The determination of the pressure–viscosity coefficient of a lubricant through an accurate film thickness formula and accurate film thickness measurements

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Abstract: The pressure–viscosity coefficient is an indispensable property in the elastohydrodynamic (EHD) lubrication of hard contacts, but often not known. A guess will easily lead to enormous errors in the film thickness. This article describes a method to deduct this coefficient by adapting the value of the pressure–viscosity coefficient until the differences between accurate film thickness approximation values and accurate film thickness measurements over a wide range of values are at a minimum. Eleven film thickness approximation formulas are compared in describing the film thickness of a test fluid with known value of the pressure–viscosity coefficient. The measurement method is based on spacer layer interferometry. It is concluded that for circular contacts the newer more versatile expressions are not better than some older approximations, which are limited to a smaller region of conditions, and that the older fits are as least as appropriate to find the pressure–viscosity coefficient of fluids, in spite of the limited data where they have been based on.

Keywords: elastohydrodynamic lubrication, pressure–viscosity coefficient, film thickness measurement, film thickness formula, film thickness equation, approximation

1 INTRODUCTION: FINDING ALPHA

In many engineering applications involving lubrication a film thickness computation or estimate is required. The computation of heavily loaded elastohydrodynamically lubricated (EHL) contacts requires the value of the pressure–viscosity coefficient (α) to be known. A guess will easily lead to gross errors, while film thickness is roughly related to $\alpha^{1/2} \div \alpha^{3/4}$, and $\alpha$ is within a range of about 10–40 GPa$^{-1}$. Which value should be chosen?

In general, $\alpha$ depends on the lubricant at hand, and on the pressure, temperature, and shear rate in the contact, see Bair [1]. EHL contacts have very high pressures (of the order of 1 GPa) and high shear rates (up to over $10^6 \text{ s}^{-1}$). Values for $\alpha$ can be found for a few oils only, and for conditions that differ from the given ones (see, for example, Jones et al. [2], Larsson et al. [3], or Taylor [4]). The other option is to perform high pressure–viscosity measurements, but this way is only reserved for researchers who have access to such equipment, or for the ones who can afford to assign others to do it. High-pressure viscometers (up to and over 1 GPa) are rare, as is the experience to operate them. High-pressure viscometers can be found among others in Luleå and Lund, see Jacobson [5], and Atlanta, see Bair [1].

The aim of this work is to find a procedure employing other, more widely used equipment that yields good estimates for $\alpha$.

2 THE IDEA

Optical film thickness measurements allow an accurate determination of film thickness in heavily loaded EHL contacts, down to 1 nm accuracy (see among others Spikes and Cann [6], Luo and Wen [7], and Hartl et al. [8]). Quite a number of tribology research laboratories in the world have such a device. These rigs are operated at conditions almost similar to practice, like in ball bearing contacts. The numerical simulation of
which acronyms will be used as an index.

3.2 Assumptions and definitions

The film thickness formulas below are based on common fluid film lubrication theory (see, for example, Venner and Lubrecht [9] or Hamrock et al. [12]):

(a) isothermal and steady state conditions;
(b) a Barus or Roelands viscosity–pressure relationship, see under section 3.4;
(c) the fluid is either incompressible or obeys a Dowson and Higginson [13] pressure–density relationship, see under section 3.4;
(d) the deformation of the solids is linear elastic;
(e) thin film lubrication is described by Reynolds’ equation, hence low Re numbers and thin films ($h \ll R$);
(f) the pressure–viscosity coefficient depends on temperature and pressure only. It is assumed that when the controlled temperature and load are constant, this coefficient is constant too.

The contact is considered as a short elliptical contact when the major contact ellipse axis is perpendicular to the direction of entrainment, and a long elliptical contact when the major contact ellipse axis is along the direction of entrainment. According to Johnson [14] four film thickness regimes can be distinguished in full film EHL:

(a) IR: the isoviscous rigid (hydrodynamic) regime: no deformation nor pressure–viscosity effects;
(b) VR: the piezoviscous rigid regime: variable viscosity, insignificant deformation;
(c) IE: the isoviscous elastic regime: constant viscosity, elastic deformation;
(d) VE: the piezoviscous elastic regime: variable viscosity, elastic deformation;

which acronyms will be used as an index.
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The term hard EHL contacts is used for contacts where both solid bodies have a high modulus of elasticity. In linear elastic EHL problems the elastic properties of the solids can be expressed by one parameter instead of four, the reduced or equivalent modulus of elasticity

$$\bar{E}_r = \left\{ \frac{1}{2} \left[ \left( \frac{1 - \nu_1^2}{E_1} \right) + \left( \frac{1 - \nu_2^2}{E_2} \right) \right] \right\}^{-1}$$

3.3 Experimental set-up

The measurements of the central film thickness were performed on a PCS Instruments EHL ultra thin film (UTF) measurement system, which is described in Johnston et al. [15] (see Fig. 1). The PCS ultra device uses a smooth steel ball on a flat optical disc, which implies that it has a circular contact geometry. The ball is a super finished Cr steel ball. The rig is applied in rolling motion mode, i.e. the optical glass disc drives the ball, which is supported by three small ball bearings. The two at the left are slightly higher in position than the right one, to eliminate contact spin under pure rolling. More disc and ball details are listed in Table 1.

Two lubricants were tested: a reference oil, HVI60, and a base oil, which is a blend with unknown value for the pressure–viscosity coefficient. Values from Choo et al. [16] have been used for the viscosity data of HVI60, see Table 2. These values have been confirmed by many experiments and numerical calculations. The UTF device was used in its standard mode, i.e. the film thickness in the contact centre is determined. The test conditions are also listed in Table 2.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Ball and disc data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Steel ball</td>
</tr>
<tr>
<td>Diameter</td>
<td>$19.05 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>$E$ modulus</td>
<td>$2.07 \times 10^{11}$ Pa</td>
</tr>
<tr>
<td>Poisson coefficient</td>
<td>0.29</td>
</tr>
<tr>
<td>RMS surface roughness</td>
<td>3.5 nm</td>
</tr>
</tbody>
</table>

3.4 The pressure–viscosity relationship

The choice of the pressure–viscosity relationship is vital in film thickness calculations. However, it should be emphasized that the aim of the current study is to determine a value for the pressure–viscosity coefficient appropriate for numerical calculations, where presently only two models prevail. In both the numerical calculations, as well as in the derivation of the approximation film thickness formulas, many assumptions are made. One should be aware of the limitations, and some models will actually be used at the border of, or even outside, their regimes. A total of 11 film thickness formulas will be employed to find those which fit the experimental data best, by fitting the value of $\alpha$.

EHL theory learns that the film thickness is mainly determined by the conditions in the inlet of the contact, see Ertel [17], and Dowson and Higginson [13]. Film pressures in the inlet are lower than that in the narrowest part of the film, the conjunction zone. So for film thickness calculations, the pressure–viscosity model at modest pressures is particularly important. Traction is mainly controlled in the conjunction zone, and therefore the pressure–viscosity behaviour at high pressures is important for traction modelling.

Hence a pressure–viscosity model that may only be used up to a pressure of 0.2 GPa may yield reasonable
good film thickness results for maximum film pressures of 0.4 GPa. This explains the success of the simple pressure–viscosity model that was employed in the beginning of EHL computer calculations, the relation from Barus [18], see Dowson and Higginson [13]

\[ \eta = \eta_0 e^{\alpha p} \]  

But at high film pressures of the order of 1 GPa, encountered in many highly loaded EHL contacts, the Barus equation is no longer applicable. The most popular one by far in EHL work is the Roelands [19] pressure–viscosity equation. Bair [1] questions the claimed accuracy of the Roelands equation, by showing that it is incorrect from about 0.5 GPa or higher, depending on the fluid considered, and proposes alternatives. But the popularity of the Roelands equation implies that many of the film thickness approximations from the literature have been based on it, and as there are no other pressure–viscosity relationships being used, only Barus- and Roelands-based equations will be utilized.

Ten Napel et al. [20] and Moes and Bosma [21] were probably the first to employ one of the models suggested by Roelands [19] for EHL work. According to Venner [22] and many other contributions, the Roelands relationship can be written in a form resembling the simple exponential equation (2)

\[ \eta = \eta_0 e^{[\ln(\alpha p)]/(1 + p/p_0)^{2-1})} \]  

where the values for the reference viscosity \( \eta_R \) and reference pressure \( p_R \), as defined by Roelands [19], are \( \eta_R = 6.315 \times 10^{-5} \text{ Pa s} \) and \( p_R = 1.98 \times 10^8 \text{ Pa} \).

