align*}
\ddot{x}_1 &= c_1d_{s1} \\
\ddot{x}_2 &= c_2d_{s2} \\
\ddot{x}_3 &= \bar{f}_3(.) - \mu_1d_{s1} - \mu_2d_{s2} + c_3d_{s3},
\end{align*}
\]

where \(d_{s1}, d_{s2}\) and \(d_{s3}\) represent the specified dynamics for both oscillators and the beam. Notice that the beam is still coupled to the dynamics of both oscillators via \(\mu_1\) and \(\mu_2\) even without any specified dynamics for the beam on \(d_{s3}\) and that the coupling factors \(\mu_1\) and \(\mu_2\) are not free to choose within the specified dynamics. To fully decouple the oscillators and the beam and to introduce specified dynamics on the beam the input \(d_{s3}\) is chosen to be:
\[
d_{s3} = \bar{d}_{s3} + \frac{1}{c_3} \left( \mu_1d_{s1} + \mu_2d_{s2} - \bar{f}_3(.) \right),
\]

where \(\bar{d}_{s3}\) and \(\bar{f}_3(.)\) represent the specified dynamics for the beam and the a priori beam cancelation respectively, resulting in the following equations of motion:
\[
\begin{align*}
\ddot{x}_1 &= c_1d_{s1} \\
\ddot{x}_2 &= c_2d_{s2} \\
\ddot{x}_3 &= c_3\bar{d}_{s3}.
\end{align*}
\]

Using this situation any type of coupling between oscillators and beam can be introduced and the introduction of different coordinate systems becomes possible, for example rotational coordinates to implement Huijgens’ experiments.

### 3.3.5 Simulation results

The presented simulation result in Figure 3.15 uses model compensation without input decoupling for both oscillators and the unknown part of the beam dynamics is added to the observer. The “blue” and “red” oscillator both have the same specified dynamics and extra damping is introduced on the beam what results in the following equations of motion:
\[
\begin{align*}
\ddot{x}_1 &= -\omega_{1x}^2\Delta x_1 - \beta_{1x}\Delta \dot{x}_1 \\
\ddot{x}_2 &= -\omega_{2x}^2\Delta x_2 - \beta_{2x}\Delta \dot{x}_2 \\
\ddot{x}_3 &= \sum_{i=1}^{2} \mu_i \left[ \omega_{ix}^2\Delta x_i + \beta_{ix}\Delta \dot{x}_i \right] - \omega_{3x}^2x_3 - 2\zeta_3\omega_3\dot{x}_3 - \beta_{3x}\dot{x}_3 + \\
&= \sum_{i=1}^{2} \mu_i \left[ \omega_{ix}^2\Delta x_i + \beta_{ix}\Delta \dot{x}_i \right] - \omega_{3x}^2x_3 - 2\zeta_3\omega_3\dot{x}_3 - \beta_{3x}\dot{x}_3 - 2\zeta_3\omega_3\dot{x}_3 - \beta_{3x}\dot{x}_3,
\end{align*}
\]
where \( \omega_s = 2\pi [\text{rad/s}], \ i = 1, 2, \beta_{is} = 0 \ [\text{Nsm}^{-1}], \ i = 1, 2 \) and \( \beta_{is} = 4 \ [\text{Nsm}^{-1}] \). The coordinates \( \Delta x_1, \Delta x_2 \) and \( x_3 \) are defined according to the definitions used in section 3.1. Extra noise is added to all position signals in the model to more approximate reality. Both oscillators are brought with a PID controller to their initial position \( (x_{0 \text{ blue}} = -5 \ [\text{mm}], x_{0 \text{ red}} = -4 \ [\text{mm}]) \) and released at \( t = 20 \ [\text{s}] \). The figure clearly shows a correct tracking of all position and velocity signals and shows a noise reduction on all position signals. The tracking error stays within the order of 1 percent for all three subsystems. The noise reduction effect is comparable with the presented result in Figure 3.10 and therefore is not presented here explicitly. The resulting frequencies of both oscillators are 1 [Hz] and no damping is present what matches with the specified dynamics. The amplitude changes are due to the coupling of the oscillators and the beam.

### 3.3.6 Robustness

The designed observer and cancelation strategy is robust for parameter uncertainties as long as the discontinuous terms are capable to cope with the errors occurring in the signals, depending on the parameters \( k_{i,3} \). The robustness analysis is similar to analysis done in section 3.2.2.

### 3.4 Conclusions

At simulation level the separate identifiers are able to identify the non-identified dynamics and are robust for model uncertainties. The noise level of the reconstructed position signal is reduced by a factor 10 and the velocity signal is estimated correctly. No comments can be given about noise levels on the
velocity signals, this needs to be done in the experimental mode. A tradeoff needs to be made between noise reduction and observer speed, this is done in the next chapter, supported by experiments.

The full state observer is capable to reconstruct all system states and is able to identify the unknown dynamics in the system at a level which properly suits for the purpose of introducing new dynamics. The observer reduces the noise level on all position signals with a factor 10. The exact tuning of the 3DOF observer parameters and the experimental results are shown in chapter 5.
Chapter 4

Self sustained oscillators

Using the approach discussed in chapter 3 it is possible to implement a self sustained oscillator on the experimental setup which exactly behaves as desired. This chapter describes two types of driven oscillators used on the experimental setup, a translational linear one and a rotational non linear one. The design of the oscillators and their driving mechanisms are based on isolated situations, so the beam in the setup is fixed to not influence the behavior of the oscillators.

4.1 Driving mechanism

Although the specified dynamics can theoretically be chosen in such a way that no damping is present, the remaining system dynamics after full dynamics compensation can contain some damping and stiffness. The remaining stiffness causes the resulting frequency to be slightly different than the specified frequency. The remaining damping (positive as well as negative) causes an unstable amplitude of the oscillations. To compensate for the last phenomena a driving mechanism is needed to stabilize the systems oscillation at a decided amplitude level. In order to create a driving mechanism in the sense of Huijgens, the driven system needs phase freedom in order to make synchronization with an other system possible. This causes that linear controllers are not suited to control this motion because they need a reference trajectory what results in a locked phase. Apart of this, it costs an exceptional lot of effort to control a system with a linear controller near its eigenfrequency. Appendix E describes the design of a linear controller for a single linear oscillator and describes the problems with this type of controllers. The driving mechanisms described in the following sections are all energy based. The introduced driving mechanisms are not able to cope with the possibly appearing frequency differences.

4.2 Linear oscillator

To match with Huijgens’ experiments the specified dynamics of the linear oscillator in this chapter has a frequency of \( \omega_s = 2\pi \,[rad] \) and zero damping, \( \beta_s = 0 \,[Ns^{-1}] \), represented by:

\[
\ddot{x} = -\omega_s x - \beta_s \dot{x} + \frac{1}{c} s,
\]

(4.1)

where \( c \,[ms^{-2}V^{-1}] \) is the motor constant and \( s \) is the internal control signal which is going to be determined in this chapter. The control goal is to keep the oscillation at a certain amplitude what can be represented by an energy level. The actual energy level of the system is represented as follows:

\[
E = \frac{1}{2} \omega_s^2 x^2 + \frac{1}{2} \dot{x}^2,
\]

(4.2)

where \( E \) presents the actual energy of the system scaled by the mass \( m \) composed out of the potential and kinetic energy. The reference energy level is presented in (4.3) where \( x_{ref} \) is the desired amplitude level at
zero velocity.

\[ E^* = \frac{1}{2} \omega_s^2 x_{ref}^2, \]  \hspace{1cm} (4.3)

### 4.2.1 Controller design

Lyapunov theory is used to design the control signal where the following Lyapunov candidate is used:

\[ L = \frac{1}{2} (E - E^*)^2. \]  \hspace{1cm} (4.4)

The function \( L \) represents the squared energy error between the current and the desired energy level which is a positive definite function. To guarantee stability of the closed loop system the time derivative of \( L \) needs to be negative (semi-)definite according to:

\[ \dot{L} = (E - E^*)(\dot{E}) \]
\[ = (E - E^*)( -\beta_s \dot{x}^2 + c s \dot{x}) \leq 0. \]  \hspace{1cm} (4.5)

To achieve this goal the following control signal is proposed:

\[ s = \frac{1}{c} (\beta_s \dot{x} - k_e (E - E^*) \dot{x}) \]  \hspace{1cm} (4.6)

which results in the following time derivative of \( L \):

\[ \dot{L} = -k_e (E - E^*)^2 \dot{x}^2 \]  \hspace{1cm} (4.7)

where \( k_e \) represents the controller gain. Function (4.7) satisfies the Lyapunov stability criteria \( \forall k_e > 0 \) to be negative definite \( \forall \dot{x} \neq 0 \) and LaSalle’s principle is used to prove stability at the origin of the system and to prove the global asymptotical stability of the system. The set where \( \dot{L} = 0 \)

\[ S = \{(x_1, x_2) | \dot{L} = 0 \} \]  \hspace{1cm} (4.8)

which is equal to:

\[ S = \{(x_1, x_2) | x_2 = 0 \} \]  \hspace{1cm} (4.9)

contains no trajectories except for the trivial solution \((x_1, x_2) = 0\). Hence, global asymptotical stability is proven.

### 4.2.2 Robustness

The theory in the previous section assumes that the damping in the controller signal is exactly known, what is practically untrue and causes errors. Hence the controller signal is rewritten as:

\[ s = \frac{1}{c} \left( \hat{\beta}_s \dot{x} - k_e (E - E^*) \dot{x} \right), \]  \hspace{1cm} (4.10)

where \( \hat{\beta}_s \) represents the estimated damping. Recalculating the Lyapunov derivative results in:

\[ \dot{L} = -k_e (E - E^*)^2 \dot{x}^2 - (E - E^*) \Delta \beta \dot{x}^2, \]
\[ = \left( -k_e (E - E^*) - \Delta \beta \right) (E - E^*) \dot{x}^2 \]  \hspace{1cm} (4.11)

where the introduced estimation error is given by \( \Delta \beta = \beta_s - \hat{\beta}_s \). This function is not negative definite any more and has a zero at either:

\[ (\dot{x} = 0) \text{ or } (E = E^*) \text{ or } (E = E^* - \Delta E), \]  \hspace{1cm} (4.12)

where \( \Delta E = \frac{\Delta \beta}{k_e} \). Combining this information in a map results in Figure 4.1 where the curve represents the Lyapunov candidate function for \( \dot{x} = 0 \). The arrows in the figure show the sign of \( \dot{L} \). The first and the second solution are marginally stable solutions. Only the third solution is a stable solution and all trajectories end up here. Depending on the controller parameter \( k_e \) the position of the third solution on the curve can be determined. In the limit, when \( k_e \to \infty \), a bifurcation takes place and solution 2 and 3 merge. Depending if \( \hat{\beta}_s \) is overestimated or underestimated solution 3 appears at the left or the right side of solution 2.
4.2. LINEAR OSCILLATOR

Figure 4.1: Graphical representation of the stability analysis for the proposed controller $s$ in (4.10). The presented situation shows the situation where $\hat{\beta}_s$ is underestimated.

4.2.3 Experimental results

The designed controller (4.10) is implemented as a part of the specified dynamics on the setup. Depending on the remaining dynamics after cancelation the experimental setup behaves approximately according to the specified dynamics. This is graphically represented in Figure 4.2 where the left bar shows the ideal specified dynamics. The center and right bar present the resulting dynamics of the experimental setup including the remaining dynamics after cancelation, $\epsilon(.)$. Obviously the bar at the center is the more preferable situation where $\epsilon(.)$ is small resulting in a small frequency and damping deviation compared to the specified dynamics itself. To minimize $\epsilon(.)$ the combination of the Luenberger observer and the sliding mode identifier needs to be fine tuned on the experimental setup. These steps are described in section 4.3.3 and only the final results are presented here including the designed controller (4.10).

Consider Figure 4.3 which presents the measured positions of the “blue” and “red” oscillator on the first row. These positions behave as expected, a frequency of 1 [Hz] with a constant amplitude. The second row presents two comparisons made. The first one, in green, presents the difference between the observed and the measured signals, the second one presents the error between the ideal specified behavior and the observer behavior. As the figure shows the error between observed and ideal is growing in time where the other error remains limited. This is clarified by the fact that due to some remaining dynamics $\epsilon(.)$ a
slight frequency difference is present. The presented error is reflecting the increasing phase difference due to this frequency difference. Over a time span of 100 seconds the real behavior shows a phase lag of 7.2 milliseconds compared to the ideal behavior resulting in a frequency difference of 70 micro Hertz which is neglected. The last row presents the energy levels of both oscillators. The occurring error is due to noise, especially in the velocity signal.

![Graphs showing position and energy levels of Blue and Red oscillators](image)

Figure 4.3: Experimental results for linear self sustained stand alone oscillators using the fine tuned observer in combination with the sliding mode identifier and designed controllers with $k_{e1h} = k_{e2h} = 200$.

### 4.3 Huijgens’ oscillator

To come more close to the oscillators used by Huijgens this section describes the design of a self sustained non linear oscillator, a driven pendulum. The equation of motion is given by:

$$\ddot{\phi}_i = \frac{1}{m_i l_i^2} \left( c_i u_i - d_i \dot{\phi}_i - m_i g l_i \sin(\phi_i) \right), \quad (4.13)$$

where $m_i$, $l_i$, $d_i$ and $g$ respectively represent the mass, length and damping of the pendulum and the gravitational acceleration. The pendulum is driven by input $u_i$ with its corresponding motor constant $c_i$. The frequency of the pendulum can be approximated by $\omega_i = \sqrt{\frac{g}{l_i}}$ and is chosen to be $2\pi \text{ [rad]}$, in the sense of Huijgens’ experiments and no damping is present: $d_i = 0 \text{ [Nms/rad]}$.

### 4.3.1 Controller design

To keep the oscillations within a desired amplitude a controller is added to the system. The controller design is based on Lyapunov theory and the same Lyapunov candidate is proposed:

$$L = \frac{1}{2} (E - E^*)^2, \quad (4.14)$$
where $E$ represents the actual energy level of the pendulum:

$$E = m_i g l_i (1 - \cos(\phi_i)) + \frac{1}{2} m_i l_i^2 \dot{\phi}_i^2,$$

and $E^*$ represents the desired energy level. The controller $u_i$ needs to be designed in such a way that it guarantees:

$$\dot{L} = (E - E^*) \dot{E} = (E - E^*) \left( u_i c_i \dot{\phi}_i - d_i \dot{\phi}_i^2 \right) \leq 0.$$  (4.16)

Hence, by defining $u_i$ as:

$$u_i = \frac{d_i}{c_i} \dot{\phi}_i - k_e h \frac{(E - E^*)}{c_i} \dot{\phi}_i,$$  (4.17)

(4.16) becomes negative semi-definite. LaSalle’s theorem is used to prove the global asymptotical stability. The set where $\dot{L} = 0$

$$S = \{(x_1, x_2) \mid \dot{L} = 0\}$$  (4.18)

which is equal to:

$$S = \{(x_1, x_2) \mid x_2 = 0\}$$  (4.19)

contains no trajectories except for the trivial solution $(x_1, x_2) = 0$. Hence global asymptotical stability is proven.

