Internal-Model-Based Design of Repetitive and Iterative Learning Controllers for Linear Multivariable Systems

DICK DE ROOVER†
SC Solutions Inc.
3211 Scott Boulevard
Santa Clara, CA 95054
USA
roover@scsolutions.com

OKKO H. BOSGRA
Mechanical Engineering Systems and Control Group
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
o.h.bosgra@wbmt.tudelft.nl

MAARTEN STEINBUCH
Faculty of Mechanical Engineering, Systems and Control Group
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven The Netherlands
m.steinbuch@wfw.wtb.tue.nl

Abstract

Repetitive and iterative learning control are two modern control strategies for tracking systems in which the signals are periodic in nature. This paper discusses repetitive and iterative learning control from an internal model principle point of view. This allows the formulation of existence conditions for multivariable implementations of repetitive and learning control. It is shown that repetitive control can be realized by an implementation of a robust servomechanism controller that uses the appropriate internal model for periodic disturbances. The design of such controllers is discussed. Next it is shown that iterative learning control can be implemented in the format of a disturbance observer/compensator. It is shown that the resulting control structure is dual to the repetitive controller, and that both constitute an implementation of the internal model principle. Consequently, the analysis and design of repetitive and iterative learning control can be generalized to the powerful analysis and design procedure of the internal model framework, allowing to trade-off the convergence speed for periodic-disturbance cancellation versus other control objectives, such as stochastic disturbance suppression.

†Author to whom correspondence should be addressed. Tel: 408.486.6060 Ext 34. Fax: 408.486.6083.
1 Introduction

In practice, many tracking systems have to deal with periodic reference and/or disturbance signals, for example computer disk drives, rotating machine tools, or robots that have to perform their tasks repeatedly. It is well known that any periodic signal can be generated by an autonomous system consisting of a time-delay element inside a positive feedback loop. Therefore, in view of the internal model principle, [5], it might be expected that accommodation of these periodic signals can be achieved by duplicating this model inside a feedback loop. In the literature, two types of compensators can be found which accomplish this: the repetitive controller, see for example [10, 8, 20, 19], and the iterative learning controller, see for example [3, 4, 12].

Although it has been recognized that both schemes differ in the way periodic compensation is performed, [8, 9], still the impression exists that both schemes are equivalent. However, in a recent paper it was shown that the schemes are not equivalent but are related by duality, which is a consequence of the difference in location of the internal model inside the compensator, [17]. It was shown that a repetitive controller has the structure of a servo compensator—with the internal model located at the system output—while a learning controller has the structure of a disturbance observer, with the internal model located at the system input.

In this paper we use the general framework given in [17] to set up a general framework for the synthesis of (MIMO) repetitive and learning controllers. It is shown that a number of existing repetitive and learning control schemes can be put into this framework according to specific modifications in the internal model.

The remainder of this paper is organized as follows. In the next section we describe existing repetitive and learning control approaches. In Section 3 we give a discussion of the properties of existing approaches and we define the robust periodic control problem. In Section 4 we show how the repetitive and learning control problem can be formulated and solved in an internal-model-based framework, which allows the joint formulation of periodic-disturbance rejection and other — equally important — control objectives. This is illustrated with an example of a MIMO aircraft model in Section 5.

Throughout this paper, \( \mathbb{R} \) denotes the field of real numbers. Let \( n_u \) denote the dimension of the vector \( u \), then \( \mathbb{R}^n \) denotes the set of all \( n_u \)-vectors with elements in \( \mathbb{R} \). Likewise, \( \mathbb{R}^{n_u \times n_y} \) denotes the set of all \( n_u \times n_y \) matrices with elements in \( \mathbb{R} \), and \( I_n \) denotes the \( n_u \times n_u \) identity matrix. Furthermore, \( z \) denotes the discrete-time delay operator, and \( \mathbb{R}(z) \) denotes the set of all rational functions with real coefficients in \( z \). Let \( M \in \mathbb{R}^{n \times m} \) then \( \rho(M) \) denotes the rank of \( M \), and \( \rho(M) \leq \min\{n, m\} \). Discrete time is indicated by \( t_k, k = \ldots, -1, 0, 1, \ldots \). Signals as a function of discrete time are indicated either as \( x(t_k) \) or as \( x_k \) for shorthand notation \( x_f(t_k) \) and \( x_r, k \) for signals with subscripts, respectively. A recursion or iteration is indicated as \( x^i, i = 1, 2, \ldots \).
2 Existing Repetitive and Learning Control Algorithms