All work can be expressed in a generalized relationship, suggested by Blok [23]

\[ \alpha^* = \frac{1}{p_{as}} = \left( \int_0^{\infty} \frac{\eta_0}{\eta(p)} dp \right)^{-1} \]  

which is the inverse of Blok’s [23] asymptotic fictitiously isoviscous pressure \( p_{as} \) and is called the reciprocal asymptotic isoviscous pressure by Bair [24]. Here \( \eta(p) \) denotes the pressure-dependent dynamic viscosity, and \( \eta_0 \) the value at ambient pressure. For the Barus model \( \alpha^* = \alpha \). This allows all results for different pressure–viscosity relationships to be represented by one parameter \( \alpha^* \), and to be interpreted in established groups and formulas. This parameter is to be seen as an effective value of \( \alpha \) over the pressure range encountered, and is the relevant pressure–viscosity coefficient to characterize the film formation in EHL contacts, rather than \( \alpha \). Bair [24] corroborates this by comparing experimentally determined film thickness values for seven lubricants with calculated ones, based on Hamrock and Dowson’s [25] central film thickness formula, and the \( \alpha \) and \( \alpha^* \) values for the lubricants. Most lubricants showed significant departure from exponential pressure–viscosity behaviour.

Hence, the pressure–viscosity coefficients resulting from the analysis presented in this article must be interpreted as the value of \( \alpha^* \), the reciprocal asymptotic isoviscous pressure. In this case, the maximum value for the pressure is not infinity, but bounded by the maximum Hertzian pressure \( p_{Hz} \).

Moes [10] shows that for the Roelands [19] equation \( \alpha^* \) can be expressed as a generalized incomplete gamma function, \( \alpha^* = \alpha^*(z, \alpha_0 p_R) \), and suggests that for mineral oils, where \( (z/\alpha_0 p_R) \to 0 \), the following approximation can be used

\[ \alpha^* \approx \frac{\alpha_0}{1 + ((1 - z)/(\alpha_0 p_R))} \]  

If it is assumed that at ambient pressure \( p = 0 \) the slopes of the Barus (2) and Roelands (3) equations are equal, then the Roelands parameter \( z \) can be expressed in the Barus parameters \( \eta_0 \) and \( \alpha \)

\[ z = \frac{(\alpha p_R)}{\ln \left( \frac{\eta_0}{\eta_R} \right)} \]  

and this value for \( z \) has been used in numerical calculations, see section 3.7.

3.5 The pressure–density relationship

The analytical solutions (more on this in section 3.6) usually employ an incompressible fluid. Almost all of the numerical work on EHL film thickness employs one single compressibility law, an empirical relationship first used by Dowson and Higginson [13]

\[ \frac{\rho}{\rho_0} = \frac{p_{Hz} + 1.34 p}{p_{Hz} + P} \]  

In this relationship \( \rho \) is the density at hand, \( \rho_0 \) the atmospheric density, \( p \) the pressure, and \( p_{Hz} \)
the reference pressure, $p_{\text{ref}} = 5.9 \times 10^6$ Pa. The fluid compressibility affects the pressure peak (spike) and the central film thickness appreciably, and has a minor effect on the minimum film thickness. If all other lubrication conditions are identical, but only different $\alpha$ values are employed, different pressures will be the result, and hence different compressibility effects will emerge \cite{26}. All approximation formulas for compressible fluid films below use equation (7). As this investigation is aiming at finding the best fitting existing film thickness approximation to experimental data, rather than at developing a new film thickness equation, this is accepted with some inconvenience.

3.6 Approximation formulas for the central film thickness in elliptical EHL contacts

Hamrock and Dowson’s \cite{25} empirical power-law film thickness formula for elliptical EHL contacts has been used by many investigators. But is this the most accurate equation for assessing the value of $\alpha^*$? Since 1977 other approximations have been suggested. The following 11 candidate central film thickness formulas have been selected from the literature:

(a) Archard and Cowking \cite{27} for circular hard contacts;
(b) Hamrock and Dowson \cite{25,28} for circular and short elliptical contacts;
(c) Hamrock et al. \cite{12} and Hamrock \cite{29} for circular and short elliptical contacts;
(d) Chittenden et al. \cite{11} for circular and long elliptical contacts (and arbitrary entrainment direction);
(e) Hooke \cite{30} for arbitrary elliptical contacts;
(f) Sutcliffe \cite{31} for arbitrary elliptical contacts;
(g) Greenwood \cite{26} for circular contacts;
(h) Venner \cite{22}, and Venner and Ten Napel \cite{32}, for circular contacts;
(i) Nijenbanning et al. \cite{33} for circular and short elliptical contacts;
(j) Venner and Lubrecht \cite{9} for circular contacts;
(k) Moes \cite{10} for arbitrary elliptical contacts.

These film thickness approximations are defined in different sets of non-dimensional groups. The most commonly used sets are listed in Appendix 3. Nowadays, a set of non-dimensional groups $H$, $L$, and $M$ is preferred in most EHL research

\begin{align*}
H &= \left( \frac{h}{R_e} \right) \left( \frac{E_r}{2\eta u} \right)^{1/2} \\
M &= \left( \frac{F}{E_r R_e^2} \right) \left( \frac{2\eta u}{R_e} \right)^{3/4} \\
L &= \left( \alpha E_r \right) \left( \frac{2\eta u}{R_e} \right)^{1/4} \\
\omega &= \left( \frac{R_i}{R_e} \right)
\end{align*}

(7)

where $H$ stands for the non-dimensional film thickness, $M$ for the non-dimensional load, $L$ for the lubricant’s pressure–viscosity coefficient, and $\omega$ for the curvature radius ratio. All approximation formulas (1) to (11) have been written in terms of $H$, $L$, and $M$, see Appendix 4.

Figure 2(a) shows a sectional plot of the non-dimensional film thickness $H$ as a function of $M$ and $L$. The transition line from the VR to the VE lubrication regime is taken from Moes \cite{10}, p. 174\textsuperscript{4}. In addition, the numerical data used by Hamrock and Dowson \cite{25} and by Chittenden et al. \cite{11} has been added. The figure shows that the data points from \cite{11} and \cite{25} are all well into the VE regime. Figure 2(b) is a contour plot of the data from the sectional plot in Fig. 2(a). Figure 2(b) employs the same data as Fig. 2(a), and additionally provides the Greenwood \cite{26} line that marks the average conditions of all the experimental and numerical data that he analysed.

All these approximation formulas fall into four types:

(a) (semi) analytical formulas, or asymptotic solutions, for a specific lubrication regime;
(b) interpolation formulas, usually power law expressions fitted on numerical results for a specific lubrication regime;
(c) interpolation formulas, usually power law expressions fitted on both numerical and experimental results;
(d) general formulas, based on (a) and/or (b), which allow a smooth interpolation between the various lubrication regimes.

Archard and Cowking’s \cite{27} formula is of type (a). It is a classic one, and has been chosen for completeness. It employs a Barus pressure–viscosity model and an incompressible fluid.

The family of Hamrock and Dowson \cite{12,25,28,29} formulas are of type (b). They are often used as a reference, so they should be evaluated. The Chitten- den et al. \cite{11} formulas may be seen as a member

\textsuperscript{4}Actually, compressibility effects should result in at least one more non-dimensional group; it is therefore remarkable that this influence is not reflected in any of the film thickness formulas.

\textsuperscript{1}The phrase ‘lubrication regimes’ shall be used for the Johnson \cite{14} film thickness regimes.
Fig. 2  (a) Non-dimensional central film thickness $H$ as a function of load parameter $M$ and lubricant parameter $L$. Legend: —— contour lines for $L = 0, 1, 1.2, 5, 10, 25, 50, 100, 250,$ and 500; −−−−−−−− VR/VE transition; ♦ Hamrock and Dowson [25] data; Δ Chittenden et al. data. (b) Non-dimensional central film thickness $H$ contour map for a circular contact. Legend: —— contour lines for $H = 1–12$; −−−−−−−− VR/VE transition; ··········· Greenwood’s curve (see text); ♦ Hamrock and Dowson [25] data; Δ Chittenden et al. data. (c) Non-dimensional central film thickness $H = H(M, L)$ for a circular EHL contact. Legend: ♦ Hamrock and Dowson [25] data; Δ Chittenden et al. data; —— non-dimensional Moes [10] film thickness; ··········· non-dimensional Hamrock and Dowson [25, 28] film thickness
of this family and it claims a smaller inaccuracy than the others. It is expected that it may be the best of (2)–(4). The Hamrock family uses the Roelands model and the fluid is compressible according to the Dowson and Higginson model.