### 4.3.2 Parameter tuning and coordinate transformation

To implement the rotational pendulum on the masked translational system a coordinate transformation is introduced. The stroke of the translational oscillators, $5$ [mm], is used as horizontal projection of the rotational motion according to transformation:

$$x_i = l_i \sin(\phi_i)$$

$$\dot{x}_i = l_i \dot{\phi}_i \cos(\phi_i),$$  (4.20)

resulting in the inverse transformation:

$$\phi_i = \arcsin \left( \frac{x_i}{l_i} \right) \quad \forall \quad |x_i| \leq l_i$$

$$\dot{\phi}_i = \frac{1}{\sqrt{1 - \frac{x_i^2}{l_i^2}}} \frac{\dot{x}_i}{l_i} \quad \forall \quad |x_i| < \sqrt{l_i}.$$  (4.21)

The maximum angle defined is $\pm 45[deg]$ resulting in a pendulum length of $7$ [mm]. By using this pendulum length in combination with the regular gravitational acceleration, $g = 9.81 \, [ms^{-2}]$, results in a pendulum frequency of $\omega_i = \sqrt{\frac{g}{l_i}} \approx 12\pi \, [rad/s]$. To achieve a pendulum frequency of $1$ [Hz] a virtual gravitational acceleration is used: $g_h = 0.2763 \, [ms^{-2}]$. The option to upscale all system states is not used because of increasing noise levels, especially near the origin. The mass, damping and motor constant of the pendulum are chosen arbitrarily to be $m_i = 1 \, [kg]$, $d_i = 0 \, [Nms/rad]$ and $c_i = 1 \, [radm^2s^{-2}V^{-1}]$ respectively.

The angular acceleration is determined by the specified dynamics according to (4.13) and is transformed to the translational coordinates according to:

$$\ddot{x}_i = l_i \ddot{\phi}_i \cos(\phi_i) - l_i \dot{\phi}_i^2 \sin(\phi_i).$$  (4.22)
4.3.3 Experimental results and fine tuning of the cancelation mechanism

As stated in the previous section the experimental setup can only behave according to the specified dynamics when the cancelation dynamics is achieved with a certain accuracy. This section describes the steps to tune the sliding mode identifier in combination with the Luenberger observer. All graphical results can be found in appendix C.1. The non linear rotational pendulum is used as specified dynamics for the “blue” and the “red” oscillator according to (4.13) using the parameters stated in section 4.3.2. The beam is fixed in the origin of the experimental setup. The first 10 seconds of all shown results cannot be compared because the experimental setup is brought to the initial position, were the ideal behavior starts at this position immediately.

Consider Figure 4.4 which shows the experimental results when only the a-priori knowledge is used for the cancelation of the original dynamics, where the specified dynamics are plugged in according to (4.13) and where the startup angle of the pendula equals -0.44 [rad]. The figure clearly shows, for the angular displacement as well as the angular velocity, a remaining damping component in both oscillators and shows a frequency mismatch, corresponding to an incorrect stiffness estimation in the masking dynamics. Theoretically these dynamical differences can be solved by adding the estimation obtained with the sliding mode identifier to the observer or by subtracting these dynamics from the system. The initially used sliding mode parameters are given in Table 3.2 and the experimental result is presented in Figure C.1. A small negative resulting damping for the “red” oscillator and a small positive resulting damping for the “blue” oscillator remains. The frequency mismatch is reduced to 0.5 milliHertz. The remaining error is mainly caused by two phenomena: measurement noise and the fact that the sliding mode signals are filtered, causing an information loss.

By increasing the cut-off frequency $\omega_c$ of these filters more dynamics are captured but the negative
The side effect is that the control signals contain more high frequent components. A tradeoff is found at $f_{ci} = 150\pi \text{[rad]}$ and by adding the signal introduced in equation 3.21 with $k_{i4} = 250$ to compensate for the filter losses. The result is presented in Figure C.2. These adjustment result in the expected behavior for the “blue” oscillator but not for the “red” one.

The performance can be improved further by lowering the full state observer gains $k_{i5}$. Figure C.3 shows the error between the ideal and real behavior. The increasing error is declared by a small frequency difference, $75 \text{[µHz]}$ for the “blue” oscillator and $0.5 \text{[mHz]}$ for the “red” oscillator. As the figure shows, the amplitude error is within the margin of 2 per mille.

To cope with the remaining damping errors and disturbances the proposed controller of section 4.3.1 is added to the system, resulting in the behavior presented in Figure 4.5. The figure shows the error between the ideal and real behavior. The increasing error component in the “blue” oscillator is disappeared now so the frequency difference is almost completely compensated. The “red” oscillator still contains a frequency error of $0.12 \text{[mHz]}$. The presented amplitude error, apart of phase shifts, is in the order of 2 per mille.

![Figure 4.5: Final experimental results of tuning process including energy controller(4.3.1) using $k_{e1b} = k_{e2b} = 200$. Both pendula are released at $0.44 \text{[rad]}$ and the controller reference amplitude is $0.29 \text{[rad]}$, corresponding to an energy level of $0.08 \text{[mJ]}$.](image)

The performance of the “red” oscillator is slightly different than the performance of the “blue” one. This can be clarified by considering Figure 4.6 which shows the measured position signals when both oscillators are in rest. The “blue” zero-signal shows white noise but the “red” zero-signal shows a pattern with a positive mean value causing an incorrect measured position and a corresponding overestimated velocity. Due to this the masking dynamics overcompensates the system, resulting in a negative damping component shown in Figure C.2. The zero-signal can not be compensated because its amplitude, phase and frequency are not constant over several experiments.
4.4 Conclusions

The in chapter 3.2 designed cancelation mechanism is able to compensate almost all dynamics and in combination with the energy controllers the experimental results hardly differ the specified behavior. The frequency difference is hardly measurable and the resulting damping is zero. The remaining difference is a phase difference between the experimental results and the specified behavior, especially for the “red” oscillator.
Chapter 5

Linear synchronization

This chapter describes a synchronizing system based on two interacting translational linear oscillators, designed in chapter 4. Firstly the model is described whereafter several synchronization scenarios are described: the free moving oscillators, the situation where one oscillator is driven and a scenario in the sense of Huijgens where both oscillators are driven. In all scenarios the beam can freely move, apart of the actuation needed for the internal dynamics modification.

5.1 Linear translational model

All experiments presented in this chapter use the full dynamics compensation including the input decoupling as described in chapter 3.3.4 resulting, under ideal circumstances before introducing the specified dynamics, in the following closed loop system dynamics:

\[
\begin{align*}
\ddot{x}_1 &= c_1 d_{s1} \\
\ddot{x}_2 &= c_2 d_{s2} \\
\ddot{x}_3 &= c_3 \ddot{d}_3,
\end{align*}
\]

(5.1)

where \(c_i\) represents the motor constants and \(d_{si}\) the specified dynamics. The specified dynamics contain two identical oscillators, coupled to the beam by parameter \(\mu_i\) according to:

\[
\begin{align*}
\ddot{x}_1 &= -\omega_s^2 \Delta x_1 + u_{1h} c_{1h} \\
\ddot{x}_2 &= -\omega_s^2 \Delta x_2 + u_{2h} c_{2h} \\
\ddot{x}_3 &= -\omega_b^2 x_3 - 2 \zeta_b \omega_b \dot{x}_3 + u_{3h} c_{3h} + \sum_{i=1}^{2} \mu_{si} \left[ \omega_s^2 \Delta x_i - u_{ih} c_{ih} \right],
\end{align*}
\]

(5.2)

where \(\omega_s = 2\pi \ [rad]\), \(\omega_b = 9.7369 \ [rad]\) and \(\zeta_b = 0.0409\) represent the specified undamped eigen-frequency for both oscillators, the specified undamped eigen-frequency of the beam and the dimensionless damping of the beam. The motor constants of the specified dynamics and the control inputs within the specified dynamics, representing the escapement mechanism, are denoted by \(c_{ih} = 1\) and \(u_{ih}\). All states \(x_i\) are defined with respect to the fixed world and \(\Delta x_i = x_i - x_3\), \(\forall i = 1, 2\) denotes the relative position of the oscillators with respect to the beam. Both oscillators are free of damping.
5.2 Experimental results and stability analysis

5.2.1 Synchronization of free moving oscillators and parameter tuning

This scenario presents simulation and experimental results when no driving mechanisms are present within the specified dynamics resulting in the following closed loop dynamics:

\[
\begin{align*}
\ddot{x}_1 &= -\omega_s^2 \Delta x_1 \\
\ddot{x}_2 &= -\omega_s^2 \Delta x_2 \\
\ddot{x}_3 &= -\omega_{bs}^2 x_3 - 2\zeta_{bs}\omega_{bs} \dot{x}_3 + \sum_{i=1}^{2} \mu_{si} \left[\omega_s^2 \Delta x_i \right],
\end{align*}
\]

(5.3)

where all parameters values are defined as in section 5.1. The coupling parameters \(\mu_{si}\) are varied during the experiments. This scenario is used to tune and validate the 3DOF sliding mode identifier, resulting in the following sliding mode identifier and observer parameters:

<table>
<thead>
<tr>
<th>Blue oscillator</th>
<th>Red oscillator</th>
<th>Beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_{11} = 10)</td>
<td>(k_{21} = 10)</td>
<td>(k_{31} = 10)</td>
</tr>
<tr>
<td>(k_{12} = 20)</td>
<td>(k_{22} = 20)</td>
<td>(k_{32} = 20)</td>
</tr>
<tr>
<td>(k_{13} = 0.5)</td>
<td>(k_{23} = 0.5)</td>
<td>(k_{33} = 0.5)</td>
</tr>
<tr>
<td>(\omega_{c1} = 150\pi \text{ [rad]})</td>
<td>(\omega_{c2} = 150\pi \text{ [rad]})</td>
<td>(\omega_{c3} = 150\pi \text{ [rad]})</td>
</tr>
<tr>
<td>(k_{14} = 250)</td>
<td>(k_{24} = 250)</td>
<td>(k_{34} = 250)</td>
</tr>
<tr>
<td>(k_{15} = 0.3)</td>
<td>(k_{25} = 0.3)</td>
<td>(k_{35} = 0.1)</td>
</tr>
</tbody>
</table>

Table 5.1: Parameters for sliding mode identifier and state observer tuned in experimental mode.

Appendix D.1 shows the proof, copied from [22], for the stability of the anti-phase synchronization manifold for the non driven translational experimental setup. This proof is used to prove the stability of the anti-phase synchronization manifold for system (5.3) in particular. It results that for most of the initial conditions the solution of system (5.3) converges to the anti-phase synchronization manifold. Especially for all in phase initial conditions this does not hold and the system damps out and no synchronization appears.

Two experimental results for the non driven linear scenario are presented. Figure 5.1 presents the first result and shows the effect of input decoupling to introduce equal coupling strengths \(\mu_{s1} = \mu_{s2} = 0.0411\). The “blue” oscillator starts at 5 [mm] and the “red” one starts in the origin. The figure shows the expected anti-phase synchronizing behavior of both oscillators. The effect of equal coupling strengths is clearly visible in the left plot on the second row, presenting the amplitude behavior of both oscillators, where both amplitudes converge to the same value. The center plot shows that the phase difference between both oscillators converges to \((\pi)\), meaning anti-phase motion.

The third row in the Figure 5.1 shows the errors, in pairs of two, which are determined to qualify the performance of the identification and compensation process. The yellow lines show the difference between the real and observed position which is clearly only noise. The increasing error between the ideal and the observed position is declared by a small frequency difference causing an increasing phase shift.
Figure 5.1: Experimental results for synchronization of free moving linear oscillators using equal coupling strengths: $\mu_{s1} = \mu_{s2}$.
Figure 5.2 shows the results for the second experiment which is done under the same circumstances as the previous one but now the original coupling strengths are used: $\mu_{s1} = 0.0411$ and $\mu_{s2} = 0.0578$. This results in an anti-phase synchronous motion but with different amplitudes for the “blue” and “red” oscillator. The errors as defined for the previous experiment are not presented here but show the same behavior as in the previous experiment.

5.2.2 Driven oscillators

The synchronization process which appears in the free moving scenario costs energy and so causes a change of the amplitude levels of both oscillators. This scenario adds an escapement like mechanism to one, and both, oscillators to maintain the energy level resulting in the following system dynamics:

$$
\ddot{x}_1 = -\omega_s^2 \Delta x_1 - k_{e1b}(E - E^*) \Delta \dot{x}_1 \\
\ddot{x}_2 = -\omega_s^2 \Delta x_2 - k_{e2b}(E - E^*) \Delta \dot{x}_2 \\
\ddot{x}_3 = -\omega_{bs}^2 x_3 - 2\zeta_{bs}\omega_{bs}\dot{x}_3 + \sum_{i=1}^{2} \mu_{si} [\omega_s^2 \Delta x_i + k_{eih}(E - E^*) \Delta \dot{x}_i],
$$

(5.4)

where $k_{e1b}$ and $k_{e2b}$ are representing the controller gains, which can be compared with the power of the, replaced, escapement mechanisms. The implemented controllers are equal to the ones used in chapter 4.2 for the stand alone driven oscillators but in this scenario the beam is used as reference instead of the fixed world resulting in relative coordinates in the controller expressions in stead of absolute coordinates and the controllers are based on the total system energy $E$ according to:

$$
E = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} k_1 \Delta x_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} k_2 \Delta x_2^2 + \frac{1}{2} m_3 \dot{x}_3^2 + \frac{1}{2} k_3 x_3^2,
$$

(5.5)

where $m_i = 1 [kg]$, $\forall i = 1, 2$ and $k_i = (2\pi)^2 [Nm^{-1}]$, $\forall i = 1, 2$ represent the mass and stiffness of both oscillators respectively. The mass, stiffness and damping of the beam are denoted by $m_3 = 25 [kg]$, $k_3 = 2370 [Nm^{-1}]$ and $d_3 = 19.9 [Nsm^{-1}]$ respectively. The reference energy level is set to
\(E^* = 0.987 \text{ [mJ]}\). Using the given parameter values this energy level corresponds to an amplitude of 5 [mm] for both oscillators and no motion of the beam.