Let the system to be controlled be given by a discrete-time time-invariant state-space realization \( \{A, B, C, D\} \) having transfer function matrix

\[
P(z) = C(zI - A)^{-1}B + D.
\]

In iterative learning control, the response of the system over a finite horizon of \( N \) samples is considered for a fixed initial state:

\[
\begin{align*}
x(t_{k+1}) &= Ax(t_k) + Bu(t_k), \quad x(0) = x_0 \\
y(t_k) &= Cx(t_k) + Du(t_k).
\end{align*}
\]

Without loss of generality, \( x_0 \) is taken as zero. The finite-interval response of (1) is:

\[
\begin{bmatrix}
y(0) \\
y(1) \\
\vdots \\
y(N-1)
\end{bmatrix}
\begin{bmatrix}
D & 0 & 0 & \cdots & 0 \\
CB & D & 0 & \cdots & 0 \\
CAB & CB & D & \cdots & 0 \\
CA^{N-2}B & CA^{N-3}B & CA^{N-4}B & \cdots & D
\end{bmatrix}
\begin{bmatrix}
u(0) \\
u(1) \\
\vdots \\
u(N-1)
\end{bmatrix},
\]

or:

\[
y = Gu. \tag{3}
\]

As repeated trials are considered in repetitive control, we write \( y^i = Gu^i \) for the \( i^{th} \) trial (iteration), \( i = 0, 1, 2, \ldots \). Note that each trial assumes the same initial condition for the system to be controlled. Let

\[
r = \begin{bmatrix}
r(0) \\
r(1) \\
\vdots \\
r(N-1)
\end{bmatrix},
\]

be the reference vector over the time horizon of one trial, and let similarly \( e^i = r - y^i \) be the error vector. Then a prototype update law that implements iterative learning control by updating the past inputs on the basis of the past error is:

\[
u^{i+1} = u^i + Le^i, \tag{4}
\]

where \( L \) is a matrix of appropriate dimensions. Using (3) this can be rewritten to:

\[
u^{i+1} = u^i + L(r - Gu^i) = (I_N - LG)u^i + Lr,
\]

and the recursion for the error is:

\[
e^{i+1} = (I_N - GL)e^i.
\]

If the recursion algorithm converges for \( i \to \infty \), then

\[
LGu^\infty = Lr,
\]
which implies $L e_r = 0$. Several approaches exist in the literature to design $L$ such that guaranteed and controlled convergence conditions exist, see e.g., [1].

In repetitive control one assumes that the reference signal $r(t_k)$ is periodic with period $N$, i.e., $r(t_{k+N}) = r(t_k)$, $k = 0, 1, \ldots$. A prototype algorithm for repetitive control is to select the input $u(t_{k+N})$ as:

$$ u(t_{k+N}) = u(t_k) + R(z)e(t_k), \quad k = \ldots, 0, 1, 2, \ldots, $$(5)

where $R(z)$ is a rational matrix in the shift operator $z$. This implies:

$$ y(t_{k+N}) = y(t_k) + P(z)R(z)e(t_k), $$

or

$$ y(t_k) = P(z)R(z)\Phi(z)e(t_k), $$

where $\Phi(z) := (z^N I_y - I_y)^{-1}$. This relation is represented in Figure 1. Considering $r(t_k)$ as external input:

$$ y(t_{k+N}) = [I_y - P(z)R(z)]y(t_k) + P(z)R(z)r(t_k). $$

One difference between iterative learning control and repetitive control is the fact that iterative learning control assumes a fixed initial condition for the system at the beginning of each period, whereas repetitive control assumes the system to have an initial condition at the beginning of a period that is the result of the previous periods. Thus iterative learning considers motions such as a repeated pick and place operation of a robot, whereas repetitive control considers periodicities such as those occurring in rotating equipment having constant speed of rotation.