Hooke [30] provides expressions of type (a) for the central and minimum film thickness, based on a Barus exponential viscosity model and an incompressible fluid. His work is mainly on the minimum film thickness, especially on its shift from the side lobes to the centre line at the end of the contact, along the entrainment axis (e.g. when the ellipticity ratio increases). It contains a central film thickness approximation. His later work [34, 35] addresses the IE/VE and VR/VE transitions and the influence on the minimum film thickness. The Sutcliffe [31] formula is of type (a). It is based on Hooke's [30] analysis and [31] suggests that it is more accurate. As a consequence, Sutcliffe's analysis is based on a Barus viscosity equation and an incompressible fluid.

In the proper sense, Greenwood [26] does not provide a best-fit line for film thickness, but his study suggests a simple formula which is also considered here. It covers many experimental and numerical data from that time, hence this is a type (c) expression. He identified that, for his choice of non-dimensional numbers, most values from the literature are close to that time, hence this is a type (c) expression. It is repeated here that Moes [10] states that he fitted his formula on numerical results obtained at $0 \leq L \leq 25$ and $5 \leq M \leq 1000$, which almost coincides with the whole area in Fig. 2, so it may be assumed that when the conditions of the measurement are far away from the results by Hamrock and Dowson or Chittenden et al., the Moes family may be the better one.

3.7 Calculations

The minimization of the error estimate was done in Excel, using the Solver tool, see Billo [36]. This tool allows the determination of roots and extremes for multiple regression coefficients. In this case it is only one ($\alpha$). One constraint was enforced, i.e. $\alpha > 1 \times 10^{-12}$ Pa$^{-1}$. The result is sensitive in the starting value, and it was ascertained that the RMSE was at a minimum.

Several measurements were numerically simulated by running multigrid software for circular contacts from Venner [37]. This employs the Roelands viscosity relationship, where $z$ is determined through equation (6), and Dowson and Higginson's compressibility model from equation (7).

4 RESULTS

First the results for the reference lubricant, HVI60, will be presented, and next the three series of an oil blend with unknown pressure–viscosity coefficient. Figure 3 shows the range of these measurements in the $H = H(M, L)$ diagram. To be able to calculate the $L$ value for the base oil A, the estimate of $\alpha$ has been used, see section 4.2. It can be concluded that the experimental conditions of all experiments reported in this article are well into the VE regime, but also that they lie far outside the region which was explored at the time when Greenwood [26] published his best-fit line.

4.1 The reference lubricant: HVI60

Two series of measurements, performed in 2006 and 2008 but under the same conditions and at the same PCS test rig, are used. The target value for the $\alpha$ value of HVI60 at 40 °C is 19.8 GPa$^{-1}$, see Table 2.

The 2006 series was performed at 20 N load and in a speed range from 0.0297 to 2.29 m/s, thereby creating film thicknesses in the range 14–297 nm. Figure 4 shows the central film thickness versus rolling speed, for the measurements as well as for a few multigrid calculations. It is seen that in this case the calculation follows the experiment accurately, and that the film thickness varies with speed as a power law. Most of the 11 models are power-law models, so it may be expected that in this case the best fits can get close to the experiment. The solutions for this series of experiments are

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1These measurement data were kindly supplied by Kees Venner from Twente University, Enschede.
Fig. 3  Non-dimensional central film thickness $H = H(M, L)$ map for a circular contact. Legend: —— contour lines for $H = 1–12$; ····· VR/VE transition; ······· Greenwood’s curve (see text); + HVI60 at 40 °C (Series 2006); ○ HVI60 at 40 °C (Series 2008); ▲ oil A at 30 °C; ◇ oil A at 40 °C; □ oil A at 60 °C.

Fig. 4  Central film thickness behaviour for HVI60 at 20 N and 40 °C, series 2006. Legend: —♦— multigrid calculations; ···×··· measurements.

shown in Table 3. Figures 5(a) and (b) show the deviations of all models and the multigrid calculation for $\alpha = 19.8$ GPa$^{-1}$. When the optimized $\alpha$ value is applied, Fig. 5(a) transforms into Fig. 5(c), as an example.

To test the validity of the method if measurements are used which deviate appreciably from a power-law behaviour, another measurement series was used. Series 2008 was performed at 20 N load and in a speed range from 0.0048 to 1.587 m/s, thereby creating film thicknesses in the range 7–233 nm. Figure 6 presents the film thickness versus rolling speed, for the measurements as well as for some multigrid calculations.

It can be seen clearly that at low speeds and very low film thickness values the behaviour of the central film thickness deviates from the calculations, which designates non-Newtonian behaviour, see also Spikes [38]. This may affect the value found for $\alpha^*$, and it might be better to refrain from using the data obtained at the lowest speeds, say below 17 nm (0.029 m/s). If the full measurement range is used, Table 4(a) is the
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Table 3 Results for HVI60 series 2006, measurements 1–24, linear film thickness values

<table>
<thead>
<tr>
<th>For linear film thickness values model</th>
<th>Estimate for $\alpha^*$ (Pa$^{-1}$)</th>
<th>Standard deviation in lin $h_{\text{cent}}$</th>
<th>Correlation $R^2$</th>
<th>Deviation of 1.98E-08 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Archard and Cowking</td>
<td>2.84E-08</td>
<td>6.85E-09</td>
<td>0.9940</td>
<td>43.2</td>
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<td>2 Hamrock and Dowson</td>
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<td>1.72E-09</td>
<td>0.9996</td>
<td>-8.4</td>
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<tr>
<td>3 Hamrock et al.</td>
<td>1.83E-08</td>
<td>1.77E-09</td>
<td>0.9996</td>
<td>-7.5</td>
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<td>4 Chittenden et al.</td>
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<tr>
<td>5 Hooke</td>
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<td>7.64E-09</td>
<td>0.9926</td>
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<td>6 Sutcliffe</td>
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<td>7 Greenwood</td>
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<tr>
<td>11 Moes</td>
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<td>1.16E-09</td>
<td>0.9998</td>
<td>-23.0</td>
</tr>
</tbody>
</table>

result, and if the measurement range is limited to 0.029–1.587 m/s, the results of Table 4(b) are obtained. Figure 7(a) provides the deviation from the experimental results using $\alpha = 19.8\text{ GPa}^{-1}$ for the entire speed range and for the six models from Fig. 5(a), while Fig. 7(b) shows the results for the limited speed range and the five models from Fig. 5(b). It is seen that the deviations are less pronounced.

4.2 The base oil A with unknown $\alpha$

Base oil A is a blend of two other base oils, and the pressure–viscosity value is unknown. Using the same analysis as in section 4.1, where the RMSE value of the film thickness is minimized by changing the pressure–viscosity coefficient, the results for 30, 40, and 60 °C, respectively, were obtained. Table 5 shows the results. In these experiments the load was 50 N (contact pressure 0.71 GPa) and the speeds varied from 0.01 to 2.22 m/s.

5 DISCUSSION

5.1 Requirements for a good estimate

An estimate of $\alpha$ is considered good when:

(a) the deviation in the result is less than 3 per cent (which is of the order of $0.5 \times 10^{-9} \text{ Pa}^{-1}$);
(b) the standard deviation for the film thickness prediction is better than $3 \times 10^{-9} \text{ m}$ for the linear evaluation, and better than 0.025 for the logarithmic evaluation (see below on linear and logarithmic scales);
(c) the correlation is better than 0.998.

5.2 Linear film thickness scales

The results presented by Tables 3, 4(a), and 4(b) show that model (b) type of approximations have good results, the Chittenden et al. [11] model being the best in terms of estimate and correlation. All results from models employing a Barus pressure–viscosity model [30, 31] are far from the target and cannot be recommended, except Greenwood’s [26]. This is remarkable, since the approximation is used well away from the regime where it was obtained, see the Greenwood line in Fig. 3. After all, this model is more an observation of non-dimensional behaviour, rather than a mathematical–physical model.

Although the results for the (d) type of models, the Moes family of formulas, score very well on criteria (2) and (3), they do not meet the most important criterion (1): they are some 20 per cent too low in $\alpha$ and can therefore not be recommended for determining $\alpha$.