The stability proof consists of three parts and is presented in appendix D.2.1. The first part analyses the nominal system without driving mechanisms whereafter the second part is based on the theory on vanishing perturbations \([12]\) and determines an upper bound for \(k_{e1h} \) and \(k_{e2h}\). Within this boundary the controller inputs are not affecting the stability of the anti-phase synchronization manifold of the free moving system described in section D.2.1. The third part of the proof shows that the total system energy level converges to the desired energy level, \(E \to E^*\), under the assumption of anti-phase synchronization out of the first and second part of the proof. The proof results in the following boundary for \(k_h\):

\[
0 < k_h = k_{e1h} = k_{e2h} < \frac{\lambda_{\min}(I)}{2\lambda_{\max}(P)} \frac{||\sigma||_2}{||\sigma_2 - 2\sigma_4||_2} E^* \tag{5.6}
\]

where \(\sigma\) and \(P\) are defined in appendix D.2.2. For the used parameter settings the bound results in:

\[
0 < k_h < 0.5273 \tag{5.7}
\]

Simulating system (5.4) over the full grid of initial conditions leads to the expected behavior but due to the conservative estimation of the controller gains the controller is extremely slow. Depending on the initial condition it takes up to 8000 seconds to reach the desired energy level. The process to reach anti-phase synchronization itself is much faster and takes approximately 10 percent of this time. Figure 5.3 shows one simulation result as illustration of the slow controller behavior. The controller gain \(k_h = 2\) is used and the system starts in almost in-phase motion \(x_0_{\text{blue}} = -5\) [mm] and \(x_0_{\text{red}} = -4\) [mm]. The figure clearly shows that the synchronization process is faster (500 [s]) than the process to reach the desired energy level (2500 [s]). The worst case estimation made for \(k_h\) in (5.6) is really conservative, especially caused by the fact that \(E - E^*\) and \(\sigma_2 - 2\sigma_4\) are physically excluding each other in the current setting as long as \(E < E^*\). For example: when \(||E - E^*||_2 = E^*\) is maximal, the norm \(||\sigma_2 - 2\sigma_4||_2\) can not be
maximal as well and the other way around. This effect gives rise to increase the value of $k_h$ on simulation level to check the effects. The value can be increased to approximately $k_h = 1500$ before the anti-phase synchronization manifold is not stable any more over the full grid of initial conditions. Increasing $k_h$ further causes in-phase synchronization when the initial conditions are close to in-phase. This process is shown for two different values of $k_h$ in Figure 5.4. The presented results can not be proven for the moment.

![Figure 5.4: Simulation result using system (5.4). Left: $k_h=3000$ and right: $k_h=6000$. Anti-phase synchronization occurs when the value 1 is presented, in-phase synchronization when -1 is presented.](image)

For one initial condition, almost in-phase startup, the experimental results are presented in Figure 5.5 and Figure 5.6. Both fully match the expected result when using the energy based controller. The system ends up in a stable anti-phase motion with equal oscillation amplitudes at 5 [mm] and the beam fully damped out. Figure 5.5 shows the result when only the “blue” oscillator is driven, Figure 5.6 shows the result when both oscillators are driven.
5.2. EXPERIMENTAL RESULTS AND STABILITY ANALYSIS

Figure 5.5: Experimental results using a driven “blue” oscillator, $k_e1h = 200$, and equal coupling strengths $\mu_{i1} = 0.0411$. Both oscillators are released nearly in phase and end up in anti-phase.

Figure 5.6: Experimental results when driving both oscillators using $k_e1h = k_e2h = 200$ and use equal coupling strengths $\mu_{i1} = 0.0411$. Both oscillators are released nearly in phase and end up in anti-phase.
5.3 Discussion

The boundary determined for the controller gains is extremely conservative and although no formal proof can be given for the moment, simulation as well as experimental results show that the practical boundary value for the controller gains is approximately $k_h = 1500$. Within these boundary for almost all initial conditions the desired energy level is reached in anti-phase synchronization. It also appears that the gains $k_{e1h}$ and $k_{e2h}$ do not need to be equal and even by making one of the gains zero results in the desired behavior of the system. These results are approximately equal on simulation level and on experimental level.

To come more close to Huijgens’ original setup the driving mechanisms should not control based on the total system energy but based on Hamiltonian functions representing a sort of energy per oscillator. These Hamiltonian function converge to the real energy of an oscillator under the assumption that anti-phase synchronization occurs. An example of these controllers can be:

$$
\begin{align*}
    u_1 &= -k_{e1h}(H_1 - H_1^*)\Delta \dot{x}_1 \\
    u_2 &= -k_{e2h}(H_2 - H_2^*)\Delta \dot{x}_2
\end{align*}
$$

with the corresponding Hamiltonian functions:

$$
\begin{align*}
    H_1 &= \frac{1}{2} \omega_s^2 \Delta x_1^2 + \frac{1}{2}\Delta x_1^2 \\
    H_2 &= \frac{1}{2} \omega_s^2 \Delta x_2^2 + \frac{1}{2}\Delta x_2^2
\end{align*}
$$

At simulation level and on experimental level these controllers are able to meet the control goals but the stability of these controllers can not be proven for the moment.

5.4 Conclusion

This chapter aimed to present some synchronization experiments based on coupled linear translational (driven) oscillators. The experimental results are matching the ideal simulation results for all scenarios. For the non driven scenario a proof for stability of the anti phase synchronization manifold is given. For the driven scenario a proof is given on the stability of the synchronization manifold in combination with achieving a certain energy level in the system. The given proof is based on the theory on “vanishing disturbances” which results in a boundary for the controller gains. Experiments and simulations show that the boundary is extremely conservative.

After analyzing the linear case the next chapter introduces the non linear rotational model which comes close to Huijgens’ original experimental setup.
Chapter 6

Huijgens’ synchronization

This chapter describes a synchronizing system based on two interacting non linear rotational oscillators, which are designed in section 4.3. Firstly the model is described whereafter two synchronization scenarios are described: the free moving pendula scenario and a scenario in the sense of Huijgens where both pendula are driven. In both scenarios the beam is freely moving, apart of the actuation needed for the internal dynamics modification. The chapter ends by presenting some experimental results where extra disturbances are introduced on the experimental setup to validate the robustness of the introduced sliding mode compensation mechanism.

6.1 Non linear rotational model

Consider Figure 6.1 which shows a simplified model of Huijgens’ clocks setup used approximately 350 years ago. The coordinates used to describe the position and velocity of both pendula and the beam are:

\[
\begin{pmatrix}
\phi_1 & \dot{\phi}_1 & \phi_2 & \dot{\phi}_2 & x_3 & \dot{x}_3
\end{pmatrix}
\]

(6.1)

where \(\phi_i \ [rad], \ i = 1, 2\) and \(\dot{\phi}_i \ [rads^{-1}], \ i = 1, 2\) represent the angle and angular velocity respectively of both pendula and \(x_3\) and \(\dot{x}_3\) represent the position and velocity of the beam respectively. The length, mass and damping of both pendula is defined by \(l_i \ [m], \ m_i \ [kg]\) and \(d_i \ [Nmsrad^{-1}]\) respectively and the gravitational acceleration is represented by \(g \ [ms^{-2}]\). The equations of motion are derived in detail in
Appendix B.2 and result in the following equations of motion:

\[
\ddot{\phi}_1 = \frac{1}{m_1 l_1^2} \left[ u_{h1} c_1 - d_1 \dot{\phi}_1 - m_1 \dot{x}_3 \cos \phi_1 - m_1 g l_1 \sin \phi_1 \right]
\]

\[
\ddot{\phi}_2 = \frac{1}{m_2 l_2^2} \left[ u_{h2} c_2 - d_2 \ddot{\phi}_2 - m_2 \dot{x}_3 \cos \phi_2 - m_2 g l_2 \sin \phi_2 \right]
\]

\[
\ddot{x}_3 = \frac{1}{B} \left[ -d_3 \dot{x}_3 - k_3 x_3 + u_{h3} c_3 - \sum_{i=1}^{2} \frac{\cos \phi_i}{l_i} (u_{hi} c_i - d_i \dot{\phi}_i - m_i g l_i \sin \phi_i) - m_i l_i \dot{\phi}_i^2 \sin \phi_i \right],
\]

where \( B = (m_1 + m_2 + m_3) - m_1 \cos^2 \phi_1 - m_2 \cos^2 \phi_2 \). Both pendula and the beam are driven by the inputs \( u_{hi} \ [V], \forall \ i = 1, 2, 3 \) and the corresponding motor constants \( c_{hi}, \forall \ i = 1, 2, 3 \). The beam is considered to be a rigid body.

To implement system (6.2) as specified dynamics on the experimental setup, which consists of only translational oscillators, a coordinate transformation is necessary and the masking structure is slightly changed.

### 6.1.1 Coordinate transformation, parameter tuning and specified dynamics

To implement the rotational Huijgens’ system on the translational experimental setup a coordinate transformation is necessary for both oscillators. The transformation is similar to the one introduced in section 4.3.2, extended with the beam coordinates. The horizontal projection of the rotational motion of the oscillators is defined by:

\[
x_i = x_3 + l_i \sin (\phi_i)
\]

\[
\dot{x}_i = \dot{x}_3 + l_i \sin (\phi_i) \dot{\phi}_i
\]

\[
\ddot{x}_i = \ddot{x}_3 + l_i \ddot{\phi}_i \cos (\phi_i) - l_i \dot{\phi}_i^2 \sin (\phi_i),
\]

where \( B = (m_1 + m_2 + m_3) - m_1 \cos^2 \phi_1 - m_2 \cos^2 \phi_2 \). Both pendula and the beam are driven by the inputs \( u_{hi} \ [V], \forall \ i = 1, 2, 3 \) and the corresponding motor constants \( c_{hi}, \forall \ i = 1, 2, 3 \). The beam is considered to be a rigid body.
what results in the following coordinate transformations:

\[ \phi_i = \arcsin \left( \frac{\Delta x_i}{l_i} \right) \quad \forall \ |\Delta x_i| \leq l_i \]
\[ \dot{\phi}_i = \frac{1}{\sqrt{1 - \left( \frac{\Delta x_i}{l_i} \right)^2}} \quad \forall \ |\Delta x_i| < \sqrt{l_i}, \]

(6.4)

where \( \Delta x_i = x_i - x_3, \ \forall \ i = 1, 2. \) The same angle and stroke criteria are used as presented in chapter 4 resulting in a pendulum length of \( l_i = 7 \ [mm] \) and a virtual gravitational acceleration: \( g_h = 0.2763 \ [ms^{-2}] \). Figure 6.2 presents a graphical representation of the coordinate transformation which is introduced. The red dashed line presents the linear approximation which is valid as long as \( |\phi| < 0.3 \ [rad] \) and \( |x_i| < 2 \ [mm] \).

6.2 Experimental results and stability analysis

6.2.1 Synchronization of free moving pendula and parameter tuning

This scenario presents the experimental results for Huijgens’ synchronization when both pendula are moving without any driving mechanism according to model:

\[ \ddot{\phi}_1 = \frac{1}{m_1 l_1^2} \left[ -m_1 l_1 \dot{x}_3 \cos \phi_1 - m_1 g l_1 \sin \phi_1 \right] \]
\[ \ddot{\phi}_2 = \frac{1}{m_2 l_2^2} \left[ -m_2 l_2 \dot{x}_3 \cos \phi_2 - m_2 g l_2 \sin \phi_2 \right] \]
\[ \ddot{x}_3 = \frac{1}{B} \left[ -d_3 \dot{x}_3 - k_3 x_3 + u_{h3} c_3 - \sum_{i=1}^{2} \frac{\cos \phi_i}{l_i} \left( -m_i g l_i \sin \phi_1 \right) - m_i l_i \dot{\phi}_i^2 \sin \phi_1 \right], \]

(6.5)

It is expected that almost all initial conditions of system (6.5) converge to the anti-phase synchronization manifold. For completeness the formal proof (by Pogromsky et al. 2003, [20]) is provided in Appendix D.3. Especially all exact in phase initial conditions are expected to damp out and no anti-phase synchronization appears.

Two experimental results of the free moving pendula situation are presented here in Figure 6.3 and 6.4 to show the difference between starting exactly in phase, and starting nearly in phase. The expectation for the first scenario is that the system damps out without reaching the anti-phase synchronization manifold. The second scenario should reach the anti-phase synchronization manifold with a small amplitude, due to the almost in-phase startup. As both figures show, the system behaves as expected for both initial conditions. From the first figure can be concluded that a driving mechanism is necessary to make in-phase synchronization possible in steady state. Both scenarios show that the dynamics compensation in combination with the introduction of the specified dynamics works within a certain accuracy and the dynamics mismatch, \( \epsilon \), can be neglected. In case of a large dynamics mismatch after compensation the in-phase startup situation would probably never have ended in a fully damped out steady state and probably the nearly in-phase startup would have damped out.
Figure 6.3: Experimental results for Huijgens’ synchronization of free moving pendula when starting exactly in phase, using system parameters: $m_3 = 50$, $d_3 = 20$, $k_3 = 1$, $k_{e1h} = k_{e2h} = 0$, $g_h = 0.2763$, $\phi_{0\text{ blue}} = -0.79$ [rad].

Figure 6.4: Experimental results for Huijgens’ synchronization of free moving pendula when starting nearly in phase, using system parameters: $m_3 = 50$, $d_3 = 20$, $k_3 = 1$, $k_{e1h} = k_{e2h} = 0$, $g_h = 0.2763$, $\phi_{0\text{ blue}} = -0.79$ [rad] and $\phi_{0\text{ red}} = -0.75$ [rad].
6.2.2 Driven pendula, the real Huijgens experiment!

The synchronization process which takes place within the specified Huijgens’ dynamics costs energy. The more close to in-phase the pendula are released the more energy is lost during the process to reach the anti-phase synchronization state. This phenomena is clearly shown in Figures 6.3 and 6.4 where the first result presents an energy loss of 100 percent, due to the in-phase start, and the second one of almost 100 percent, due to nearly in phase start. To compensate for these energy losses escapement like mechanisms are introduced, similar to the ones presented in section 4.3 and the ones used in the previous chapter.

The proposed controllers are given by:

\[
\begin{align*}
    u_1 &= -k_{e1h}(E - E^*)\dot{\phi}_1 \\
    u_2 &= -k_{e2h}(E - E^*)\dot{\phi}_2,
\end{align*}
\]  

(6.6)

where \(E^*\) is defined as the reference level and \(E\) is defined as the total system energy as:

\[
E = \sum_{i=1}^{2} \frac{m_i}{2} \left( \dot{x}_i^2 + I_i^0 \dot{\phi}_i^2 + 2x_i \ddot{\phi}_i \cos(\phi_i) \right) + \frac{m_3 \dot{x}_3^2}{2} + \sum_{i=1}^{2} m_1 g h_i \left(1 - \cos(\phi_i)\right) + \frac{k_3 x_3^2}{2}. \tag{6.7}
\]

For all initial conditions where system (6.2) can be linear approximated and as long as the driving mechanism not pushes the amplitude of the oscillators in the non linear area, depicted in Figure 6.2, the stability of the anti-phase synchronization manifold is guaranteed and the energy level of the system converges to the reference level as long as the controller gains \(k_{e1h}\) and \(k_{e2h}\) are within the following boundary:

\[
0 < k_h = k_{e1h} = k_{e2h} < \frac{\lambda_{\min}(I)}{2\lambda_{\max}(P)} \frac{||\sigma||_2}{||\sigma_2 - 2\sigma_4||_2^2 E^*}. \tag{6.8}
\]

where \(\sigma\) and \(P\) are defined in appendix D.4 For the current system parameters the bound results in:

\[
0 < k_h < 0.77. \tag{6.9}
\]

The proof is provided by appendix D.4 and is based on the same technique as presented for the driven linear translational scenario but now extended with the linearization of system (6.2) which is valid as long as \(|\phi_1| < 0.3 \text{ [rad]}\) and \(|\phi_2| < 0.3 \text{ [rad]}\).