In many applications one neglects the effects of the fixed initial condition in iterative learning. Then the update law (4) assumes the form:

$$ u(t_{k+1}) = u(t_k) + L(z)e(t_k), $$(6)

where $L(z)$ is a rational matrix in the shift operator $z$. Then:

$$ u(t_k) = \Phi(z)L(z)e(t_k), $$
and this relation is shown in Figure 2. Note that in Figure 1 and Figure 2 the block $\Phi(z)$ has the internal representation as given in Figure 3, where $z^{-N}I$ represents a delay of one period of $N$ samples. If $\Phi(z)$ is regarded as a causal dynamic system, it will have $N$ state variables in each channel. Note that in repetitive control (Figure 1) there are $n_y$ channels as $\Phi(z)$ operates in the output space, whereas in learning control (Figure 2) $\Phi(z)$ has $n_u$ channels as it operates in the input space. The characteristic polynomial of $\Phi(z)$ is $z^N - 1$, $n_y$ or $n_u$ times repeated. This characteristic polynomial has all its roots evenly distributed on the unit circle. Asymptotic stability of the loop in Figure 1 or Figure 2 implies that all these roots have to be moved inside the unit circle by the action of the feedback loop.

The stability analysis suggested in the literature proceeds with isolating the delay chain of the internal model (Figure 3) in an equivalent system representation, see e.g., [8]. For the repetitive controller of Figure 1 this is shown in Figure 4. Since the delay chain has magnitude equal to one the small gain theorem can be used, which states that the following condition is sufficient for the equivalent system to be stable:

$$\| I_y - P(z) R(z) \|_i < 1,$$

for some induced $i$-norm. Equation (7) motivates to choose the repetitive control feedback as $R(z) = P^{-1}(z)$, i.e., equal to the (right) inverse of the system $P(z)$. Consequently, the dimension of $R$ is now determined by the system $P$, and not by the number $N$ of the period length anymore.

In the literature on iterative learning control, schemes like (6) are called past error feedforward, see for example [1, 14, 12]. An alternative is to use current-error feedback,
Figure 4: Equivalent system representation of repetitive control system.

see e.g., [13, 6, 7], where:

\[ u(t_{k+N}) = u(t_k) + L(z)e(t_{k+N}). \]

Now

\[ u(t_k) = (I_u - z^{-N}I_u)^{-1}L(z)e(t_k), \] (8)

where the transfer function \( \Phi'(z) = (I_u - z^{-N}I_u)^{-1} \) has internal representation as shown in Figure 5.

Figure 5: Internal representation of \( \Phi'(z) \).

In this case a sufficient condition for stability of the system (8) is:

\[ \| (I_u - L(z)P(z))^{-1} \|_i < 1, \] (9)

which leads to high-gain solutions for \( L(z) \), whereas a sufficient condition for stability of the system (6) is:

\[ \| I_u - L(z)P(z) \|_i < 1. \] (10)

This shows the advantage of past-error feedforward schemes over current-error feedback schemes if one is only interested in stabilization of the loop, i.e., in convergence of the recursion: In practical situations neither Equation (9) nor Equation (10) can be realized, and frequency weighted norms have to be introduced. The frequency up to which (10) holds in practical situations, is in general two to three times larger than the frequency up to which (9) is valid, see for example [16, 15].
3 Discussion and Problem Definition

3.1 Discussion of Properties of Repetitive and Learning Controllers

The existing repetitive and learning control schemes have several limitations that will be discussed.