5.3 Logarithmic film thickness scales

It can be seen in, e.g. Figs 5(c) and 7(b), that the deviations in film thickness prediction from the measurement are large in the low speed (low thickness) regime, when the optimum $\alpha$ value is employed. If the minimization procedure is carried out on the logarithmic values of the film thickness, the regime of low speeds (low $L$ and high $M$ values) is more emphasized. This is a regime further away from the conditions of the numerical results, which form the basis of the Hamrock family of formulas, and moves towards the IE asymptote ($L = 0$) (see Fig. 2(a)). At first sight the results of the type (d) ‘Moes’ models improve remarkably, see Table 6, and actually are the best. This may be expected, because these formulas contain the IE asymptote. But only at the sacrifice of lower correlation and larger standard deviation, the other criteria are not met and the good regression outcome may be considered as a coincident hit. In series 2006, see Table 3, criterion (1) is far away, the other two are on target. It must be concluded that using logarithmic values optimization does not bring any advantages. This leads to the conclusion that the best model is the one by Chittenden et al. [15], as long as the film thickness measurement shows a power-law behaviour.
Fig. 5  (a) Deviations of calculated central film thickness from the measured values, versus entrainment speed (series 2006), for $\alpha = 19.8$ GPa$^{-1}$. Legend (from top to bottom): $\triangle$ Sutcliffe; $\square$ Moes; $\circ$ Hamrock and Dowson; $*$ Venner and Lubrecht; $\blacktriangle$ Chittenden et al.; $\times$ Hooke; $\blacklozenge$ Archard and Cowking. (b) Deviations of calculated central film thickness from the measured values, versus entrainment speed (series 2006), for $\alpha = 19.8$ GPa$^{-1}$. Legend (from top to bottom): $*$ Nijenbanning et al.; $\square$ Venner; $\times$ Hamrock et al.; $\odot$ Greenwood; $+$ FMG calculation. (c) Deviations of calculated central film thickness from the measured values, versus entrainment speed, using the optimized $\alpha$ value with each model (series 2006). Legend (from top to bottom): $\triangle$ Sutcliffe; $\square$ Moes; $\circ$ Hamrock and Dowson; $*$ Venner and Lubrecht; $\blacktriangle$ Chittenden et al.; $\times$ Hooke; $\blacklozenge$ Archard and Cowking.
The determination of the pressure–viscosity coefficient of a lubricant

Fig. 6 Central film thickness behaviour for HVI60 at 20 N and 40 °C, series 2008. Legend: —♦— multigrid calculations; ···×··· measurements

Table 4(a) Results for HVI60 series 2008, measurements 1–31, linear film thickness values

<table>
<thead>
<tr>
<th>For linear film thickness values model</th>
<th>Estimate for (\alpha^*) (Pa(^{-1}))</th>
<th>Standard deviation in lin (h_{centr})</th>
<th>Correlation (R^2)</th>
<th>Deviation of 1.98E-08 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Archard and Cowking</td>
<td>2.92E-08</td>
<td>6.05E-09</td>
<td>0.9920</td>
<td>47.4</td>
</tr>
<tr>
<td>2 Hamrock and Dowson</td>
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<td>2.34E-09</td>
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<td>-7.7</td>
</tr>
<tr>
<td>3 Hamrock et al.</td>
<td>1.83E-08</td>
<td>2.76E-09</td>
<td>0.9982</td>
<td>-7.5</td>
</tr>
<tr>
<td>4 Chittenden et al.</td>
<td>1.96E-08</td>
<td>2.76E-09</td>
<td>0.9982</td>
<td>-1.0</td>
</tr>
<tr>
<td>5 Hooke</td>
<td>2.50E-08</td>
<td>6.59E-09</td>
<td>0.9905</td>
<td>26.3</td>
</tr>
<tr>
<td>6 Sutcliffe</td>
<td>1.42E-08</td>
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<tr>
<td>7 Greenwood</td>
<td>1.82E-08</td>
<td>2.22E-09</td>
<td>0.9988</td>
<td>-7.9</td>
</tr>
<tr>
<td>8 Venner</td>
<td>1.57E-08</td>
<td>2.17E-09</td>
<td>0.9989</td>
<td>-20.6</td>
</tr>
<tr>
<td>9 Nijenbanning et al.</td>
<td>1.50E-08</td>
<td>2.72E-09</td>
<td>0.9982</td>
<td>-24.2</td>
</tr>
<tr>
<td>10 Venner and Lubrecht</td>
<td>1.59E-08</td>
<td>2.54E-09</td>
<td>0.9985</td>
<td>-19.6</td>
</tr>
<tr>
<td>11 Moes</td>
<td>1.50E-08</td>
<td>2.72E-09</td>
<td>0.9982</td>
<td>-24.1</td>
</tr>
</tbody>
</table>

Table 4(b) Results for HVI60 series 2008, measurements 9–31, linear film thickness values

<table>
<thead>
<tr>
<th>For linear film thickness values model</th>
<th>Estimate for (\alpha^*) (Pa(^{-1}))</th>
<th>Standard deviation in lin (h_{centr})</th>
<th>Correlation (R^2)</th>
<th>Deviation of 1.98E-08 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Archard and Cowking</td>
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<td>47.2</td>
</tr>
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<td>2 Hamrock and Dowson</td>
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<td>1.72E-09</td>
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</tr>
<tr>
<td>3 Hamrock et al.</td>
<td>1.83E-08</td>
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<td>-7.5</td>
</tr>
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<td>4 Chittenden et al.</td>
<td>1.96E-08</td>
<td>2.21E-09</td>
<td>0.9989</td>
<td>-1.1</td>
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<tr>
<td>5 Hooke</td>
<td>2.51E-08</td>
<td>6.80E-09</td>
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<td>-8.0</td>
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<tr>
<td>8 Venner</td>
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<tr>
<td>9 Nijenbanning et al.</td>
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<td>0.9988</td>
<td>-24.4</td>
</tr>
<tr>
<td>10 Venner and Lubrecht</td>
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<td>1.93E-09</td>
<td>0.9991</td>
<td>-19.4</td>
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<tr>
<td>11 Moes</td>
<td>1.53E-08</td>
<td>1.91E-09</td>
<td>0.9991</td>
<td>-24.6</td>
</tr>
</tbody>
</table>

5.4 The comparison of experimental with theoretical results

All figures show that with very thin films, type (d) Moes family formulas are closer to the measurement than model (b) Hamrock family approximations, but at films thicker than 30–40 nm the type (b) is superior. This is an amazing conclusion, because the Hamrock and Dowson [25] and related approximations are based on a relatively small regime within the \((L, M)\)
plane (see Figs 2(b) or 2(c)). The most versatile fit of the Moes family, the Moes [10] formula, is not inferior to the other ones [9, 22, 33] for a circular contact, which is a good thing. But in this case the only issue that matters is that the fit should be as good as possible for circular contacts. The physical basis of the Moes fits is very much appealing, and the swift and smooth transition between lubrication regimes is to be commended, but probably the Moes formulas encounter problems in the transition regimes from one to another asymptote to make them the best. Here is left room for improvement, and because much more accurate numerical data are available over a much wider range in the \((L, M)\) field than in 1985, an improved (and maybe less versatile) approximation formula is very welcome.

A comparison of Tables 4(a) and 4(b) also learns that the results are hardly affected by omitting the data at very thin films. This can be attributed to the small contribution to the total error (RMSE) of the model deviations at small film thicknesses. It also implies
Table 5  Results for base oil A at 30, 40, and 60 °C for linear film thickness optimization