Three experimental results are presented here to show the effects of different initial conditions and different reference energy levels. All other parameters are constant during the experiments. Figure 6.5 presents the results when the system is released within the linear area, \(\phi_{0 \text{ blue}} = -0.29 \text{ [rad]}\) and \(\phi_{0 \text{ red}} = -0.14 \text{ [rad]}\), and the reference energy level \(E^*\) corresponds with an amplitude for both oscillators of \(\phi_{\text{ref blue}} = \phi_{\text{ref red}} = 0.29 \text{ [rad]}\). The experimental setup behaves as expected and reaches a steady state anti-phase synchronization. Changing the initial conditions, without changing the reference energy level, does not change the final steady state situation. This is presented in Figure 6.6 where the system is released at \(\phi_{0 \text{ blue}} = 0.79 \text{ [rad]}\) and \(\phi_{0 \text{ red}} = 0.14 \text{ [rad]}\), for the “blue” pendulum in the non linear area. The situation changes when the desired energy level ends up in the non linear area of the system which is presented in Figure 6.7. The system is released in the area where it can be linear approximated, at \(\phi_{0 \text{ blue}} = -0.29 \text{ [rad]}\) and \(\phi_{0 \text{ red}} = -0.14 \text{ [rad]}\) and the desired energy level corresponds to an amplitude for both oscillators of \(\phi_{\text{ref blue}} = \phi_{\text{ref red}} = 0.79 \text{ [rad]}\), clearly not within the linear area. The results is no anti-phase synchronization but in-phase synchronization.
Figure 6.5: Experimental results for dual driven Huijgens’ synchronization using system parameters: $m_3 = 50$, $d_3 = 20$, $k_3 = 1$, $k_{v1h} = k_{v2h} = 0$, $g_h = 0.2763$, $\phi_0\text{ blue} = -0.29$ [rad], $\phi_0\text{ red} = -0.14$ [rad], $k_{v1h} = k_{v2h} = 0.005$, $\phi_{ref\text{ blue}} = 0.29$ [rad] and $\phi_{ref\text{ red}} = 0.29$ [rad].

Figure 6.6: Experimental results for dual driven Huijgens’ synchronization using system parameters: $m_3 = 50$, $d_3 = 20$, $k_3 = 1$, $k_{v1h} = k_{v2h} = 0$, $g_h = 0.2763$, $\phi_0\text{ blue} = -0.79$ [rad], $\phi_0\text{ red} = -0.75$ [rad], $k_{v1h} = k_{v2h} = 0.005$, $\phi_{ref\text{ blue}} = 0.29$ [rad] and $\phi_{ref\text{ red}} = 0.29$ [rad].
6.2. EXPERIMENTAL RESULTS AND STABILITY ANALYSIS

Figure 6.7: Experimental results for dual driven Huijgens’ synchronization using system parameters: $m_3 = 50, d_3 = 20, k_3 = 1, k_{e1h} = k_{e2h} = 0, g_h = 0.2763, \phi_0^{\text{blue}} = -0.29 \text{ [rad]}, \phi_0^{\text{red}} = -0.14 \text{ [rad]}, k_{e1h} = k_{e2h} = 0.005, \phi_{\text{ref blue}} = 0.79 \text{ [rad]} \text{ and } \phi_{\text{ref red}} = 0.79 \text{ [rad]}.
6.3 Experimental robustness

All experiments presented till now are done under “normal” circumstances which means no extra disturbances are introduced in the system and the introduced sliding mode identifier only has to cope with the uncertainties present in the original system. To challenge the sliding mode identifier two experiments are presented where extra disturbances are added by means of extra weights placed on both oscillators. Figure 6.8 presents the nominal results, without any extra setup disturbances, and Figure 6.9 presents the experimental results when the “blue” oscillator is disturbed by an extra weight of 306 [g] and the “red” oscillator by 257[g]. This is a relative increase of approximately 50 percent. Both experimental results are done using the following parameters: $m_3 = 50 \, [kg]$, $d_3 = 20 \, [Nsm^{-1}]$, $k_3 = 1 \, [Nm^{-1}]$. The oscillators are released at: $x_{0 \text{blue}} = -0.29 \, [rad]$ and $x_{0 \text{red}} = -0.14 \, [rad]$. Only the “blue” oscillator is driven using gain $k_{eh} = 0.005$ and has a reference amplitude of $x_{ref} = 0.61 \, [rad]$. Both oscillator have a frequency of $1 \, [Hz]$, a mass of $1 \, [kg]$ and no damping. Comparing the nominal result and the disturbed result shows that the error between real and observed signal increases by a factor of two what can be declared by the fact that the sliding mode observer needs to put more effort in the control signal to compensate for the introduced disturbance, causing more “noise” on the measured position signal. This noise is filtered by the observer causing a larger difference between the observed and clean signal versus the real signal. The difference between the observed and real behavior is not only a small frequency mismatch, causing a phase difference, but is also a small amplitude error. The energy plot shows a clear difference too. The nominal systems energy increases faster than the energy level of the disturbed system. This can be caused by the amplitude difference in between the observer and ideal system. This amplitude observation error is approximately 2 percent. Apart from these differences, the steady state behavior of the disturbed system is equal to the non disturbed system.
Figure 6.9: Experimental results when increasing the weight of both oscillators by approximately 50 percent.
6.4 Discussion

Although no formal proof can be given for the full non-linear system a couple of remarkable results appeared during the experiments. It seems that the type of synchronization which is occurring is mainly dependent on the product of the controller gains and the reference energy level. If this product is large in-phase synchronization seems to be the steady state solution and a fairly small product ends up in anti-phase synchronization.

Apart from these parameters it seems that the mass ratio between beam and pendula can result in different synchronization regimes. A relatively heavy beam compared to the pendula masses results in anti-phase and in-phase synchronization. A relatively light beam mainly results in anti-phase synchronization.

As long as the rotational system is used within the linear area it behaves equal to the linear translational system and high gains are needed to manage in-phase synchronization in this area.

To come more close to Huijgens’ original experiments controllers are implemented based on Hamiltonian functions, presenting a sort of energy per pendulum. The controllers have the following structure:

\[
\begin{align*}
    u_1 &= -k_{c1b}(H_1 - H_1^*)\dot{\phi}_1 \\
    u_2 &= -k_{c2b}(H_2 - H_2^*)\dot{\phi}_2,
\end{align*}
\]

where \(H_i^*\) presents the reference level and \(H_i\) is defined as:

\[
\begin{align*}
    H_1 &= m_{1h}g_1l_{1h}(1 - \cos(\phi_1)) + \frac{1}{2}m_{1h}l_{1h}^2\dot{\phi}_1^2 \\
    H_2 &= m_{2h}g_2l_{2h}(1 - \cos(\phi_2)) + \frac{1}{2}m_{2h}l_{2h}^2\dot{\phi}_2^2.
\end{align*}
\]

The results using these controllers can be compared with the described type of controller but no formal proof can be provided for stability of the anti-phase synchronization manifold nor the fact that the controllers converge to their reference levels.

It seems that the speed of the process to reach anti-phase synchronization is dependent on the amount of initial energy present in the system but no mathematical proof can be given at the moment.

6.5 Conclusion

The experimental results are behaving as expected. The introduced driving mechanisms have the intended result but no formal proof can be given for the moment which is valid for all initial conditions and all reference energy levels. The most remarkable conclusion is that it seems that key factor for the resulting synchronization manifold is the product of the driving mechanisms reference energy level and the gains of the driving mechanisms.
Chapter 7

Conclusions and recommendations

7.1 Conclusions

This report presents the results of the second research project done by using the experimental setup developed at the department of Mechanical engineering of the Eindhoven University of Technology. The ultimate goal for this research project is to reconstruct Huijgens synchronizing experiments on the experimental setup. To achieve this goal a model is developed for the experimental setup whereafter the internal dynamics of the experimental setup are compensated in order to create the possibility to introduce desired dynamics on the setup. The results are described in detail by the following paragraphs. Speaking about “phase one” or “the first phase” means: the first phase of the synchronization study using the experimental setup done by Rijlaarsdam [22].

Setup modeling

The model of the setup, derived in the first phase, is modified and is presented in this report. A fundamental change is made in how to tread input signals and how several motor constants are being used. The modified model gives a correct representation of the system concluded from the fact that parameters used in simulation mode can directly be used on the experimental setup without causing different behavior on the experimental setup.

Identification structure

Because of the presence of unknown dynamics in the experimental setup an identification structure is developed which is capable to identify most of the unknown dynamics of the setup. The identifier is based on a sliding mode identifier [23] in combination with the theory on “equivalent” control [27].

Because the full state vector is not available in the measured signal a full state Luenberger observer is introduced to recover the state vector of the system parallel to the sliding mode identifier. A positive side effect of this observer is that it reconstructs the position signal whereby reducing the noise level of the signal.

Due to the translational structure of the setup the inputs are coupled. Input decoupling is introduced to make it possible to introduce new coordinate systems on the setup, for example rotational coordinates.

By introducing this identification structure the calibration done during the setup initialization, in phase one, is not necessary any more.
Quality of the implemented identification structure including the Luenberger observer

The robustness of the sliding mode identifier is challenged in two scenarios. The first one showed that the identifier is robust when the a priori knowledge is reduced by approximately 20 percent. The second scenario, by using Huijgens’ dynamics, showed that the identifier is robust for extra disturbances introduced on the experimental setup.

Several sets of specified dynamics are introduced on the experimental setup and all sets show appropriate results. First of all stand alone (driven) linear and nonlinear oscillators are implemented on the setup by using a fixed beam situation. By using the compensation strategy it is possible to implement uncontrolled oscillator dynamics with an accuracy of approximately 99.9 percent on the natural frequency and 99 percent on the damping. These results are generated on the “blue” and the “red” oscillator. By adding energy based driving mechanisms the damping accuracy can be improved to approximately 100 percent.

Linear synchronization

Several successful experiments are presented for the scenario with two (driven) linear translational oscillators. A prove is presented for the introduced driving mechanism based on the total system energy. The prove results in a boundary for the controller gains which is really conservative. Simulation and experimental results show that the boundary values can be increased without making the anti-phase synchronization manifold instable for the feasible initial conditions on the experimental setup.

Huijgens’ synchronization

In the scenario where both pendula are driven, it seems that the reached synchronization manifold is depending on the reference value for both driving mechanisms in combination with the controller gain. When both driving mechanisms control to a final amplitude level within the linear area of the setup, anti-phase synchronization occurs. When both control to a level within the non-linear area in-phase synchronization occurs. In the case that one controller stays within the linear area and one tries to reach the non linear area, no clear synchronization can be detected. It seems that this is independent of the initial conditions.

Generally speaking the experimental results show that the system is more willing to go in-phase synchronization when the mass ratio of the beam and the pendula is decreasing.

Experimental setup and infrastructure

Based on all experiment done during this research project the conclusion can be stated that the physical properties of the experimental setup do not need to change, as stated after phase one. The conclusion stated in phase one, that the actuators are not powerful enough, found its origin in the motor constant modeling error. This problem is solved by using the modified model presented in the report.

The quality of the compensation strategy is fully dependent on the quality of the measured signals. The input signals of the TUeDACS are not clean. Grounding all inputs results in a noisy signal and for one input even a sawtooth kind of signal appears. Despite of this the results of the experiments are appropriate but can be improved.

7.2 Recommendations

The most important result of this research phase is that the experimental setup can be used to implement all kind of synchronizing dynamical systems. To finalize this thesis a set of recommendations is given with a clear focus on future experiments to get more insight in synchronizing behavior.
7.2. RECOMMENDATIONS

Beam model

Huijgens observed that the distance between both pendula was a key factor in the type of synchronization occurring. The distance presents a coupling strength between both pendula. The model used in this thesis also contains a coupling strength but this one presents the coupling between each pendulum and the beam and therefore can not be used to influence the distance between both oscillators. To model the beam by using finite element methods can be a part of future studies to incorporate this effect.

Dynamics compensation

To improve the quality of the compensation mechanism two things can be part of future work. The first one is to improve the quality of the measured signals by the TUEDAC. The second thing is to re-identify all system parameters making use of the model presented in this thesis. The coupling parameters $\mu_i$ are a little suspicious because the identifier of the beam dynamics relatively needs a lot of effort to identify the dynamics which are assumed to be quit linear.

Synchronization investigations

Using the specified (Huijgens’) dynamics on the experimental setup a part of future work can be to determine parameter sets which show in-phase or anti-phase behavior and to get more insight in the properties of synchronization. A part of this can be to investigate the difference between the results generated on the experimental setup and the results generated with the new FEM beam model.

Stability analysis

The stability of the introduced driving mechanisms, within the specified dynamics of the oscillators, needs to be analyzed when using the oscillators in a combined, synchronizing, scenario. The stand-alone driven oscillators are stable and fulfill the control goal to reach a certain amount energy but no general guarantees are given in this report for stability and resulting energy level of the synchronizing scenario. This analysis is dependent on the type of synchronization manifold where a scenario ends so this needs to be included in the analysis.

Driving mechanisms

To come more close to Huijgens’ original experiments future work can be to replace the energy based driving mechanisms by more escapement like mechanisms and to investigate the influence on the synchronizing behavior of the system.
Bibliography


Appendix A

Escapement mechanism

A.1 Pendulum clock

The principles of the pendulum clock used in Huijgens’ setup do not differ a lot from the principles of pendulum clocks nowadays. The clock can be separated in three parts. A driving mechanism (escapement), a timekeeping mechanism (escapement and pendulum) and a mechanism to convert the oscillating motion to an interpretable time. These three parts are shown in Figure A.1. The escapement has a double function, it is part of the driving mechanism and also part of the timing mechanism. Basically the mass, at the start of the chain, directly drives the hands of the clock with a continuous rotation. Due to the escapement mechanism the motion of the gear train is made periodical with a frequency equal to the frequency of the pendulum. The next sections describe the escapement mechanism and the pendulum in detail.

Concept

The escapement mechanism contains of two parts. One wheel on the driving axes between the two gearboxes in Figure A.1 and a sort of fork statically connected to the pendulum. Figure A.2 and A.3 show these two parts in two different states. The working of the escapement is explained based on these two states, the lock-state in Figure A.2 and the driving state in Figure A.3. Starting in the lock state, the pendulum in the full right position and starts moving to the left, Figure A.2. When the gear train is free from the escapement it starts moving clockwise due to the driving force. In the meanwhile the pendulum gets in its full left position where the left tooth of the escapement-fork meets the rotating wheel of the escapement, the driving state is reached as shown in Figure A.3. Due to the rotating wheel the pendulum is pushed away and accelerated in the counter clockwise direction until the fork loses contact with the gear train on the left side. After half a period the gear train is blocked again by the fork which moves in between two teeth of the escapement wheel and the cycle starts again.
Construction

The fork, shown in Figure A.2 and A.3, seems to stop against the center of the wheel. In reality this is not the case, the fork stops before hitting the center of the wheel what causes a smooth sinusoidal motion of the pendulum. In other words: the maximum reachable amplitude of the pendulum is never reached during operation and the pendulum never stops or gets locked, only the gearbox gets locked. To guarantee the timekeeping mechanism the fork in combination with the escapement wheel is constructed in a way that the wheel can never freely rotate more than one tooth per period.