- Repetitive and learning control are assumed to contribute to the performance of a control system by identifying the most useful feedforward input signal that suppresses the periodic disturbances. In this paper the recursion of the repetitive or learning part of the controller will be represented as feedback. Convergence of this recursion then translates into stability of the complete feedback system. In this case the feedback system can be thought of to contain a compensator \( C(z) \) used for disturbance suppression, and in addition a repetitive or learning part used for suppressing the periodic disturbances. In general, the repetitive or learning part will influence the sensitivity function of the complete loop and thus will modify the disturbance suppressing properties of the controller \( C(z) \). In this respect, fast convergence of the recursion is not necessarily the best solution. Fast convergence of a repetitive controller might imply that stochastic disturbances occurring in the past are effectively contributing to the formation of the periodic disturbance waveform. In that case the feedforward signal is corrupted by stochastic disturbances and affects the sensitivity function of the loop negatively. Rather, the speed of convergence should be part of an overall feedback control design where a single goal of disturbance suppression is pursued. The internal model approach considered in the sequel provides a suitable joint design of controller \( C(z) \) and the repetitive or learning controller part.

- In many cases a controller is already available for the plant \( P(z) \), and a learning controller is added to the existing scheme, see Figure 6. Then a sufficient small-gain-based condition for stability is:

\[
\| I_u - L(z)(I_y + P(z)C(z))^{-1}P(z) \|_i < 1,
\]

i.e., \( P(z) \) is left multiplied by the output sensitivity function, which itself is stable if \( C(z) \) stabilizes \( P(z) \). This shows that in principle it is still possible to design \( L(z) \)
given $C(z)$. However, $L(z)$ has to be chosen such that it operates within the stability margin of the loop closed by $C(z)$. It must be concluded that better results could be expected if $L(z)$ and $C(z)$ are designed simultaneously on the basis of a common design goal (i.e., common control performance).

- The discussion of using past-error feedforward or current-error feedback should be discussed also from the point of view that one should utilize the form of the error feedback that eliminates the periodic disturbances and simultaneously has the best performance in terms of an overall control performance criterion.

In the following section it will be shown that an internal-model-based framework can be formulated that allows the joint design of $C(z)$ and a learning or repetitive controller having favourable properties with respect to a single performance criterion. First, the robust periodic control problem will be formulated.

### 3.2 The Robust Periodic Control Problem

Any periodic signal can be generated by an autonomous system consisting of a time-delay inside a positive feedback loop with appropriate initial conditions, see Figure 5. For example, a discrete time periodic signal of length $N$ can be generated by:

\[
\begin{align*}
x_w(t_{k+1}) &= A_w x_w(t_k), \quad x_w(t_0) = x_{w0} \\
w(t_k) &= C_w x_w(t_k),
\end{align*}
\]

with

\[
A_w = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix} \in \mathbb{R}^{N \times N}
\]

\[
C_w = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]

Note that the spectrum of $A_w$ consists of $N$ roots equally spaced on the unit disk, i.e., these are the roots of $\det(zI - A_w) = z^N - 1$. Next, consider a discrete time linear time-invariant (LTI) plant $P(z) \in \mathbb{R}(z)^{y \times u}$ with input signal $u = u_p + w_u \in \mathbb{R}^u$, and output signal $y = Pu \in \mathbb{R}^y$, according to Figure 7. Given a desired periodic output signal $r(t_k) = r(t_k+N), t_k = 0, \Delta T, 2\Delta T, \ldots$, with $\Delta T$ denoting the sampling time. Let $e = r - y$ be the tracking error. Then we define the robust periodic control problem as:

**Definition 3.1** The robust periodic control problem is to find a feedback compensator $C(z)$ for the system $P(z)$ such that:

1. The resulting compensated system is exponentially stable.

2. The tracking error $e$ tends to zero asymptotically, for all periodic references $r$ and periodic disturbances $w_u$ satisfying (11)\(^1\).