<table>
<thead>
<tr>
<th>Model</th>
<th>Estimate for α (GPa)</th>
<th>Standard deviation in lin h_{cent}</th>
<th>Correlation R^2</th>
<th>Estimate for α (GPa)</th>
<th>Standard deviation in lin h_{cent}</th>
<th>Correlation R^2</th>
<th>Estimate for α (GPa)</th>
<th>Standard deviation in lin h_{cent}</th>
<th>Correlation R^2</th>
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</thead>
<tbody>
<tr>
<td>Archard and Cowking</td>
<td>3.79E-08</td>
<td>2.10E-08</td>
<td>0.9943</td>
<td>3.35E-08</td>
<td>2.44E-08</td>
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<td>3.21E-08</td>
<td>6.94E-09</td>
<td>0.9972</td>
</tr>
<tr>
<td>Hamrock and Dowson</td>
<td>3.24E-08</td>
<td>1.94E-08</td>
<td>0.9947</td>
<td>2.55E-08</td>
<td>2.16E-08</td>
<td>0.9929</td>
<td>2.30E-08</td>
<td>7.12E-09</td>
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<td>Hamrock et al.</td>
<td>3.15E-08</td>
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<td>0.9954</td>
<td>2.50E-08</td>
<td>1.99E-08</td>
<td>0.9941</td>
<td>2.24E-08</td>
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<td>0.9973</td>
</tr>
<tr>
<td>Chittenden et al.</td>
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<td>1.83E-08</td>
<td>0.9953</td>
<td>2.99E-08</td>
<td>1.30E-08</td>
<td>0.9976</td>
<td>2.50E-08</td>
<td>6.41E-09</td>
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<tr>
<td>Hooke</td>
<td>3.22E-08</td>
<td>2.25E-08</td>
<td>0.9936</td>
<td>2.85E-08</td>
<td>2.64E-08</td>
<td>0.9911</td>
<td>2.77E-08</td>
<td>7.62E-09</td>
<td>0.9966</td>
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<tr>
<td>Sutcliffe</td>
<td>2.31E-08</td>
<td>1.60E-08</td>
<td>0.9965</td>
<td>1.71E-08</td>
<td>3.63E-08</td>
<td>0.9789</td>
<td>1.73E-08</td>
<td>5.63E-09</td>
<td>0.9981</td>
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<td>Venner</td>
<td>2.64E-08</td>
<td>1.76E-08</td>
<td>0.9957</td>
<td>2.32E-08</td>
<td>1.21E-08</td>
<td>0.9980</td>
<td>1.93E-08</td>
<td>6.66E-09</td>
<td>0.9972</td>
</tr>
<tr>
<td>Nijenbanning et al.</td>
<td>2.54E-08</td>
<td>1.74E-08</td>
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<td>2.22E-08</td>
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<td>0.9981</td>
<td>1.85E-08</td>
<td>6.35E-09</td>
<td>0.9974</td>
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<tr>
<td>Venner and Lubrecht</td>
<td>2.58E-08</td>
<td>1.74E-08</td>
<td>0.9958</td>
<td>2.26E-08</td>
<td>1.18E-08</td>
<td>0.9981</td>
<td>1.85E-08</td>
<td>6.30E-09</td>
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<td>Moes</td>
<td>2.55E-08</td>
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<td>0.9958</td>
<td>2.22E-08</td>
<td>1.16E-08</td>
<td>0.9981</td>
<td>1.85E-08</td>
<td>6.30E-09</td>
<td>0.9975</td>
</tr>
</tbody>
</table>

Table 6  Results for HVI60 series 2008, measurements 1–31, log film thickness values

<table>
<thead>
<tr>
<th>For log film thickness values model</th>
<th>Estimate for α^∗ (Pa^{-1})</th>
<th>Standard deviation in log h_{cent}</th>
<th>Correlation R^2</th>
<th>Deviation of 1.98E-08 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Archard and Cowking</td>
<td>4.26E-08</td>
<td>2.88E-01</td>
<td>0.9530</td>
<td>114.9</td>
</tr>
<tr>
<td>2 Hamrock and Dowson</td>
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<td>1.74E-01</td>
<td>0.9784</td>
<td>16.5</td>
</tr>
<tr>
<td>3 Hamrock et al.</td>
<td>2.41E-08</td>
<td>1.89E-01</td>
<td>0.9753</td>
<td>33.1</td>
</tr>
<tr>
<td>4 Chittenden et al.</td>
<td>2.63E-08</td>
<td>1.80E-01</td>
<td>0.9753</td>
<td>33.1</td>
</tr>
<tr>
<td>5 Hooke</td>
<td>3.75E-08</td>
<td>3.03E-01</td>
<td>0.9493</td>
<td>89.4</td>
</tr>
<tr>
<td>6 Sutcliffe</td>
<td>1.88E-08</td>
<td>2.07E-01</td>
<td>0.9715</td>
<td>-4.9</td>
</tr>
<tr>
<td>7 Venner</td>
<td>1.96E-08</td>
<td>1.69E-01</td>
<td>0.9794</td>
<td>9.6</td>
</tr>
<tr>
<td>8 Nijenbanning et al.</td>
<td>1.97E-08</td>
<td>1.95E-01</td>
<td>0.9741</td>
<td>-0.6</td>
</tr>
<tr>
<td>9 Venner and Lubrecht</td>
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<td>1.92E-01</td>
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<tr>
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<td>1.94E-01</td>
<td>0.9741</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

It is also observed that a good approximation formula, as seen from a statistical viewpoint, is not a guarantee that it also is the best tribological tool. Alongside statistical arguments, the accuracy of the result is a most decisive factor.

5.5 The minimum film thickness

Some of the references from the 11 central film thickness approximations in section 3.6 also provide expressions of the minimum film thickness. To demonstrate that this is a much more difficult issue, Fig. 8 is shown as one example out of a few more figures.

Moes [10] (Moes, H., 19 September 2006, personal communication) provides a suggestion to attain at a minimum film thickness formula. He states that the ratio of the minimum to the central film thickness varies between 0.65 and 0.81, and he reasons that a value of 3/4 will be quite adequate. As can be seen in Lubrecht [51] and Venner [22] the ratio between the central and minimum film thickness value can be much higher than 4/3, e.g. over 3.0 at M values of 10^3, as is also found by Hooke [30]. Figure 8 therefore shows that this suggestion does not yield a proper fit. Also, it illustrates that it is difficult to predict the minimum film thickness through an approximation formula, and if so, type (b) formulas like Hamrock and Dowson [25] and later derivatives are the best available.
6 CONCLUSIONS AND RECOMMENDATIONS

1. Central film thickness approximate formulas can be used to determine the pressure–viscosity coefficient $\alpha^*$, by fitting them to accurate central film thickness measurements.

2. Eleven approximation formulas for the central film thickness in EHL circular contacts have been compared in the assessment of $\alpha^*$. In the measurement range of this study the formula from Chittenden et al. [11] proved to be the best, and related formulas as from Hamrock and Dowson [25] and Hamrock et al. [12] are close. The Chittenden et al. formula for central film thickness can be recommended for estimating the value of the pressure–viscosity coefficient of a lubricant through an interferometric device with proper accuracy.

3. The validity of the Chittenden et al. [11] formula transcends the area where it was originally designed for.

4. The Moes [10] formulas are the most versatile and general ones available, but in the range of the measurements they lack the accuracy of the Chittenden approximation.

5. The conclusions are valid for relatively low values of $L$. It is recommended to perform film thickness measurements for fluids with high $L$ values, as traction fluids, and check whether the conclusions still hold.

6. More and better numerical data in a wide range of $L$ and $M$ values, available at present, provide a basis for a better approximation formula, which yields an increased accuracy in the determination of $\alpha^*$.

ACKNOWLEDGEMENTS

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lubrication with curved surfaces under high load and relative motion)\(^1\), 1984, 92 pp.


APPENDIX 1

Notation

- $a_t$: help variable, after Sutcliffe [31]
- $a_{uz}$: semi-width of the Hertzian contact ellipse (generally in entrainment $x$ direction) (m)
- $b_{uz}$: semi-width of the Hertzian contact ellipse (generally normal to entrainment $y$ direction) (m)
- $c$: transition parameter for isoviscous and piezoviscous fluid behaviour, after Sutcliffe [31]
- $E$: modulus of elasticity (N/m$^2$)
- $E_r$: equivalent modulus of elasticity, see equation (1) (N/m$^2$)
- $F$: normal load (N)
- $g_E$: non-dimensional number for elasticity effects after Johnson [14]†
- $g_v$: non-dimensional number for pressure–viscosity effects after Johnson [14]†
- $G$: non-dimensional number for materials, after Dowson and Higginson [13]†
- $h$: film thickness, in general $h = h(x, y, t)$ (m)
- $h'$: non-dimensional number for minimum film thickness, after Johnson [14]†
- $h_c$: central film thickness (film thickness at $x = y = 0$) (m)
- $H$: non-dimensional number for film thickness, after Dowson and Higginson [13]†
- $H_0$: non-dimensional number for film thickness, after Blok [39]†
- $k$: ellipticity ratio, $k = b_{uz}/a_{uz}$
- $L$: non-dimensional number for the pressure–viscosity coefficient, after Blok [39]†
- $M$: non-dimensional number for the load, line contacts, after Blok [39]†
- $n$: non-dimensional extent of the inlet zone, after Hooke [30]
- $N$: non-dimensional number for the load at elliptical contacts, after Moes [10]†
- $p$: fluid film pressure (N/m$^2$)
- $p_R$: reference fluid pressure according to Roelands [19] (N/m$^2$)
- $R$: radius of curvature of a body near the nominal contact point (m)
- $R_e$: effective radius of curvature in the direction of entrainment (m)
- $R_t$: effective radius of curvature perpendicular to the direction of entrainment (m)
- $\dot{s}$: approximation parameter for the film thickness, after Moes [10]†
- $\ddot{u}$: mean entrainment or rolling velocity in the $x$ direction, $u = 1/2 (u_1 + u_2)$ (m/s)
- $U$: non-dimensional number for speed, after Dowson $u$. Higginson [13]†
- $W$: non-dimensional number for the load, after Dowson $u$. Higginson [13]†
- $x$: coordinate in the direction of entrainment, along the lubricant film (m)
- $y$: coordinate perpendicular to the direction of entrainment, along the lubricant film (m)
- $z$: coordinate across the lubricant film (m)
- $\alpha$: pressure–viscosity coefficient (m$^2$/N)
- $\dot{\gamma}$: shear velocity in the fluid (s$^{-1}$)
- $\eta$: dynamic viscosity of the fluid (Ns/m$^2$)
- $\eta_0$: dynamic viscosity of the fluid at ambient pressure, $p = 0$ (Ns/m$^2$)
- $\eta_R$: reference viscosity of the fluid according to Roelands [19] (Ns/m$^2$)
- $\lambda'$: film thickness factor, after Sutcliffe [31]
- $\nu$: Poisson modulus
- $\sigma_{Ht}$: Hertzian (maximum dry contact) pressure value (N/m$^2$)
- $\tau$: shear stress in the fluid (N/m$^2$)
- $\phi$: Archard side leakage factor (see section A4.1 in Appendix 4)
- $\omega$: effective radii of curvature ratio (crowning ratio), $\omega = R_t/R_e$