Energy and control

Each period the system loses energy due to friction. This energy is supplied to the pendulum in the driving state when the wheel is pushing the fork. Although the driving system looks like an open loop system without any feedback the system has to be closed loop, with feedback. This can be concluded from the fact that the system is stable and that the amplitude of the pendulum keeps within boundaries although several inaccuracies are present in the mechanical system. Everything in the escapement mechanism is position triggered and not time triggered. In the most ideal situation the position based triggering becomes periodical.

Modeling

Due to the complex and highly non linear design of the escapement mechanism it is not possible to create an accurate model which can be used for calculations.

A.1.1 Pendulum

Not only the escapement mechanism is highly non linear, the pendulum is non linear too. Figure A.4 shows the frequency behavior of 5 different pendula. As long as the amplitude of the pendulum keeps within $0.2\text{rad}$ the frequency of the pendulum can be assumed to be amplitude independent and calculated according to equation (A.1) where $T$ is the period of the pendulum, $l$ is its length and $g$ is the gravitational
acceleration. When increasing the amplitude over $0.2 [\text{rad}]$, the pendulum's frequency decreases up to 15% when the amplitude reaches $0.5\pi$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{pendulum_freq_vs_amp.png}
\caption{Amplitude - frequency relationship pendulum.}
\end{figure}

\[ T = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \] \hspace{1cm} (A.1)
Appendix B

Euler-Lagrange derivations

This chapter describes the Euler-Lagrange derivation of the equations of motion used for the model according to

\[
\left[ \frac{d}{dt} (T'_{q_\dot{q}}) - T'_{\dot{q}} + V'_{\dot{q}} \right] T = Q^{nc} + W\lambda, \tag{B.1}
\]

where \( T \) and \( V \) represent the kinetic and potential energy respectively. All non-conservative forces are depicted in the \( Q \) vector, the vector \( W \) presents all coordinate constraints and \( \lambda \) represents the Lagrange multipliers. Finally \( q \) represents the generalized coordinate vector.

B.1 Euler-Lagrange applied on the experimental setup

Consider the model depicted in Figure B.1, where \( x_i \) are the absolute coordinates with respect to a fixed reference frame. To simplify the derivation of the Euler-Lagrange equation a set of relative coordinated \( q_i \) is used. Equation B.2 defines the coordinate transformation.

\[
x = Tq, \quad T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{B.2}
\]

Using the relative coordinates \( q_i \), the total amount of kinetic energy is given by:

\[
T(q) = \frac{1}{2} q^T \begin{bmatrix} m_1 & 0 & m_1 \\ 0 & m_2 & m_2 \\ m_1 & m_2 & m_1 + m_2 + m_3 \end{bmatrix} q = \frac{1}{2} q^T M q, \tag{B.3}
\]

where \( M \) represents the mass matrix of the system. To determine the potential energy and the non-conservative forces all springs and dampers are assumed to be linear, \( k_i \approx \kappa_i \) and \( d_i \approx \beta_i \). The resulting
potential energy, stored in the springs is given by:

\[
V(q) = \frac{1}{2} q^T K q,
\]

where \( K \) represents the stiffness matrix. The non-conservative forces are depicted in the following equation:

\[
Q^{nc} = \begin{bmatrix}
F_1(t) - b_1 \dot{q}_1 \\
F_2(t) - b_2 \dot{q}_2 \\
F_3(t) - F_1(t) - F_2(t) - b_3 \dot{q}_3
\end{bmatrix},
\]

where \( F_i \) represents the resulting force on a mass. These forces are generated by the voltage inputs \( u_i \) multiplied by the motor constants \( c_i \) so (B.1) can be rewritten to:

\[
Q^{nc} = \begin{bmatrix}
u_1(t)c_1 - b_1 \dot{q}_1 \\
u_2(t)c_2 - b_2 \dot{q}_2 \\
u_3(t)c_3 - u_1(t)c_1 - u_2(t)c_2 - b_3 \dot{q}_3
\end{bmatrix}.
\]

Using the Euler-Lagrange equation the equation of motion appears in the following, familiar form:

\[
M \ddot{q} + B \dot{q} + K q = F(t),
\]

where \( B \) represents the damping matrix and \( F \) the actuation forces according to:

\[
B = \begin{bmatrix}
b_1 & 0 & 0 \\
0 & b_2 & 0 \\
0 & 0 & b_3
\end{bmatrix}
\quad F(t) = \begin{bmatrix}
u_1(t)c_1 \\
u_2(t)c_2 \\
u_3(t)c_3 - u_1(t)c_1 - u_2(t)c_2
\end{bmatrix}.
\]

Rewriting this resulting equation of motion in the absolute coordinates \( x_i \) results in the following set of equations:

\[
\begin{align*}
\ddot{x}_1 &= \frac{1}{m_1} ((x_3 - x_1)k_1 + (\dot{x}_3 - \dot{x}_1)d_1 + u_1c_1) \\
\ddot{x}_2 &= \frac{1}{m_2} ((x_3 - x_2)k_2 + (\dot{x}_3 - \dot{x}_2)d_2 + u_2c_2) \\
\ddot{x}_3 &= \frac{1}{m_3} (-x_3K - \dot{x}_3D + x_2k_2 + \dot{x}_2d_2 + x_1k_1 + \dot{x}_1d_1 + u_3c_3 - u_1c_1 - u_2c_2),
\end{align*}
\]

where \( K = \sum_{i=1}^{3} k_i \) and \( D = \sum_{i=1}^{3} d_i \). Defining \( \Delta x_i = x_i - x_3 \), \( \omega_i = \sqrt{\frac{k_i}{m_i}} \) \( \text{[rad/s]} \) and \( \zeta_i = \frac{d_i}{2\omega_i m_i} \) \( \text{[-]} \) for the subsystems \( i = 1, 2 \) as the displacement with respect to the beam, the undamped eigenfrequency and dimensionless damping results in the next set of equations:

\[
\begin{align*}
\ddot{x}_1 &= -\omega_1^2 \Delta x_1 - 2\zeta_1 \omega_1 \Delta \dot{x}_1 + c_1 u_1(t) \\
\ddot{x}_2 &= -\omega_2^2 \Delta x_2 - 2\zeta_2 \omega_2 \Delta \dot{x}_2 + c_2 u_2(t) \\
\ddot{x}_3 &= -\omega_3^2 \Delta x_3 - 2\zeta_3 \omega_3 \Delta \dot{x}_3 + \sum_{i=1}^{2} \mu_i \left[ \omega_i^2 \Delta x_i + 2\zeta_i \omega_i \Delta \dot{x}_i - c_i u_i(t) \right] + c_3 u_3(t),
\end{align*}
\]

where \( \mu_i = \frac{m_i}{m_3} \) \( \text{[-]} \) is the dimensionless coupling strength and the parameters \( c_i \) \( \text{[ms}^2 \text{V}^{-1}] \) are the amplifier and motor constants. These equations are equal to the presented equations in chapter 3.

**B.1.1 Short notations**

(B.10) can be written in a more generalized way according to:

\[
\begin{align*}
\ddot{x}_1 &= f_1(.) + \gamma_1(.) + \eta_1(.) \\
\ddot{x}_2 &= f_2(.) + \gamma_2(.) + \eta_2(.) \\
\ddot{x}_3 &= f_3(.) + \gamma_3(.) - \mu_1 \gamma_1(.) - \mu_2 \gamma_2(.) + \eta_3(.)
\end{align*}
\]

where \( \mu_i = \frac{m_i}{m_3} \) \( \text{[-]} \) is the dimensionless coupling strength and the parameters \( c_i \) \( \text{[ms}^2 \text{V}^{-1}] \) are the amplifier and motor constants. These equations are equal to the presented equations in chapter 3.
where \( f_1(.) \) represents the known dynamics of the system, shown in (B.12) where \( \beta_i = 2\zeta_i\omega_i, \gamma_i \) represents the unknown dynamics and \( \eta_i \) represents the input signals, shown in (B.13).

\[
\begin{align*}
  f_1(.) &= -\omega_1^2 \Delta x_1 - \beta_1 \Delta \dot{x}_1 \\
  f_2(.) &= -\omega_2^2 \Delta x_2 - \beta_2 \Delta \dot{x}_2 \\
  f_3(.) &= -\omega_3^2 x_3 - \beta_3 \dot{x}_3 + \mu_1 \omega_1^2 \Delta x_1 + \mu_1 \beta_1 \Delta \dot{x}_1 + \mu_2 \omega_2^2 \Delta x_2 + \mu_2 \beta_2 \Delta \dot{x}_2
\end{align*}
\]  

(B.12)

\[
\begin{align*}
  \eta_1(.) &= c_1 u_1(t) \\
  \eta_2(.) &= c_2 u_2(t) \\
  \eta_3(.) &= c_3 u_3(t) - \mu_1 c_1 u_1 - \mu_2 c_2 u_2
\end{align*}
\]  

(B.13)

For analysis purposes (B.11) is written in relative coordinates according to:

\[
\begin{align*}
  \Delta \ddot{x}_1 &= f_1(.) - f_3(.) + (1 + \mu_1) \gamma_1(.) + \mu_2 \gamma_2 - \gamma_3(.) + \eta_1(.) - \eta_3(.) \\
  \Delta \ddot{x}_2 &= f_2(.) - f_3(.) + \mu_2 \gamma_1 + (1 + \mu_2) \gamma_2(.) + \eta_2(.) - \eta_3(.) \\
  \Delta \ddot{x}_3 &= f_3(.) + \gamma_3(.) - \mu_1 \gamma_1(.) - \mu_2 \gamma_2(.) + \eta_3(.)
\end{align*}
\]  

(B.14)

where \( \Delta x_i = x_1 - x_3, \ i = 1, 2 \) and \( \Delta x_3 = x_3 \) what results in the following short notation:

\[
\begin{align*}
  \Delta \ddot{x}_1 &= \tilde{f}_1 + \tilde{\eta}_1 + (1 + \mu_1) \gamma_1 + \mu_2 \gamma_2 - \gamma_3 \\
  \Delta \ddot{x}_2 &= \tilde{f}_2 + \tilde{\eta}_2 + \mu_2 \gamma_1 + (1 + \mu_2) \gamma_2 - \gamma_3 \\
  \Delta \ddot{x}_3 &= \tilde{f}_3 + \tilde{\eta}_3 + \gamma_3 - \mu_1 \gamma_1 - \mu_2 \gamma_2
\end{align*}
\]  

(B.15)

where \( \tilde{f}_i = f_i - f_3, \ i = 1, 2, \tilde{\eta}_i - \eta_3, \ i = 1, 2, \tilde{f}_3 = f_3 \) and \( \tilde{\eta}_3 = \eta_3 \).

**B.2 Euler-Lagrange applied on Huijgens’ setup**

A simplified model of Huijgens’ setup is depicted in Figure B.2. The same procedure can be followed to retrieve the equations of motion. The generalized coordinate vector \( q \) is defined as:

\[
q = [\ x_1 \ y_1 \ x_2 \ y_2 \ x_3 \ \phi_1 \ \phi_2 \ ]
\]  

(B.16)
These equations are written in the closed form, assuming that

\[ T(q) = \frac{1}{2} m_3 \ddot{x}_3^2 + \frac{1}{2} m_1 (\ddot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\ddot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} J_1 \dot{\phi}_1^2 + \frac{1}{2} J_2 \dot{\phi}_2^2 \]

(B.17)

\[ V(q) = \frac{1}{2} k_3 x_3^2 + m_1 g y_1 + m_2 g y_2 \]

(B.18)

where \( J_i, i = 1, 2 \) represents the moment of inertia of a pendulum. Coordinate constraints are present in the system which are described by the following set of equations:

\[
\begin{align*}
    x_1 &= x_3 + l_1 \sin \phi_1 \\
    y_1 &= -l_1 \cos \phi_1 \\
    x_2 &= x_3 + l_2 \sin \phi_2 \\
    y_2 &= -l_2 \cos \phi_2,
\end{align*}
\]

(B.19)

what can be represented in the matrix \( W \) according to:

\[
W = \begin{bmatrix}
    1 & 0 & 0 & 0 & -1 & -l_1 \sin (\phi_1) & 0 \\
    0 & 1 & 0 & 0 & 0 & l_1 \cos (\phi_1) & 0 \\
    0 & 0 & 1 & 0 & -1 & 0 & -l_2 \sin (\phi_1) \\
    0 & 0 & 0 & 1 & 0 & 0 & l_2 \cos (\phi_2)
\end{bmatrix}.
\]

(B.20)

The non-conservative forces/torques applied to the system are represented in the \( Q_{nc} \) matrix according to:

\[
Q_{nc} = \begin{bmatrix}
    0 & 0 & 0 & u_3 c_3 - d_3 x_3 & u_3 c_1 - d_1 \dot{\phi}_1 & u_2 c_2 - d_2 \dot{\phi}_2
\end{bmatrix}^T,
\]

(B.21)

where \( c_i \) are the motor constants and \( d_i \) are the damping coefficients. The assumption is made that the torques on the pendula not influence the beams motion. The Euler-Lagrange equation can be written in the following set of equations:

\[
\begin{align*}
    m_1 \ddot{x}_1 &= \lambda_1 \\
    m_1 \ddot{y}_1 + m_1 g &= \lambda_2 \\
    m_2 \ddot{x}_2 &= \lambda_3 \\
    m_2 \ddot{y}_2 + m_2 g &= \lambda_4 \\
    m_3 \ddot{x}_3 &= u_3 c_3 - k x_3 - \lambda_1 - \lambda_3 \\
    J_1 \ddot{\phi}_1 &= u_1 c_1 - d_1 \dot{\phi}_1 - \lambda_1 l_1 \cos \phi_1 - \lambda_2 l_1 \sin \phi_1 \\
    J_2 \ddot{\phi}_2 &= u_2 c_2 - d_2 \dot{\phi}_2 - \lambda_3 l_2 \cos \phi_2 - \lambda_4 l_2 \sin \phi_2,
\end{align*}
\]

(B.22)

what, after eliminating of \( \lambda_i \), results in (B.23), which is equal to the presented Huygens equations in Rijlaarsdam [22].

\[
\begin{align*}
    (J_1 + m_1 l_1^2) \ddot{\phi}_1 + m_1 l_1 \dddot{x}_3 \cos \phi_1 + m_1 g l_1 \sin \phi_1 &= u_1 c_1 - d_1 \dot{\phi}_1 \\
    (J_2 + m_2 l_2^2) \ddot{\phi}_2 + m_2 l_2 \dddot{x}_3 \cos \phi_2 + m_2 g l_2 \sin \phi_2 &= u_2 c_2 - d_2 \dot{\phi}_2 \\
    m_3 \dddot{x}_3 + \sum_{i=1}^{2} m_i l_i (\ddot{\phi}_i \cos \phi_i - \dot{\phi}_i^2 \sin \phi_i) + m_3 \dddot{x}_3 &= -k x + u_3 c_3.
\end{align*}
\]

(B.23)

These equations are written in the closed form, assuming that \( J_i = 0 \), given by:

\[
\begin{align*}
    \ddot{\phi}_1 &= \frac{1}{m_1 l_1^2} \left[ u_1 c_1 - d_1 \dot{\phi}_1 - m_1 l_1 \dddot{x}_3 \cos \phi_1 - m_1 g l_1 \sin \phi_1 \right] \\
    \ddot{\phi}_2 &= \frac{1}{m_2 l_2^2} \left[ u_2 c_2 - d_2 \dot{\phi}_2 - m_2 l_2 \dddot{x}_3 \cos \phi_2 - m_2 g l_2 \sin \phi_2 \right] \\
    \dddot{x}_3 &= \frac{1}{B} \left[ -d_1 \dddot{x}_3 - k_3 x_3 + u_3 c_3 - \sum_{i=1}^{2} \frac{\cos \phi_i}{l_i} (u_i c_i - m_i g l_i \sin \phi_i) - m_i l_i \phi_i^2 \sin \phi_i \right],
\end{align*}
\]

(B.24)
where $B = (M + 2m) - m \cos^2 \phi_1 - m \cos^2 \phi_2$. 
Appendix C

Additional graphical results

C.1 Fine tuning steps sliding mode identifier and state observer

This section presents some extra graphical information belonging to the step by step tuning process described in chapter 4.