\(^1\)Note that there is no fundamental difference between an error resulting from a disturbance at the output or from a reference input $r$. We will only consider $r$. 
Figure 7: General periodic control problem; $u_p$ and $w_u$ denote a periodic control signal and periodic input disturbance, respectively.

3. Properties 1. and 2. are robust, i.e., they also hold in case the dynamics of $P$ are perturbed.

The solution to the robust periodic control problem is provided by the internal model principle ([5]) which states:

**Internal Model Principle:** Suppose that the controller $C(z)$ in Figure 7 contains in each channel a realization of the disturbance generating system, driven by the error $e(z)$. Further, let the controller $C(z)$ be such that the feedback connection of $C(z)$ and $P(z)$ is internally stable. Then $C(z)$ solves the robust periodic control problem.

As repetitive and learning control attempt to solve the (robust) periodic control problem, it follows that the internal model principle provides a solution for repetitive and learning control. Moreover, the internal model principle can be formulated in the format of a servocompensator where the disturbance model is realized in each channel of the output space, or in a dual format where the disturbance model is realized in each channel of the input space. The first format corresponds to the manner in which repetitive control is implemented. The second format utilizes the structure of a disturbance observer and corresponds with iterative learning control. Both structures will be discussed in the next section.

4 Synthesis in an Internal-Model-Based Framework

This section extends the approach given in [17]. Introducing disturbance dynamics in the controller and assuming the feedback system to behave asymptotically allows for state feedback of the disturbance dynamics as those states are available for feedback. This would lead to current-error feedback instead of past-error feedforward. We act according to the following points of view:

- We assume that the feedback controller must have a loop feedback control performance in addition to the requirements of asymptotic rejection of periodic disturbances. In this case there is an advantage in applying current-error feedback instead of past-error feedforward as the latter introduces a delay of one period in each channel of $C(z)$. This delay only obstructs the stabilization of the disturbance dynamics at the cost of control performance.
As the state of the disturbance dynamics in $C(z)$ is available for feedback, it is worthwhile to apply state feedback of this state for the purpose of stabilization and control performance enhancement.

In addition, control performance will be realized by using an estimated-state-feedback compensator added to the servo part containing the disturbance model. The state variables of the disturbance model act as memory variables for periodic errors.

Instead of concentrating on the stabilization of the memory variables, as is automatically done in classical approaches to repetitive and learning control, we here apply feedback to the memory variables with feedback loop control performance as underlying goal.

### 4.1 Repetitive Control

Define:

\[
A_w := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{N \times N}, \quad B_w := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{N \times 1}
\]

\[
C_w := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1 \times N}.
\]

and

\[
A_r := \begin{bmatrix} A_w & 0 \\ \vdots & \ddots & \vdots \\ 0 & A_w \\ \vdots \end{bmatrix}, \quad B_r := \begin{bmatrix} B_w & 0 \\ \vdots & \ddots & \vdots \\ 0 & B_w \\ \vdots \end{bmatrix}, \quad C_r := \begin{bmatrix} C_w & 0 \\ \vdots & \ddots & \vdots \\ 0 & C_w \end{bmatrix},
\]

where each diagonal block is repeated $n_y$ times. In a similar way, define \{A_l, B_l, C_l\} where each diagonal block is repeated $n_u$ times. Then

\[
C_r(\mathbb{I}_{N_y} - A_r)^{-1}B_r = z^{-N} \mathbb{I}_y(\mathbb{I}_y - z^{-N} \mathbb{I}_y)^{-1},
\]

\[
C_l(\mathbb{I}_{N_u} - A_l)^{-1}B_l = z^{-N} \mathbb{I}_u(\mathbb{I}_u - z^{-N} \mathbb{I}_u)^{-1}.
\]