Indices

- $c$: central value (at the centre, at $x = y = 0$)
- $e$: in the direction of entrainment
- $IE$: isoviscous-elastic
- $IR$: isoviscous-rigid
- $max$: maximum value
- $min$: minimum value
- $R$: reference value
- $t$: in the transversal direction
- $VE$: piezoviscous-elastic
- $VR$: piezoviscous-rigid
- $x,y,z$: in the direction of $x, y, z$, respectively, $x$ is generally in the entrainment direction
- $0$: under ambient conditions
- $1,2$: solid body number

---

†See Appendix 3 for the definition of EHL non-dimensional numbers and transformations between sets of numbers.
APPENDIX 2: A SHORT REVIEW OF NON-DIMENSIONAL GROUPING

The use of non-dimensional groups is essential in finding approximations to numerical and experimental results. Blok [39, 43, 44] and Koets [45] mapped experimental and numerical minimum film thickness solutions for EHL line contacts in a survey diagram, by employing three non-dimensional groups. This early but advantageous use of non-dimensional EHL groups was almost unnoticed. Dowson and Higginson [46, 47] increased the interest in, and use of, non-dimensional grouping in EHL work. Dowson and Higginson [13] employed dimensional analysis to conclude that the non-dimensional film thickness $H^*$ depends on three other groups

$$H^* = H^*(U, W, G)$$ (8)

They tied their numerical results for highly loaded EHL line contacts together in their well-known formula for the non-dimensional minimum film thickness $H^*$

$$H^* = 1.6 \frac{G^{0.6} U^{0.7}}{W^{0.13}}$$ (9)

$U$ is the only group depending on the rolling speed, $W$ is a load dependent group, and $G$ a materials group, hence a total of four non-dimensional groups (including $H$). Note that this formula holds for a certain regime, where both $g_E$ and $g_V$ have high values. A definition of the groups is provided in Appendix 3. Formula (9) allowed an accurate prediction, far better than existing theories at that time were able to, for this regime.

By employing vectorial dimensional analysis, Blok [39] reasoned that three groups instead of four would suffice

$$H = H(M, L)$$ (10)

where $M$ is the only group depending on the load (Blok, H., 1995, personal communication), and $L$ contains the pressure–viscosity coefficient. The choice of these groups implies that the rolling speed appears in all three groups $H, M, L$, see Appendix 3. The reduction to three groups allowed Moes [40] to arrive at a good curve fit for line contacts. Moes [48] also provides a systematic approach to derive the minimum set of non-dimensional groups.

Johnson [14] summarizes all EHL film thickness work until that date and by doing so he made an end to much debate on the number of groups: the minimum in EHL line contacts is 3. This allows a more condensed and generalized representation and additionally better curve fitting. By physical reasoning Johnson derived a set

$$h = h(g_E, g_V)$$ (11)

where $g_E$ represents a group that depends on the reduced elasticity, and not on the piezoviscosity, and $g_V$ is a group that depends on $a$, but not on the elasticity. Both pressure–viscosity and elasticity do not appear in the non-dimensional film thickness$. Johnson’s notation puts the same data in another perspective on the film thickness, by showing the enormous impact of the two main factors contributing to EHL lubrication: elasticity (by $E$) and pressure–viscosity (by $a$). Johnson also showed that four EHL regimes can be distinguished: IR, VR, IE, and VE, and four asymptotes can be attributed to each of them.

In the case of elliptical contacts, one more non-dimensional group appears. This is the radii of curvature ratio $\omega$, or the ellipticity ratio $k = k(\omega)$. The circular contact, having $\omega = k = 1$, was first analysed. Archard and Cowking [27] proposed an analytically derived central film thickness formula, which was good at that time, but did not fit many experimental results. Hamrock and Dowson [25, 28] came with numerical results and approximation formulas for minimum and central film thickness for short elliptical contacts, which were also corroborated by experiments. The non-dimensional formulation reads

$$H = H(W, U, G, k)$$ (12)

where $k$ is the ellipticity ratio, and the other groups $H, W, U$, and $G$ are exactly the same as those for line contacts. Moes and Bosma [21] use the same groups $M$ and $L$ as in equation (10) and add a parameter $\lambda = \omega^{-1}$

$$H = H(M, L, \omega)$$ (13)

where a caret is used to mark the circular geometry. Consistent with his line contact groups, Johnson [14] suggested a set for circular contacts (where $\omega = k = 1$)

$$h = h(g_V, g_E, \omega = 1)$$ (14)

Note that these groups differ from the corresponding ones in equation (11) for line contacts, and are therefore marked by a caret. Hamrock [29] and Hamrock

$^1$Blok [43] used non-dimensional groups $M^2$ (for the load) and $LM^{3/2} = 2^{-1/2} g_V$ (for the piezoviscosity).

$^2$Blok himself confided the author that he used the symbol ‘M’ to honour the Swiss A. Meldahl, who was working on gears in the 40s. Traceable literature citing the three Blok groups are, besides Dowson and Higginson [13], Moes [40], Blok [50] and Greenwood [26].

$^3$As Moes [40] is the first to apply these groups in the open literature, this is probably the reason why they are often designated ‘Moes’ or ‘Moes and Bosma’ groups. Blok liked to refer to them as ‘Delft groups’, and for the figure in Moes [40] as the ‘Delft Diagram’.

$^4$Note that the load and speed appear in all three groups.
et al. [12] generalized this suggestion to the elliptical case

\[ h' = h'(g'_c, g'_v, \omega) \]  

(15)

where the prime designates that the groups correspond to elliptical contacts and are different from line contacts. All groups mentioned are listed in Appendix 3, including transformations between the sets.

APPENDIX 3: NON-DIMENSIONAL GROUPS FOR ELLIPTICAL CONTACTS IN EHL LUBRICATION THEORY

A3.1 The Hamrock and Dowson [25] groups

\[ H = H(W, U, G, k) \]

\[ H = \frac{h}{Rc} \]

\[ W = \frac{F}{E_i R_c} \]

\[ U = \frac{\eta_0 \bar{u}}{E_i R_c} \]

\[ G = \alpha E_i \]

\[ k = \frac{b_{hz}}{a_{hz}} \]

A3.2 The Blok [39] or Moes [40] groups

\[ \hat{H} = \hat{H}(N, L, \omega) \]

\[ \hat{H} = \left( \frac{h}{Rc} \right) \left( \frac{E_i R_c}{2 \eta_0 \bar{u}} \right)^{1/2} \]

\[ N = \omega^{-1/2} M = \omega^{-1/2} \left( \frac{F}{E_i R_c^2} \right) \left( \frac{E_i R_c}{2 \eta_0 \bar{u}} \right)^{3/4} \]

\[ L = (\alpha E_i) \left( \frac{2 \eta_0 \bar{u}}{E_i R_c} \right)^{1/4} \]

\[ \omega = \left( \frac{R_i}{Rc} \right) \]

A3.3 The Johnson [14] or Hamrock [12, 29] groups

\[ h' = h'(g'_c, g'_v, \omega) \]

\[ h' = \left( \frac{F}{\eta_0 \bar{u} R_c} \right)^2 \left( \frac{h}{Rc} \right) \]

\[ g'_c = \left( \frac{F^8}{E_i^2 R_c^2 \eta_0 \bar{u} \alpha \sigma} \right)^{1/3} \]

Note that these definitions for elliptical contact deviate from Johnson’s [14] line contact definitions. This prompted Greenwood [26] to prefer groups independent of the geometry, namely \((\alpha \sigma_{hz})\) and \(L\). In addition, Greenwood’s groups (see section A3.4 in Appendix 3) have a load group and a speed group, which is more convenient for the experimenter.