Figure C.1: Experimental results using a priori compensation and sliding mode compensation based on the initial estimated parameters at simulation level. The parameters of the sliding mode mechanism need to be tuned to improve the result.
Figure C.2: Experimental results using a priori compensation and fine tuned sliding mode compensation. Clearly the energy levels of the oscillators are not stable yet causing increasing and decreasing amplitudes.
Figure C.3: Experimental results using a priori compensation, tuned sliding mode compensation and low full state observer gains. The energy levels are almost constant but extra controlling is needed to stabilize the oscillations amplitude.
APPENDIX C. ADDITIONAL GRAPHICAL RESULTS
Appendix D

Synchronization stability analysis

D.1 Stability of the synchronization manifold for non driven translational oscillators

Consider (D.1) which presents the coupled differential equations, representing the experimental setup, already presented by (5.3) in section 5.1.

\[
\begin{align*}
\ddot{x}_1 &= -\omega_s^2 \nu_1(q_1)q_1 \\
\ddot{x}_2 &= -\omega_s^2 \nu_2(q_2)q_2 \\
\ddot{x}_3 &= -\omega_s^2 \nu_3(q_3)q_3 - 2\zeta_s \omega_s q_3 \sigma(\dot{q}_3) + \sum_{i=1}^{2} \mu_i \left[ \omega_s^2 \nu_i(q_i)q_i \right],
\end{align*}
\]

(D.1)

where \( q_i = x_i - x_3 \) \( \forall i = 1, 2 \), \( q_3 = x_3 \), \( \omega_s = \sqrt{\frac{k_i}{m_i}} \in \mathbb{R}_{>0} \), \( \zeta_s \in \mathbb{R}_{>0} \), \( \mu_i = \frac{m_i}{m_3} \) and \( m_i \in \mathbb{R}_{>0} \) \( \forall i = 1, 2, 3 \). The functions \( \nu_i(q_i) \) and \( \sigma(q_3) \) are (nonlinear) shape functions. Function \( \nu_i(q_i) \) is chosen to be an odd, one to one, continuous function such that \( \nu_i(q_i)q_i \) has a zero only at \( q_i = 0 \). Function \( \sigma(q_3) \) is chosen to be such that \( q_3 \sigma(q_3) > 0 \) \( \forall \dot{q}_3 \neq 0 \) and \( \sigma(0) = 0 \).

To prove the global asymptotic stability of the anti-phase synchronization manifold Lyapunovs direct method is used in combination with LaSalles invariance principle.

**Theorem D.1.** Global asymptotic stability of the synchronization manifold, (Rijlaarsdam, 2008)

Consider system (D.1) and define \( S \subseteq \mathbb{R}^6 \) as the manifold \( S = \{ [x]^T \in \mathbb{R}^6 \mid x_1 = -x_2, \dot{x}_1 = -\dot{x}_2, x_3 = 0 \} \). Assume, furthermore, odd, one to one, continuous functions \( \nu_i(q_i) \), such that \( q_i \nu_i(q_i) \) has a zero only at \( q_i = 0 \). Function \( \sigma(q_3) \) is chosen to be such that \( q_3 \sigma(q_3) > 0 \) \( \forall \dot{q}_3 \neq 0 \) and \( \sigma(0) = 0 \). Finally, assume the following oscillator properties:

1. \( \nu_1(.) = \nu_2(.) \)
2. \( \nu_i(.) \) such that \( \int_0^\infty s \nu_i(s)ds \to \infty \) if \( |x| \to \infty \) \( i = 1, 2, 3 \)
3. \( \mu_1 = \mu_2 \)

The the system (D.1) will converge to \( S \) as \( t \to \infty \) for almost all initial conditions.

**Proof.** Consider system (D.1). To analyse the limit behavior of this system, consider the total energy as a candidate Lyapunov function:

\[
V = \frac{1}{2} \sum_{i=1}^{3} m_i \dot{x}_i^2 + \frac{3}{2} \sum_{i=1}^{3} \int_0^{q_i} k_i \nu_i(s) ds.
\]

(D.2)
Calculating the time derivative of $V$ along all solutions of (D.1) results in:

$$\dot{V} = \sum_{i=1}^{3} m_i \dot{x}_i \ddot{x}_i + \sum_{i=1}^{3} \frac{d}{dt} \int_{0}^{q_i(t)} k_i \nu_i(s) ds.$$  \hspace{1cm} (D.3)

The terms in the second sum may be written as:

$$\frac{d}{dt} \int_{0}^{q_i(t)} k_i \nu_i(s) ds = k_i \nu_i(q_i) \dot{q}_i \dot{q}_i,$$  \hspace{1cm} (D.4)

what leads to:

$$\dot{V} = \sum_{i=1}^{3} m_i \dot{x}_i \ddot{x}_i + \sum_{i=1}^{3} k_i \nu_i(q_i) \dot{q}_i \dot{q}_i = -2\zeta \omega_s \dot{q}_3^2 \sigma(q_3).$$  \hspace{1cm} (D.5)

Hence, $\dot{V} \leq 0$ and the system may be analyzed using LaSalle’s invariance principle.

Equation (D.1) implies that $V$ is a bounded function of time. Moreover, $x_i(t)$, is a bounded function of time and converges to a limit set where $\dot{V} = 0$. On this limit set $\dot{q}_3 = \ddot{x}_3 = 0$, according to (D.1) and thus $q_3 = x_3 = x_3^*$ is constant. Substituting this and the first two equations of (D.1) in the third equation of (D.1) yields:

$$\dddot{x}_1 + \dddot{x}_2 = -x_3^* \omega_s \mu \nu_3(x_3^*).$$  \hspace{1cm} (D.6)

Integrating (D.1) twice with respect to time yields:

$$x_1 + x_2 = -\frac{x_3^* \omega_s \mu}{\mu} \nu_3(x_3^*) t^2 + c_1 t + c_2.$$  \hspace{1cm} (D.7)

However, since both $x_1(t)$ and $x_2(t)$ are bounded functions of time, this yields $x_1^* = c_1 = c_2 = 0$. Substituting $x_3 = \dot{x}_3 = \ddot{x}_3 = 0$ in the third equation of (D.1) results in:

$$\nu_1(q_1) q_1 = -\nu_2(q_2) q_2.$$  \hspace{1cm} (D.8)

Since $\nu_i(q_i)$ is an odd function and $x_3 = 0$ on the limit set, this yields:

$$x_1 = -x_2,$$  \hspace{1cm} (D.9)

which, after differentiation with respect to time results in:

$$\dot{x}_1 = -\dot{x}_2.$$  \hspace{1cm} (D.10)

Summarizing, it has been shown that as $t \to \infty$ any solution will converge to the set where:

$$x_1 = -x_2, \quad \dot{x}_1 = -\dot{x}_2, \quad x_2 = \dot{x}_3 = 0$$  \hspace{1cm} (D.11)

D.2 Stability of the anti-phase synchronization manifold for driven translational oscillators

D.2.1 Free moving linear oscillators, the nominal system

To show the stability of the anti-phase synchronization manifold for linear non driven coupled translational oscillators according to:

$$\ddot{x}_1 = -\omega_s^2 \Delta x_1$$
$$\ddot{x}_2 = -\omega_s^2 \Delta x_2$$
$$\ddot{x}_3 = -\omega_{bs}^2 x_3 - 2\zeta \omega_{bs} \omega_s \dot{x}_3 + \sum_{i=1}^{2} \mu_{si} [\omega_s^2 \Delta x_i],$$  \hspace{1cm} (D.12)
where all parameters equally defined as in chapter 5.1 The following coordinate transformation is introduced:

\[
\begin{align*}
\sigma_1 &= x_1 + x_2 \\
\sigma_2 &= \dot{x}_1 + \dot{x}_2 \\
\sigma_3 &= x_3 \\
\sigma_4 &= \dot{x}_3.
\end{align*}
\] (D.13)

Writing system (D.12) in these new coordinates (D.13) results in the following state space representation of the system:

\[
\begin{bmatrix}
\dot{\sigma}_1 \\
\dot{\sigma}_2 \\
\dot{\sigma}_3 \\
\dot{\sigma}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-\omega_s^2 & 0 & 2\omega_s^2 & 0 \\
0 & 0 & 0 & 1 \\
\mu_s\omega_s^2 & 0 & -2\mu_s\omega_s^2 - \omega_{bs}^2 & -2\zeta_{bs}\omega_{bs}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4
\end{bmatrix}
= A\sigma.
\] (D.14)

Stability in \(\sigma\) is concluded when calculating the eigenvalues \(\lambda\) of the \(A\) matrix for the given parameter set resulting in:

\[
\begin{align*}
\lambda_{1,2} &= -0.0133 \pm 6.1139i \\
\lambda_{3,4} &= -0.3849 \pm 9.9991i
\end{align*}
\] (D.15)

which all have a negative real part causing \(\sigma_i\) to converge to zero according to:

\[
\begin{align*}
\lim_{t \to \infty} \sigma_1 &= 0 = x_1 + x_2 \\
\lim_{t \to \infty} \sigma_2 &= 0 = \dot{x}_1 + \dot{x}_2 \\
\lim_{t \to \infty} \sigma_3 &= 0 = x_3 \\
\lim_{t \to \infty} \sigma_4 &= 0 = \dot{x}_3.
\end{align*}
\] (D.16)

From the first two equations of (D.16) anti-phase synchronization, \(x_1 = -x_2\) and \(\dot{x}_1 = -\dot{x}_2\), can be concluded.

### D.2.2 Driven oscillators, the perturbed system

Appendix D.2.1 shows the stability of the anti-phase synchronization manifold for the nominal linear pair of coupled oscillators described by system D.12. To compensate for the energy losses occurring by the synchronization process driving mechanisms are added to the oscillators resulting in the following system dynamics:

\[
\begin{align*}
\ddot{x}_1 &= -\omega_s^2 \Delta x_1 - k_{e_1h} (E - E^*) \Delta \dot{x}_1 \\
\ddot{x}_2 &= -\omega_s^2 \Delta x_2 - k_{e_2h} (E - E^*) \Delta \dot{x}_2 \\
\ddot{x}_3 &= -\omega_{bs}^2 x_3 - 2\zeta_{bs}\omega_{bs} \dot{x}_3 + \sum_{i=1}^{2} \mu_{si} \left[ \omega_s^2 \Delta x_i + k_{e_ih} (E - E^*) \Delta \dot{x}_i \right],
\end{align*}
\] (D.17)

where all parameters are equally defined as shown in chapter 5. As long as \(k_h\) satisfies:

\[
0 < k_h < \frac{\lambda_{\min}(P)}{2\lambda_{\max}(P)} \frac{||\sigma||_2}{||\sigma_2 - 2\sigma_4||_2 E^*},
\] (D.18)

system (D.17) converges to the anti-phase synchronization manifold and the systems energy level \(E\) converges to \(E^*\).
Proof. The first part of the prove determines a boundary for the controller gains $k_{c1h}$ and $k_{c2h}$ based on the theory on vanishing perturbations [12]. Within these boundaries the controller inputs will not affect the stability of the anti phase synchronization manifold, proven in appendix D.2.1. As we can not explicitly calculate the bounds for $k_{c1h}$ and $k_{c2h}$ a conservative bound will be determined by analysis the worst case scenario.

As the $A$ matrix, presented in (D.2.1), is Hurwitz the Lyapunov equation:

$$PA + A^TP = -Q,$$

(D.19)

can be solved in $P$, by using $Q = QT > 0$ and results in a unique solution $P = PT > 0$. The resulting quadratic Lyapunov function $V(x) = \sigma^TP\sigma$ satisfies:

$$\lambda_{\min}(P) \|\sigma\|^2_2 \leq V(\sigma) \leq \lambda_{\max}(P) \|\sigma\|^2_2$$

(D.20)

$$\frac{\partial V}{\partial \sigma}A\sigma = -\sigma^TP\sigma \leq \lambda_{\min}(Q) \|\sigma\|^2_2$$

(D.21)

$$\|\frac{\partial V}{\partial \sigma}\| = \|2\sigma^TP\| \leq 2\|P\|_2 \|\sigma\|_2 = 2\lambda_{\max}(P).$$

(D.22)

The derivative of $V(\sigma)$ along the trajectories of the perturbed system $\dot{\sigma} = A\sigma + g(t, \sigma)$ satisfies:

$$\dot{V}(\sigma) \leq -\lambda_{\min}(Q) \|x\|^2_2 + 2\lambda_{\max}(P)\gamma \|\sigma\|^2_2,$$

(D.23)

under the restriction that:

$$\|g(t, \sigma)\|_2 \leq \gamma \|\sigma\|_2.$$  

(D.24)

The origin is asymptotically stable when $\gamma < \lambda_{\min}(Q)/2\lambda_{\max}(P)$. The bound on $\gamma$ depends on the choice of $Q$ and the ratio can be optimized when choosing $Q = I$ according to [12].