Also, define $K_r \in \mathbb{R}^{n_y \times N_y}, L_l \in \mathbb{R}^{N_u \times n_y}$. An implementation for repetitive control with current-error feedback, based upon state feedback of memory variables and estimated-state feedback of the plant $P(z) = C(zI - A)^{-1}B + D$, is:

\[
\begin{align*}
\text{System:} & \quad x_{k+1} = Ax_k + Bu_k \\
& \quad y_k = Cx_k + Du_k \\
\text{Observer:} & \quad \hat{x}_{k+1} = A\hat{x}_k + Bu_k + L\epsilon_k \\
\text{Feedback error:} & \quad \epsilon_k = r_k - y_k \\
\text{Observer error:} & \quad \delta_k = \epsilon_k + C\hat{x}_k + Du_k \\
\text{Disturbance memory:} & \quad x_{r,k+1} = A_r x_{r,k} + B_r \delta_k \\
\text{Control input:} & \quad u_k = K_r x_{r,k} + K \hat{x}_k
\end{align*}
\]
Figure 8: Repetitive controller based on internal model principle.

Here $L$ is the plant state observer gain, and $K_r$ and $K$ are the state feedback laws for disturbance memory state and plant state, respectively. The repetitive controller can be represented in the block diagram of Figure 8 where $\bar{A} := A + BK + L(C + DK)$. The structure contains the estimated-state feedback controller $C(z)$:

$$
\begin{align*}
\hat{x}_{k+1} &= \bar{A}\hat{x}_k + L e_k \\
u_k &= K\hat{x}_k
\end{align*}
$$

acting between $e_k$ and $u_k$, plus in addition the disturbance accommodating part having transfer function $K_r(zI_{Ny} - A_r)^{-1}B_r$, operating on $e_k$ and feeding an additional input in the controller $C(z)$. We observe that the memory variables are linked to the output space $e_k$. By bringing the memory variables in a structure displaying the delay $N$, and feeding into the $u_k$ signal, we can completely separate the memory variables from the feedback controller $C(z)$:

$$
K_r(zI_{Ny} - A_r)^{-1}B_r =: R'(z)(I_y - z^{-N}I_y)^{-1},
$$

where $R'(z) = R_1'z^{-1} + R_2'z^{-2} + \cdots + R'_N z^{-N}$, is one-to-one related to $K_r$ in the sense that the column $(Ni + j + 1)$ of $K_r$ equals column $(jn_y + i + 1)$ of

$$
[R_1' | R_2' | \cdots | R_N'], \quad i = 0, 1, \ldots, n_y - 1; \quad j = 0, 1, \ldots, N - 1,
$$

which directly follows from $R_{j+1}' = K_rA_r^jB_r$, $j = 0, 1, \ldots, N - 1$. The structure of the repetitive controller, written with separate memory block, and completely equivalent to Figure 8, is shown in Figure 9. The block $\tilde{G}_R(z)$ follows from the path between the output of the block $K_r(zI_{Ny} - A_r)^{-1}B_r$ and signal $u_k$ in Figure 8, and equals:

$$
\tilde{G}_R(z) = I_u + K(zI - \bar{A})^{-1}(B + LD).
$$

Figure 9 shows that internal-model-based repetitive control can be implemented in a form resembling the addition of a memory-variable block to a feedback controller $C(z)$, where
Figure 9: Internal-model-based repetitive control as add-on to $C(z)$.

$C(z)$ is the estimated-state-feedback controller defined by (13).