A3.4 The Greenwood [26] groups

For a circular contact, these groups read

\[ \hat{H} = \hat{H}(\alpha \sigma_{hz}, \bar{L}, \omega) \]

\[ \hat{H} = \left( \frac{h}{R} \right) \left( \frac{R}{\alpha \eta_0 \bar{u}} \right)^{2/3} \]

\[ \alpha \sigma_{hz}\omega = \left( \frac{3}{2 \pi^3} \right)^{1/3} \left( \frac{F \alpha^3 E_i^2}{L^2} \right)^{1/3} \]

\[ \bar{L} = (\alpha E_i) \left( \frac{\eta_0 \bar{u}}{E_i R_c} \right)^{1/4} \]

\[ \omega = 1 \]

These Greenwood groups have been used by Hooke [34, 35] and more recently by authors like Guo et al. [42] in studies into the IR–VE regimes transition. The load group can be used for any geometry, i.e. circular, elliptical, and line contact. The value for circular contact has been substituted as an example. Note that Greenwood’s group \(\bar{L}\) differs from the widely accepted Blok [39] and Moes [40] groups by a factor of 2\(^{-1/3}\), see also Table 7.

A3.5 Transformations between sets of non-dimensional groups

Transformations back and forth can be performed by employing Table 7. Note that a transformation from four to three groups is allowed, but a backwards transformation is impossible. As an example, if Table 7 is used, it can easily be shown that \(\hat{H}_c \approx 1.5\) from the preceding section A3.4 can be mapped into \(\hat{H}_c \approx 0.94 L^{2/3}\), see also section A4.6 in Appendix 4.

APPENDIX 4: NON-DIMENSIONAL EHL FILM THICKNESS FORMULAS

Below follows a detailed overview of the curve fits that have been employed in the article. All formulas have been rewritten in the \(H, M, L\) groups notation.
Transformations between the four most commonly used sets of non-dimensional groups in EHL elliptical contacts (ellipticity ratio \(\omega\))

<table>
<thead>
<tr>
<th>Johnson [14] (N', G, G, \omega = 1)</th>
<th>Moes [10] (H, N, L, \lambda)</th>
<th>Greenwood [26] (H, \alpha, H, L, \omega = 1)</th>
<th>Hamrock and Dowson ([25, 28]) (H, U, W, G, k = k(\omega))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N' = G' = G) (\omega = 1)</td>
<td>(H' = 4H M^2) (\omega = -1)</td>
<td>(H - \frac{2 \pi}{3} \left(\frac{3}{2 \pi^3}\right)^{1/3} G^2 \alpha \sigma_{\text{visc}}) (\omega = -1)</td>
<td>(H - \frac{2 \pi}{3} \left(\frac{3}{2 \pi^3}\right)^{1/3} G^2 \alpha \sigma_{\text{visc}}) (\omega = 1)</td>
</tr>
<tr>
<td>Moes [10] (H, N, L, \lambda)</td>
<td>(N = M \sqrt{\lambda})</td>
<td>(H = 2 H_L L^{-2/3})</td>
<td>(H = 2 H_L L^{-2/3})</td>
</tr>
<tr>
<td>Greenwood [26] (H, \alpha, H, L, \omega = 1)</td>
<td>(H = h' g'_{\alpha}^{-2/3})</td>
<td>(\alpha = \left(\frac{3}{2 \pi^3}\right)^{1/3} G^2 \alpha \sigma_{\text{visc}}) (\omega = -1)</td>
<td>(\alpha = \left(\frac{3}{2 \pi^3}\right)^{1/3} G^2 \alpha \sigma_{\text{visc}}) (\omega = 1)</td>
</tr>
</tbody>
</table>

A4.1 Archard and Cowking’s [27] central film thickness formula

Archard and Cowking [27] used a Barus viscosity model and considered an incompressible fluid. Their approximation for the film thickness at the beginning of the inlet (a Grubin approach) now reads

\[
\hat{H}_{\text{VE}} \approx 1.2258 \Phi^{2/3} L^{2/3} M^{-1/3} \tag{16}
\]

where \(\Phi\) represents the Archard and Cowking side-leakage factor

\[
\Phi = \left(1 + \frac{2K}{3H}\right)^{-1} \tag{17}
\]

A4.2 Hamrock and Dowson’s [12, 25, 28] VE regime formulas

Hamrock and Dowson [25, 28] defined their original non-dimensional film thickness formulas in the \(H, U, G, W\) groups, and \(k\) was used for the ellipticity. When Hamrock et al. [12] transformed them into the Johnson groups \(h', g', G, k\), and \(k\), they had to accept an approximation of a factor \(U^p\) where \(p\) is a small number. The Hamrock and Dowson approximation formula based on Hamrock et al. [12] for the central film thickness in the VE regime now reads

\[
\hat{H}_{\text{VE}} \approx 2.25\left[1 - 0.61 e^{-0.73 k} L^{0.53} M^{-0.0633}\right] \tag{18a}
\]

and for the minimum film thickness

\[
\hat{H}_{\text{VE, min}} \approx 2.13\left[1 - e^{-0.68 k} L^{0.49} M^{-0.0766}\right] \tag{18b}
\]

These equations were curve fitted on numerical solutions for a compressible fluid and a Roelands pressure–viscosity model. The claimed inaccuracy for the original equations is within 10 per cent for the central, and within 5 per cent for the minimum film thickness for the investigated range, see Hamrock and Dowson [25] and Fig. 2. The standard deviation in the factors of equations (18a) and (18b) is about 0.02. The transformations of the Hamrock and Dowson [25, 28] formulas to the Moes numbers yield slightly different factors and exponents for \(M\).

A4.3 Chittenden et al. [11] formulas

Chittenden et al. [11] investigated long and short contacts with their major axis at an angle to the direction of entrainment. The fluid is compressible and the Roelands equation has been used. Their approximation formulas almost merge into Hamrock and Dowson’s [25] for coinciding directions and differ only slightly in the constant factor. Chittenden reformulated the Hamrock and Dowson [25, 28] formulas for entrainment along the minor axis of the contact ellipse (short ellipse), and found more accurate approximations for entrainment along the major ellipse axis (long ellipse), up to the circular geometry. The central film

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1For the central film thickness formula \(p = -0.01275\), which yields a factor of 0.1342 for the range of \(U\) considered. For the minimum film thickness formula \(p = 0.00275\), giving a correction factor of 0.942. See Hamrock et al. [12], p. 532.
thickness approximation for long elliptical contacts recast in Moes numbers is

$$\hat{H}_{VE,c} \approx 1.80[1 - e^{-3.36(R_t/R_e)^1}]L^{0.49}M^{-0.073}$$  \hspace{1cm} (19a)$$

The claimed mean absolute error amounts < 4 per cent. The minimum film thickness for long elliptical contact is approximated by

$$\hat{H}_{VE,min} \approx 1.77[1 - e^{-0.96(R_t/R_e)}]L^{0.49}M^{-0.073}$$  \hspace{1cm} (19b)$$

A4.4 The Hooke [30] formulas

Hooke [30] derived formulas for both the central and the minimum film thickness in elliptical contacts. His model employs a Barus pressure–viscosity model and the fluid is incompressible. Interesting is that his analysis provides two local minimum values, which allows to assess whether the minimum occurs at the end of the contact along the entrainment axis, or at the side lobes. The criterion is the value of \((g_{3/2}^e g_{-1}^e)\) or \((L^{-1}M)\). Additionally, Hooke provides criteria for the transition between the VR and VE regime and a formula for the VR film thickness. The film thickness at the contact centre expressed in the Moes numbers now reads

$$\hat{H}_{VE,c} \approx 0.963 \left(\frac{R_t}{R_e}\right)^{0.144} L^{3/4} M^{-1/12}$$  \hspace{1cm} (20)$$

For a minimum thickness occurring along the entrainment axis at the end of the contact Hooke proposes

$$\hat{H}_{VE,min1} \approx 0.539 \left(\frac{R_t}{R_e}\right)^{0.144} L^{3/4} M^{-1/12}$$  \hspace{1cm} (21a)$$

and for a minimum at the side lobes

$$\hat{H}_{VE,min2} \approx 0.695 \left(\frac{R_t}{R_e}\right) \left(1 + 0.045 \frac{R_e}{R_t}\right)^{0.4} L M^{-1/3}$$  \hspace{1cm} (21b)$$

Conditions are sufficiently far into the VE regime when both \((g_{3/2}^e g_{-1}^e)\) and \((g_{3/2}^e g_{-1}^e)\), i.e. \((L)\) and \((L^{-1}M)\) respectively, are large. The lowest value of equations (21a) and (21b) decides where the minimum occurs. Hooke states that equation (21a) has an inaccuracy of 1 per cent for \(\omega > 1\), and equation (21b) of 5 per cent for 0.001 < \(\omega < 1000\) compared to his numerical results.