The bound defined in (D.2.2) can be satisfied by determine a boundary for $k_{c1h}$ and $k_{c2h}$. The bound on $g(t, \sigma)$ can be written as:

$$\|g(t, \sigma)\|_2 = \|-k_{c1h}(E - E^*)\Delta\dot{x}_1 - k_{c2h}(E - E^*)\Delta\dot{x}_2\|_2 \leq \gamma \|\sigma\|_2,$$

(D.25)

and assuming $k_{c1h} = k_{c2h} = k_h$ this is written as:

$$\|g(t, \sigma)\|_2 = \|-k_h(E - E^*)(\sigma_2 - 2\sigma_4)\|_2 \leq \gamma \|\sigma\|_2 < \frac{\lambda_{\min}(I)}{2\lambda_{\max}(P)} \|\sigma\|_2.$$  

(D.26)

Analyzing the worst case scenario for (D.2.2), by using $(E - E^*) = -E^*$, results in:

$$\|g(t, \sigma)\|_2 = \|k_hE^*(\sigma_2 - 2\sigma_4)\|_2 \leq \gamma \|\sigma\|_2 < \frac{\lambda_{\min}(I)}{2\lambda_{\max}(P)} \|\sigma\|_2$$

(D.27)

$$= k_hE^* \|(|\sigma_2 - 2\sigma_4|)\|_2 < \frac{\lambda_{\min}(I)}{2\lambda_{\max}(P)} \|\sigma\|_2.$$  

The bound for $k_h$ is defined by:

$$k_h < \frac{\lambda_{\min}(I)}{2\lambda_{\max}(P)} \|\sigma\|_2/E^*.$$  

(D.28)

The second part of this proof shows that system D.17 is local asymptotically stable under the assumption that the anti-phase synchronization manifold is stable, proved before under the restriction given for $k_h$ in (D.2.2). The following Lyapunov function is used for the prove:

$$L = \frac{1}{2}(E - E^*)^2,$$

(D.29)
where $E^*$ represents the reference energy level and $E$ is defined as the total system energy:

$$E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}k_1\Delta x_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k_2\Delta x_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}k_3\Delta x_3^2,$$

(D.30)

where $m_i$, $k_i$ and $d_i$ represent the mass, stiffness and damping respectively of both oscillators and the beam. Stability can be proven when the time derivative Lyapunov function is negative semi-definite hence:

$$\dot{L} = (E - E^*)\dot{E} \leq 0,$$

(D.31)

where $\dot{E}$ is defined as:

$$\dot{E} = -d_3\dot{x}_3^2 - m_1k_{c1h}(E - E^*)\Delta \dot{x}_1^2 - m_2k_{c2h}(E - E^*)\Delta \dot{x}_2^2;$$

(D.32)

resulting in the time derivative Lyapunov candidate:

$$\dot{L} = -d_3\dot{x}_3^2(E - E^*) - m_1k_{c1h}(E - E^*)^2\Delta \dot{x}_1^2 - m_2k_{c2h}(E - E^*)^2\Delta \dot{x}_2^2.$$

(D.33)

As long as $k_h$ satisfies (D.28) the anti-phase synchronization manifold is stable and (D.33) reduces to:

$$\dot{L} = -m_1k_h(E - E^*)^2\Delta \dot{x}_1^2 - m_2k_h(E - E^*)^2\Delta \dot{x}_2^2,$$

(D.34)

which is asymptotically stable as long as $k_h > 0$.

Combining the restrictions on $k_h$ from both parts of the prove results in:

$$0 < k_h < \frac{\lambda_{\text{min}}(I)}{2\lambda_{\text{max}}(P) \|\sigma\|_2} \|\sigma_2 - 2\sigma_4\|_2 E^*,$$

(D.35)

For the given system parameters the solution of the Lyapunov equation (D.2.2) is given by:

$$P = \begin{pmatrix} 674.6 & -0.5 & 856.8 & -12.5 \\ -0.5 & 18 & 31 & 23.3 \\ 856.8 & 31 & 1318.6 & 25 \\ -12.5 & 23.3 & 25 & 32 \end{pmatrix},$$

(D.36)

with the corresponding eigenvalues:

$$\lambda_{\text{min}}(P) = 0.6$$
$$\lambda_2(P) = 30.7$$
$$\lambda_3(P) = 99.6$$
$$\lambda_{\text{max}}(P) = 1912.4$$

(D.37)

The following values are used for the norms presented in (D.2.2):

$$\|\sigma\|_2 = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2} = \sqrt{0.01^2 + 0.06^2 + 0.005^2 + 0.004^2} = 0.0610$$

(D.38)

$$\|\sigma_2 - 2\sigma_4\|_2 = \sqrt{0.06^2 + 40.004^2} = 0.0605$$

(D.39)

which results in the following bound on $k_h$:

$$0 < k_h < 0.5273.$$
D.3 Stability of the synchronization manifold for non driven pendula

Consider system (6.2) which represents two pendula coupled by a beam. To analyse the stability the direct Lyapunov method is used and the total energy of the system is used as a candidate Lyapunov function as:

\[ V = \sum_{i=1}^{2} \frac{m_i}{2} (\dot{x}_i^2 + l_i^2 \ddot{\phi}_i^2 + 2x_i\dot{x}_i \dot{\phi}_i \cos(\phi_i)) + \frac{m_3 x_3^2}{2} + \sum_{i=1}^{2} m_i g h l_i (1 - \cos(\phi_i)) + \frac{k_3 x_3^2}{2}, \]  

(D.41)

which is a positive semi-definite function. The time derivative of \( V \) of the system is given by:

\[ \dot{V} = -d_3 \dot{x}_3 \leq 0, \]  

(D.42)

so LaSalles invariance principle can be used. Inequality (D.42) implies that \( V \) is a bounded function of time, so are \( x_3(t), \dot{x}_3(t) \) and \( \phi_1(t) \). Since \( \phi_i(t) \in S^1 \) every trajectory has a nonempty \( \omega \)-limit set. On this set \( \dot{V} = 0 \) and, therefore, \( \ddot{x}_3 = \dot{x}_3 = 0 \) and \( x_3 = x_3^* = \text{const} \). Since

\[ \frac{d^2}{dt^2} (\sin \phi_1(t) + \sin \phi_2(t)) = \sum_{i=1}^{2} \left( \ddot{\phi}_i(t) \cos \phi_i(t) - \dot{\phi}_i^2(t) \sin \phi_i(t) \right) \]  

(D.43)

from the third equation of (6.2) it follows that on the \( \omega \)-limit set the following relation is true:

\[ \frac{d^2}{dt^2} (\sin \phi_1(t) + \sin \phi_2(t)) = -\frac{k_3 x^*}{m l} = \text{const} \]  

(D.44)

and hence

\[ \sin \phi_1(t) + \sin \phi_2(t) = -\frac{k_3 x^*}{m l} t^2 + c_1 t + c_2 \]  

(D.45)

for some constants \( c_1 \) and \( c_2 \). The lefthand side of the last equation is a bounded function of time, thus \( x_3^* = 0 \) and \( c_1 = 0 \). From the last two equations of (6.2) in combination with (D.45) follows that on the \( \omega \)-limit set

\[ \ddot{\phi}_1(t) + \ddot{\phi}_2(t) = -\frac{g h}{l} c_2 \]  

(D.46)

and therefore

\[ \dot{\phi}_1(t) + \dot{\phi}_2(t) = -\frac{g h}{l} c_2 t + c_3 \]  

(D.47)

for some constant \( c_3 \). However, \( \dot{\phi}_1(t) \) is bounded and therefore necessary \( c_2 = 0 \). Finally from (D.45) and (D.46) we have for the \( \omega \)-limit set of any trajectory

\[ x_3(t) = 0, \quad \sin \phi_1(t) = -\sin \phi_2(t), \quad \ddot{\phi}_1(t) = -\ddot{\phi}_2(t) \]  

(D.48)

The second identity implies either

\[ \dot{\phi}_1(t) = -\dot{\phi}_2(t) \]  

(D.49)

or

\[ \phi_1(t) - \phi_2(t) = \pm \pi \]  

(D.50)

The latter implies that \( \ddot{\phi}_1(t) - \ddot{\phi}_2(t) = 0 \). However on the \( \omega \)-limit set \( \ddot{\phi}_1(t) + \ddot{\phi}_2(t) = 0 \) therefore the two pendula should be at rest. Finally, we have proved that all system trajectories tend to the set where:

\[ \phi_1(t) = -\phi_2(t), \quad \dot{\phi}_1(t) = -\dot{\phi}_2(t), \quad x_3 = 0, \quad \dot{x}_3 = 0. \]  

(D.51)
D.4 Stability of the anti-phase synchronization manifold for driven pendula

D.4.1 Free moving pendula, the nominal behavior

To prove stability for driven pendula based on the theory on “vanishing disturbances” \[12\] the first step is to prove the stability of the nominal system. The nominal system is a linearized variant of system (6.2) according to:

\[
\begin{align*}
\ddot{\phi}_1 &= -\frac{x_3}{l_1} - \omega_s^2 \phi_1 \\
\ddot{\phi}_2 &= -\frac{x_3}{l_2} - \omega_s^2 \phi_2 \\
\ddot{x}_3 &= \frac{1}{m_3} \left( -k_3 x_3 - d_3 \dot{x}_3 + m_1 g_b \phi_1 + m_2 g_b \phi_2 \right), \\
\end{align*}
\]

which is valid as long as \(|\phi_1| < 0.3 \text{ [rad]} \) and \(|\phi_2| < 0.3 \text{ [rad]} \). Applying the following coordinate transformation:

\[
\begin{align*}
\sigma_1 &= \phi_1 + \phi_2 \\
\sigma_2 &= \dot{\phi}_1 + \dot{\phi}_2 \\
\sigma_3 &= x_3 \\
\sigma_4 &= \dot{x}_3,
\end{align*}
\]

results in the following state space representation of the linearized system:

\[
\begin{pmatrix}
\dot{\sigma}_1 \\
\dot{\sigma}_2 \\
\dot{\sigma}_3 \\
\dot{\sigma}_4 \\
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 & 0 \\
(2\mu_s - 1)\omega_s^2 & 0 & \frac{2k_3}{l} & \frac{2d_3}{m_3} \\
0 & 0 & 0 & 1 \\
\mu_s g_b & 0 & -\omega_b^2 & -\frac{d_b}{m_3} \\
\end{pmatrix}
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\end{pmatrix} = A\sigma,
\]

where \(\omega_s\), \(l_s\) and \(\mu_s\) represent the specified pendula frequency, the pendula length and coupling strength respectively. The natural frequency of the beam is denoted by \(\omega_b\). The stability can be concluded by calculating the eigenvalues of matrix \(A\) resulting for the currently used settings in:

\[
\lambda_{1,2} = -0.0083 \pm 6.1558i \\
\lambda_3 = -0.3243 \\
\lambda_4 = -0.0591
\]

which all have a negative real part causing \(\sigma_i\) to converge to zero according to:

\[
\begin{align*}
\lim_{t \to \infty} \sigma_1 &= 0 = \phi_1 + \phi_2 \\
\lim_{t \to \infty} \sigma_2 &= 0 = \dot{\phi}_1 + \dot{\phi}_2 \\
\lim_{t \to \infty} \sigma_3 &= 0 = x_3 \\
\lim_{t \to \infty} \sigma_4 &= 0 = \dot{x}_3.
\end{align*}
\]

The first two equations of (D.56) show anti-phase synchronization.

D.4.2 Driven pendula, the perturbed system

One of the prerequisites for using the theory on vanishing disturbances is that the nominal system is stable in the origin, which is proven in the previous section. The disturbances are introduced in the system
by the driving mechanisms according to:

\[
\begin{align*}
\ddot{\phi}_1 &= -\frac{x_3}{l_{1h}} - \omega_s^2 \phi_1 - \frac{k_h (E - E^*)}{m_1 l_{1h}^2} \dot{\phi}_1 \\
\ddot{\phi}_2 &= -\frac{x_3}{l_{2h}} - \omega_s^2 \phi_2 - \frac{k_h (E - E^*)}{m_2 l_{2h}^2} \dot{\phi}_2 \\
\ddot{x}_3 &= \frac{1}{m_3} \left( -k_3 x_3 - d_3 \dot{x}_3 + \sum_{i=1}^{2} \left[ m_i g_i \phi_i + \frac{k_h (E - E^*)}{l_i} \dot{\phi}_i \right] \right).
\end{align*}
\]  

(D.57)

As long as \( k_h \) satisfies:

\[
0 < k_h < \frac{\lambda_{\min}(I)}{2 \lambda_{\max}(P)} \frac{||\sigma||_2}{||\sigma_2||_2 E^*},
\]

(D.58)

all trajectories of system (6.2) with initial conditions \(|\phi_1| < 0.3 \text{ [rad]} \) and \(|\phi_2| < 0.3 \text{ [rad]} \) converge to the anti-phase synchronization manifold and the system converges to the reference energy level.

**Proof.** The first part of the proof determines a boundary for the controller gains \( k_{c1h} \) and \( k_{c2h} \) based on the theory on vanishing perturbation [12]. Within these boundaries the controller inputs will not affect the stability of the anti phase synchronization manifold, proven in appendix D.4.1. As we can not explicitly calculate the bounds for \( k_{c1h} \) and \( k_{c2h} \), a conservative bound will be determined by analysis the worst case scenario.

As the \( A \) matrix, presented in (D.4.1), is Hurwitz the Lyapunov function:

\[
P A + A^T P = -Q,
\]

(D.59)

can be solved in \( P \), by using \( Q = Q^T > 0 \) and results in a unique solution \( P = P^T > 0 \). The resulting quadratic Lyapunov function \( V(x) = \sigma^T P \sigma \) satisfies:

\[
\lambda_{\min}(P) ||\sigma||_2^2 \leq V(\sigma) \leq \lambda_{\max}(P) ||\sigma||_2^2
\]

(D.60)

\[
\frac{\partial V}{\partial \sigma} A \sigma = -\sigma^T Q \sigma \leq \lambda_{\min}(Q) ||\sigma||_2^2
\]

(D.61)

\[
\left| \left| \frac{\partial V}{\partial \sigma} \right| \right| = ||2 \sigma^T P|| \leq 2 ||P||_2 ||\sigma||_2 = 2 \lambda_{\max}(P).
\]

(D.62)

The derivative of \( V(\sigma) \) along the trajectories of the perturbed system \( \dot{\sigma} = A \sigma + g(t, \sigma) \) satisfies:

\[
\dot{V}(\sigma) \leq -\lambda_{\min}(Q) ||x||_2^2 + 2 \lambda_{\max}(P) \gamma ||\sigma||_2^2,
\]

under the restriction that:

\[
||g(t, \sigma)||_2 \leq \gamma ||\sigma||_2.
\]

(D.63)

(D.64)

The origin is asymptotically stable when \( \gamma < \lambda_{\min}(Q)/2 \lambda_{\max}(P) \). The bound on \( \gamma \) depends on the choice of \( Q \) and the ratio can be optimized when choosing \( Q = I \) according to [12].