A4.5 The Sutcliffe [31] formula

Sutcliffe [31] elaborates on Hooke's [30] analysis, i.e. is using a Barus relation for the viscosity behaviour, and an incompressible fluid. He reformulates Hooke's central film thickness, which then reads essentially

$$\hat{H}_{VE,c} \approx \lambda' \left[ \frac{k^{1/2} a_1}{2 a_1} \right] \hat{R}_t^{5/4} \left[1 + (0.61 - 0.32 k) \right] \times (\hat{n}^2 - 2.2 \hat{n}) L^{3/4} M^{-1/12}$$  \hspace{1cm} (22)$$

for 0.24 < \(k < 1\) and 0 < \(\hat{n} < 1\), where

$$c = \frac{3}{2a_1} L M^{1/5}$$

$$a_1 = \frac{a}{R_e} \left( \frac{F}{E_a R_e^2} \right)^{1/3} = a_1(\omega)$$

$$\lambda' = 1.067 \left(1 + \frac{0.97}{c} \right)$$

$$\hat{n} = 0.446 a_1^{3/2} L^{1/2} M^{-1/2}$$  \hspace{1cm} (23)$$

In equation (23), \(c\) represents a parameter for the transition from the isoviscous into the piezoviscous regime, different from Hooke, and \(\hat{n}\) a parameter for the non-dimensional extent of the inlet zone, defined by Hooke [30]. Sutcliffe provides a qualitative comparison with literature data, but no quantitative information on the inaccuracy.

A4.6 The Greenwood [26] implied formula

Greenwood [26] judged that he better refrained from providing an approximation. His best-fit line is not on film thickness, but surprisingly on experimental conditions. He states that almost all but a few numerical and experimental results (at that time) can be expressed in a relationship between his groups \(\hat{L}\) (the speed group) and \(\alpha \sigma_{H\zeta}\) (the load group)

$$\hat{L} \approx 0.95 (\alpha \sigma_{H\zeta})^{0.91}$$  \hspace{1cm} (24a)$$

which can be transformed into

$$\hat{L} \approx 0.000 14 M^{1.4}$$  \hspace{1cm} (24b)$$

Essentially, this is a fit of a cloud of results that have a rather high correlation. Most of the current measurements are located far outside this area, having \(\alpha \sigma_{H\zeta}\) values of 10.4 (HV160) and 13.2 < \(\alpha \sigma_{H\zeta}\) < 18.5 (base oil A), while 1.53 < \(\hat{L} < 7.14\) and 1.64 < \(\hat{L} < 10.8\), respectively. From Greenwood's [26] results1 the central film

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1See Fig. 1 in Greenwood [26]. The Greenwood non-dimensional central film thickness is almost constant irrespective of the experimental or numerical conditions, for the 74 results collected from the literature. The average value is 1.49 with a standard deviation of 0.24. Note that Greenwood himself refrains from providing a curve fit of all the data collected.
This implies in physical terms that pressure–viscosity effects predominate elasticity effects. This equation can be rewritten as

\[ \hat{H}_c \approx 0.94L^{2/3} \]  

(25b)

This implies that the non-dimensional film thickness is independent of the value of \( M \) (the load group). Figure 2(a) shows that this behaviour is approached at high \( L \) values. Note that equation (25b), designated as Greenwood’s implied formula, closely resembles the approximation put forward by Blok [14, 50], and by Moes [10], see equation (27b).

For the higher speeds range, e.g. \( L > 4 \) when \( \alpha \sigma_{\text{hi}} \approx 10 \), the accuracy is better than 10 per cent compared with the measurements, but the error increases dramatically with decreasing \( L \) values, e.g. 65 per cent when \( L \approx 1.5 \) at \( \alpha \sigma_{\text{hi}} \approx 10 \).

**A4.7 The Moes Red Book [10] formulas**

The approximations in Moes [10] are a product of constant evolution. The first were for EHL line contacts, see among others Moes [40], later followed by circular [9, 22, 32] and elliptical contacts [33] culminating in a curve fit ‘which should fit all’, in Moes [10]. Copies of this book are rare, and it is no longer for sale. As it is essential for this article, the formulas are reproduced here. These Moes approximations are the only ones that smoothly move from one film thickness regime into another, when lubrication conditions change, making a tedious mapping procedure as in Hamrock et al. [12] superfluous. The predictions should be correct in extreme cases, as they contain the four Johnson regimes asymptotes. An additional advantage is that they should fit a wide range of short and long elliptical contacts.

Moes [10] adapted the earlier Nijenbanning et al. [33] formulas for short elliptical and circular contacts, to long contacts, to allow for elliptical contacts having their major axis along the entrainment direction. He states that a large correction term of \( 0.1\omega^{-4} \) is needed to correct for strong side leakage effects, if \( \omega < 1 \), in order to realize a smooth transition between the IR and VR regime. This is also addressed by Hamrock et al. [12], who note that in the IR and VR regimes side leakage has a strong effect on the minimum film thickness.

For fitting purposes Moes defines a non-dimensional load group \( \hat{N} \) instead of \( M \) (see also equation (7))

\[ \hat{N} = \omega^{-1/2}M = \left( \frac{R_e}{R_i} \right)^{1/2} \left( \frac{F}{E_i R_i^2} \right) \left( \frac{E_i R_e}{\eta_0 U_S} \right)^{3/4} \]  

(26)

and for circular contacts (\( \omega = 1 \)) \( N \) and \( M \) are identical. The central film thickness formula is built on the asymptotes in the four Johnson regimes.

**A4.7.1 The isoviscous rigid (IR) regime asymptote**

\[ \hat{H}_{\text{IR,c}} \approx 145(1 + 0.796\omega^{-14/15})^{-15/7}N^{-2} \]  

(27a)

**A4.7.2 The piezoviscous rigid (VR) regime asymptote**

\[ \hat{H}_{\text{VR,c}} \approx 1.29(1 + 0.691\omega^{-2/3})L^{2/3} \]  

(27b)

**A4.7.3 The soft contact (IE) regime asymptote**

\[ \hat{H}_{\text{IE,c}} \approx 3.18(1 - 0.006 \ln \omega + 0.63\omega^{-4/7})^{-14/25}N^{-2/15} \]  

(27c)

**A4.7.4 The hard contact (VE) regime asymptote**

\[ \hat{H}_{\text{VE,c}} \approx 1.48(1 - 0.006 \ln \omega + 0.63\omega^{-4/7})^{-7/20}N^{-1/12} \times L^{3/4} \]  

(27d)

**A4.7.5 The Moes [10] formula for the central film thickness**

Based on these asymptotes, the Moes [10] formula for the central film thickness reads

\[ \hat{H}_c = \left\{ \left[ \hat{H}_{\text{IR}}^{3/2} + \left( \hat{H}_{\text{IE}} + 0.1\omega^{-4} \right)^{3/4} \right]^{21/3} \right\}^{1/3} \]  

(28)

where

\[ \hat{\omega} = \frac{3}{2} \left\{ 1 + \exp \left( -1.2 \frac{\hat{H}_{\text{IE}}}{\hat{H}_{\text{IR}}} \right) \right\} \]

Moes [10] claims that the inaccuracy of his central film thickness approximation is within 10 per cent for circular contacts within a regime \( 5 \leq N \leq 1000, 0 \leq L \leq 25 \), and for contacts having \( 0.4 \leq \omega \leq 5 \) again within 10 per cent without mentioning the \( (L, M) \) range. Multigrid calculations by Venner [22] were used as a reference.

\[ \text{for a circular contact in the VR regime equation (28) yields } \hat{H}_{\text{VR,c}} \approx 0.901 L^{2.25}. \text{ Note the close resemblance to equation (25b).} \]

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