The bound defined in (D.64) can be satisfied by determine a boundary for \( k_{c1h} \) and \( k_{c2h} \). The bound on \( g(t, \sigma) \) can be written as:

\[
||g(t, \sigma)||_2 \leq ||-k_{c1h}(E - E^*) \Delta \dot{x}_1 - k_{c2h}(E - E^*) \Delta \dot{x}_2||_2 \leq \gamma ||\sigma||_2.
\]

(D.65)

and assuming \( k_{c1h} = k_{c2h} = k_h \) this is written as:

\[
||g(t, \sigma)||_2 \leq ||-k_h(E - E^*)(\sigma_2 - 2 \sigma_4)||_2 \leq \gamma ||\sigma||_2 \leq \frac{\lambda_{\min}(I)}{2 \lambda_{\max}(P)} ||\sigma||_2.
\]

(D.66)
Analyzing the worst case scenario for (D.66), by using \((E - E^*) = -E^*\), results in:
\[
\|g(t, \sigma)\|_2 = \|k_h E^* (\sigma_2)\|_2 \leq \gamma \|\sigma\|_2 < \frac{\lambda_{\min}(I)}{2\lambda_{\max}(P)} \|\sigma\|_2 \quad \text{(D.67)}
\]
\[
= k_h E^* \|\sigma_2\|_2 < \frac{\lambda_{\min}(I)}{2\lambda_{\max}(P)} \|\sigma\|_2 \quad \text{(D.68)}
\]
The bound for \(k_h\) is defined by:
\[
k_h < \frac{\lambda_{\min}(I)}{2\lambda_{\max}(P)} \|\sigma\|_2 E^*. \quad \text{(D.69)}
\]
The second part of this proof shows that system D.52 is local asymptotically stable under the assumption that the anti-phase synchronization manifold is stable, proved before under the restriction given for \(k_h\) in (D.69). The following Lyapunov function is used for the proof:
\[
L = \frac{1}{2}(E - E^*)^2, \quad \text{(D.70)}
\]
where \(E^*\) represents the reference energy level and \(E\) is defined as the total system energy:
\[
E = \frac{2}{m_1} \left( x_1^2 + l_1^2 \dot{\phi}_1^2 + 2x_1 \dot{x}_1 l_1 \cos (\phi_1) \right) + \frac{m_3x_3^2}{2} + \sum_{i=1}^{2} m_i g_i l_i \left( 1 - \cos (\phi_i) \right) + \frac{k_3 x_3^2}{2} \quad \text{(D.71)}
\]
where \(m_i, k_i\) and \(d_i\) represent the mass, stiffness and damping respectively of both oscillators and the beam. Stability can be proven when the time derivative Lyapunov function is negative (semi-) definite hence:
\[
\dot{L} = (E - E^*) \dot{E} \leq 0, \quad \text{(D.72)}
\]
resulting in the time derivative Lyapunov candidate:
\[
\dot{L} = -d_3h \ddot{x}_3^2 - k_{e1h}(E - E^*) \ddot{\phi}_1^2 - k_{e2h}(E - E^*) \ddot{\phi}_2^2, \quad \text{(D.73)}
\]
As long as \(k_h\) satisfies (D.69) the anti-phase synchronization manifold is stable and (D.33) reduces to:
\[
\dot{L} = -k_h(E - E^*) \ddot{\phi}_1^2 - k_h(E - E^*) \ddot{\phi}_2^2, \quad \text{(D.75)}
\]
which is asymptotically stable as long as \(k_h > 0\).

Combining the restrictions on \(k_h\) from both parts of the prove results in:
\[
0 < k_h < \frac{\lambda_{\min}(I)}{2\lambda_{\max}(P)} \|\sigma\|_2 E^*, \quad \text{(D.76)}
\]
For the given system parameters the solution of the Lyapunov equation (D.59) is given by:
\[
P = \begin{pmatrix}
1170 & -0.5 & -175 & -3519 \\
-0.5 & 30 & 1.8 & 32.5 \\
-175 & 1.8 & 37.9 & 557 \\
-3519 & 32.5 & 557 & 10697
\end{pmatrix}, \quad \text{(D.77)}
\]
with the corresponding eigenvalues:

\[
\begin{align*}
\lambda_{\text{min}}(P) &= 1.4 \\
\lambda_2(P) &= 14.6 \\
\lambda_3(P) &= 35.2 \\
\lambda_{\text{max}}(P) &= 11885.3 \\
\end{align*}
\] (D.78)

The following values are used for the norms presented in (D.76):

\[
\begin{align*}
||\sigma||_2^2 &= \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2} = \sqrt{0.6^2 + 4^2 + 0.0001^2 + 0.0002^2} = 4.1 \\
||\sigma_2||_2 &= \sqrt{4^2} = 4 \\
\end{align*}
\] (D.79) (D.80)

which results in the following bound on \( k_h \), using \( E^* = 8 \cdot 10^{-5} \mid J \):

\[
0 < k_h < 0.77.
\] (D.81)
Appendix E

Linear driving mechanism

E.1 Introduction

A single oscillator can be described by equation of motion (E.1). The oscillator is a single mass, \( m \), which is connected to the fixed world by a spring, \( k \), and a damper \( d \). The position, velocity and acceleration are depicted by respectively \( x \), \( \dot{x} \) and \( \ddot{x} \). The input force is denoted by \( u \).

\[
\ddot{x} = \frac{1}{m} (-d\dot{x} - kx + u)
\]  
(E.1)

The equation of motion can be rewritten to a transfer function in the frequency domain, presented by:

\[
H(s) = \frac{1}{ms^2 + ds + k}.
\]  
(E.2)

The transfer function \( H(s) \) can be made visible in a couple of ways, one is a bode plot presented in Figure E.1. The bode plot gives a visualized interpretation of the transfer function in two parts. The top part shows the amplitude response of the system, the lower part shows the phase shift of the system. Figure E.1 shows several bode diagrams for several different values of the damping \( d \) of the system. This parameter influences the shape of the diagram. The position of the resonant frequency, peak in the top plot, of the system is influenced by the mass \( m \) and the spring constant \( k \). The fixed parameter values in the presented plots are: \( m = 1[kg] \) and \( k = 4\pi^2[Nm^{-1}] \). The top plot clearly shows that decreasing the damping increases the height of the resonance peak and decreases the width of the peak. The set of parameters used for the pendulum during the controller design are: \( m = 1[kg] \), \( k = 4\pi^2[Nm^{-1}] \), and \( \xi = 0.26 \)..

The next sections start with a description of the control goal whereafter two different types of linear controllers are described. The last section compares the results and considers the effectiveness of the linear type of controller.

E.2 Control goal

The controller has to be a substitute for the original escapement mechanism. The goal of the escapement is to keep the oscillator (pendulum) running on a certain frequency and amplitude level. These goals can be translated into a reference trajectory for the controller to use, resulting in a sinusoidal trajectory with frequency \( \omega_r \) and amplitude \( A_r \), shown in (E.3). The trajectory parameters \( \omega_r = 2\pi [rad] \) and \( A_r = 1[cm] \) are used during the simulations.

\[
x_{ref} = A_r \sin \omega_r t
\]  
(E.3)

Additionally the controller has to perform in a robust way so the system parameters and the reference trajectory can slightly differ while the the controller keeps a correct performance. The error margin \( e_x \) is allowed to be 10% of the maximum of the reference trajectory. In this case this results in: \( e_x = 1[mm] \). The maximum output of the controller in steady state is set to 50% of the spring-force at maximum
amplitude of the oscillator. This results in $F_{\text{max}} = 0.2 \ [N]$.

A remark needs to be made regarding to the working principle of the controller. The original mechanism only (re)acts each period of the oscillator at a single moment and not continuously as these controllers do. The controllers designed in this chapter are continuously (re)acting in stead of once each period and stop when the reference trajectory fully meets the oscillators trajectory.

### E.3 Loop shaping

The first type of linear controller which is designed is a controller based on the SISO loop shaping procedure. Several analysis can be done during this procedure to shape the designed controller. The open loop Bode transfer can be analyzed to analyse the stability, the closed loop bode transfer can be analyzed for performance as well as stability. The root locus diagram can be used for stability analysis. The sensitivity transfer can be used to analyse the influences of noise in the system.

Stability can be guaranteed (Bode criterium) when all closed loop poles of a system are located in the left half plane of the root locus diagram. In other words this means that all closed loop eigenvalues of the system are negative. Translating the criterium to an analysis in the bode diagram the criterium can be seen as follows: A closed-loop system is unstable if the frequency response of the open-loop transfer function has an amplitude ratio greater than one at the critical frequency. Otherwise the closed-loop system is stable.

#### Design phase 1

Figure E.2 shows 4 different closed loop Bode transfers for the system controlled with a basic P controller, $C(s) = k_p$. $k_p$ is the controller gain and the figure shows three Bode plots for three different values of $k_p$. All presented controllers are stable. The larger the controller gain becomes, the more the closed loop transfer reaches the 0dB amplitude transfer for low frequencies. This causes an almost zero-error transfer for these frequencies. An other positive effect of the increasing control gain is the increasing eigenfrequency of the closed loop system. On the contrary the closed loop system starts to get more sensitive for high frequent disturbing signals and the control effort becomes larger and the amplitude...
E.3. LOOP SHAPING

The resonance peak in the previous closed loop transfers can be suppressed by a notch filter in the controller. The notch filter filters only a small frequency band. The bandwidth of the notch filter is depending on the Q-factor of the filter. The higher the Q-factor becomes, the more narrow the bandwidth of the filter becomes. In other words, the filter becomes more specific for a certain frequency. On the other hand, the filter becomes less robust for system changes. The next figure shows three different Bode transfers, one of them is the system itself, one shows the controller Bode transfer and a the last shows closed loop Bode transfer. The closed loop Bode transfer shows that the resonance peak is suppressed by the controller. The transfer function of the controller is shown by:

$$C(s) = \frac{220}{s^2 + 2.8s + 46} \cdot \frac{1 + 13.56s + 47.20s^2}{s^2}$$

(E.4)

Figure E.4 presents the simulation results of this controller for the given reference input (E.3). The controller can track the reference signal within an error range of 1% but needs a lot of control effort for it, shown in Figure E.5. When increasing the control effort even more, by increasing the controller gain, the error can be reduced further. To accomplish the control goal, this controller needs an unacceptable lot of effort and therefore the controller cannot be used.

Design phase 3

The controllers designed in the previous paragraphs can be extended with an I-action in the origin, pole at $s = 0$, to guarantee a zero steady state error but the controllers bandwidth decreases due to this pole. Moreover, the oscillator never reaches a steady state and therefore the I-action will not have an added value for the controllers performance.
Figure E.3: Closed loop transfers, P-controller including a notch filter to suppress the resonance peak.

Figure E.4: Simulation results using controller E.4. A clear phase lag is visible.

Figure E.5: Controller effort of simulated controller E.4
Robustness

Except from the control effort, which is really bad, the closed loop Bode transfer of the controller shows that the controller is robust for all input signals up to a frequency of 10 \([\text{rad/sec}]\). On the contrary, due to the notch filter, the controller is totally non robust for system parameter changes especially for varying the systems resonance frequency. The controller can even get instable.

E.4 State feedback

Introduction

An other type of linear controller is a state feedback controller, a MISO controller. For this type of controller the state space description of the system () is given by:

\[
\dot{x} = Ax + Bu = \left( \begin{array}{c} 0 \\ -\frac{k}{m} & -\frac{d}{m} \end{array} \right) x + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) u \\
y = Cx + Du = \left( \begin{array}{cc} 0 & 1 \end{array} \right) x,
\]

(E.5)

where all system states need to be available for the controller which is assumed to be true for the moment.

Controller design

The controller calculates its output \(u_1\) based on a linear combination of the errors, \(e = r - x\), on all system states, \(x\), according to:

\[u = ce,\]

(E.6)

where \(c\) is the controller gain vector. Rewriting this in terms of the system states and reference signals leads to:

\[u = c(r - x),\]

(E.7)

resulting in:

\[u = -cx,\]

(E.8)

when the reference signal is set to zero. Combining this controller output (E.8) with the state space description (E.5) results in:

\[
\dot{x} = Ax + Bu \\
= Ax - Bcx \\
= (A - Bc)x.
\]

(E.9)

The eigenvalues, poles, of the matrix \(A - Bc\) in equation E.9 all need to be negative to guarantee a stable system. The controller gain vector \(c\) is calculated by the pole-placement method \(^1\). This method calculates the gain vector out of the system matrix, \(A\), and the desired location of the closed loop eigenvalues. The more negative the eigenvalues are chosen, the faster the closed loop system becomes and the larger the controller gain vector \(c\) becomes. Due to these increasing gain parameters, the closed loop system becomes more sensitive for noise and disturbance.

The pole-placement procedure assumes that the system is controllable. The controllability can be checked with the controllability matrix, \(R_c\), shown in equation E.10. \(A\) and \(B\) are the state space matrices of the system and \(n = 2\) is the number of states in the system. When the controllability matrix has got full rank, the system is controllable. In this case, \(R_c\) has got full rank so the system is controllable.

\[R_c = [BABA^2B \ldots A^{n-1}B]\]

(E.10)

\(^1\)Pole placement can be done in Matlab with the ‘acker’ procedure
The controller gains needed to accomplish the control goals are $c_1 = 260$ and $c_2 = 37$, where $c_1$ is the controller gain on the position error and $c_2$ is the controller gain on the velocity error. With these controller parameters the closed loop poles of the system are situated at $s = 30$ and $s = 10$. This controller needs an unacceptable lot of effort to accomplish the control goal and therefore the controller cannot be used.

E.5 LQR

Another way to design a linear controller for this system is to use the LQR algorithm. LQR, linear quadratic regulator, is based on the minimization of a cost function. This cost function contains the states and the input of the system according to:

$$J = \int_0^1 (\dot{x}'Qx + u'Ru) \, dt,$$

(E.11)

where $Q$ is the weighting matrix for the system states and $R$ is the weighting matrix for the input signals of the system (controller output). The result of the LQR design algorithm is a gain matrix, $k$. The system input becomes a linear combination of this gain matrix and the system states according to: $u = -kx$. For the $R$ matrix goes: the larger the weighting parameter, the larger the penalty on the controller output and so the controller gains are reduced. For the $Q$ matrix the effects of the weighting parameters are opposite. The larger a parameter becomes, the more important it is to reduce the control error for that state and so the control gain for that state increases. Basically this is the same type of controller as designed in the previous section but this method uses another way to derive the controller gain parameters.

Tuning the LQR algorithm until the control goals are achieved, results in a set of controller gains which can be positioned in the same range as the state feedback controller gains. Also in this case the output signals were unacceptable large and enlarging the penalty on the output signals $R$ results in not achieving the control goal. The controller is not usable.

E.6 Conclusion

Reflecting all presented controllers in this chapter on the control goal, set in section E.2 no controller can meet the control goals with realistic control parameters and with an acceptable sensitivity and robustness. High gain controllers can achieve the control goal but these controllers need an unacceptable high controller output. None of the proposed controllers can handle the resonance peak which is present in the transfer function of the system.
SYNC keeps your interest forever!!!