The classical stability analysis of repetitive control considers the feedback path around $z^{-N}I_{Ny}$. This path is defined between the signals in and out in Figure 10. From this figure it follows:

$$
\begin{align*}
\text{out}(z) &= [I + (I + P(z)C(z))^{-1}P(z)\tilde{G}_R(z)R'(z)]^{-1}\text{in}(z), \\
&= [I + \tilde{G}_K(z)R'(z)]^{-1}\text{in}(z),
\end{align*}
$$

(14)

where $\tilde{G}_K$ follows from substituting the expressions and simplifying:

$$
\tilde{G}_K(z) = [D + (C + DK)(zI - A - BK)^{-1}B].
$$

(15)

Thus the memory variables experience a feedback structure as shown in Figure 11. The feedback condition of Figure 11 can be described by the equations:

$$
\begin{align*}
x_{r,k+1} &= A_r x_{r,k} + B_r e_k \\
y_{r,k} &= K_r x_{r,k} \\
z_{k+1} &= (A + BK)z_k + By_{r,k} \\
e_k &= -(C + DK)z_k - Dy_{r,k},
\end{align*}
$$

or equivalently:

$$
\begin{bmatrix}
x_{r,k+1} \\
z_{k+1}
\end{bmatrix} =
\begin{bmatrix}
A_r & -B_r C \\
0 & A
\end{bmatrix}
\begin{bmatrix}
x_{r,k} \\
z_k
\end{bmatrix} +
\begin{bmatrix}
-B_r D \\
B
\end{bmatrix}
\begin{bmatrix}
K_r \\
K
\end{bmatrix}
\begin{bmatrix}
x_{r,k} \\
z_k
\end{bmatrix}.
$$

(16)

This shows the underlying state feedback mechanism. The feedback laws $K_r$ and $K$ are to be designed for a common control goal that assures the control performance of the feedback loop shown in Figure 11.
Figure 10: Feedback path $\text{in} \rightarrow \text{out}$ around memory delay $z^{-N}$ in IM-based repetitive control.

Figure 11: Equivalent feedback structure around memory variables in repetitive control.
Figure 12: State representation of plant \((A, B, C, D)\) with periodic input disturbance.

The implementation of the repetitive controller needs the design of an observer gain \(L\) in addition to the feedback gains \(K_r\) and \(K_c\). If \(L, K_r\) and \(K_c\) have been designed, the implementation of the control structure in Figure 8 or Figure 9 is uniquely determined.

### 4.2 Learning Control

In learning control the memory variables are linked to the input space. In [17] it has been shown that learning controllers can be formulated in an internal-model framework by dualizing the results of the repetitive controller. The underlying structure then is generated by a disturbance observer, which has been shown in [17] to be the exact dual of a servo compensator. Now we assume for the time being a periodic disturbance at the plant input, generated by a system \(\{A_l, C_l\}\) with non-zero initial condition, as shown in Figure 12. An observer for this system now results by duplicating the system and applying feedback \(L_r, L\) to the estimated states from an observer error. The estimate of the disturbance, \(\hat{d}_k\), is used to compensate the disturbance \(d_k\). In addition, estimated plant state feedback is applied through the feedback gain \(K\). Finally, the assumed error \(d\) at the plant input is replaced by an actual error resulting from the reference input \(r\). Thus the disturbance estimator compensates for an input disturbance equivalent to the control error in output space. The structure is shown in Figure 13. Observe that the memory variables in the observer directly are linked to the input space. We now have:

\[
\begin{align*}
\text{System:} & \quad \quad x_{k+1} &= Ax_k + Bu_k, \\
& \quad \quad y_k = Cx_k + Du_k, \\
\text{Observer:} & \quad \quad \hat{x}_{k+1} = A\hat{x}_k + B(u_k + \hat{d}_k) + L\epsilon_k, \\
\text{Observer error:} & \quad \quad \epsilon_k = e_k + C\hat{x}_k + D(u_k + \hat{d}_k), \\
\text{Feedback error:} & \quad \quad e_k = r_k - y_k, \\
\text{Disturbance memory:} & \quad \quad x_{l,k+1} = A_lx_{l,k} + L_l\epsilon_k, \\
& \quad \quad \hat{d}_k = C_lx_{l,k}, \\
\text{Control input:} & \quad \quad u_k = K\hat{x}_k - C_lx_{l,k}.
\end{align*}
\]
By combining several of these equations, the feedback structure underlying the scheme in Figure 13 can be made transparent. The equivalent structure is shown in Figure 14. Inspection of Figure 14 shows that the structure is dual to the repetitive structure of Figure 8. The delay structure becomes apparent by defining:

$$C_l(zI_{Nu} - A_l)^{-1}L_l =: (I_u - z^{-N}I_u)^{-1}L'(z),$$

where $$L'(z) = L'_1z^{-1} + L'_2z^{-2} + \cdots + L'_Nz^{-N},$$ defined by $$L'_{j+1} := C_lA_l^jL_l, j = 0, 1, \ldots, N-1.$$ Alternatively, $$L_l$$ and $$L'(z)$$ are related one-to-one by the fact that each row $$(Ni + j + 1)$$ of $$L_r$$ equals the row $$(jn_u + i + 1)$$ of $$L_l$$, i.e.,

$$[L'_1^T L'_2^T \cdots L'_N^T]^T, \quad i = 0, 1, \ldots, n_u - 1; j = 0, 1, \ldots, N - 1.$$

Then the structure of Figure 14 is equivalent to the structure of Figure 15. This figure directly shows duality with the repetitive controller of Figure 9. Note that the controller $$C(z)$$ equals $$C(z) = K(zI - \tilde{A})^{-1}L$$, i.e., $$C(z)$$ is the result of the design of the state feedback law $$K$$ and the state observer gain $$L$$, and equals the expression as shown previously in the repetitive controller structure of Figure 9.

An analysis of the feedback around the memory variables dual to the results in Figure 11, is shown in Figure 16, where, after some algebraic manipulation, $$\bar{G}_f(z)$$ can be computed...
Figure 14: Disturbance-observer-based learning control showing (dual) internal model principle.

Figure 15: Internal-model-based learning control as add-on to $C(z)$.
as:
\[
\tilde{G}_I(z) = [D + C(zI - A - LC)^{-1}(B + LD)].
\]

Note the duality with the expression (15). The feedback structure in Figure 16 can be realized by the system:

\[
\begin{align*}
    x_{l,k+1} &= Ax_{l,k} + L_l\epsilon_k \\
    \hat{d}_k &= C_l x_{l,k} \\
    z_{k+1} &= (A + LC)z_k + (B + LD)\hat{d}_k \\
    \epsilon_k &= Cz_k + D\hat{d}_k,
\end{align*}
\]

or equivalently:

\[
\begin{bmatrix}
    x_{l,k+1} \\
    z_{k+1}
\end{bmatrix} =
\begin{bmatrix}
    A_l & 0 \\
    BC_l & A
\end{bmatrix}
\begin{bmatrix}
    x_{l,k} \\
    z_k
\end{bmatrix} +
\begin{bmatrix}
    L_l \\
    L
\end{bmatrix}
\begin{bmatrix}
    D C_l & C
\end{bmatrix}
\begin{bmatrix}
    x_{l,k} \\
    z_k
\end{bmatrix}.
\]

Thus \(L_l\) and \(L\) must be designed such that the closed loop system (18) has favourable properties (damping, loop gain, sensitivity). Note that (18) is dual to (16).

**4.3 Design of Internal-Model-Based Repetitive and Learning Control**

The design of the repetitive controller (12) and of the learning controller (17) involves the specification of the gain matrices \(\{L, K_r, K\}\) and \(\{K, L_l, L\}\), respectively. These controllers are required to solve the robust periodic control problem. The following theorem defines necessary and sufficient conditions for the internal-model-based repetitive and learning controllers to solve the robust periodic control problem.

**Theorem 4.1** Consider the repetitive controller (12) and assume that \(L\) is chosen such that \(A + LC\) is asymptotically stable, and assume that \(\{K_r, K\}\) is chosen such that the system (16) is asymptotically stable. Then the controller (12) solves the robust periodic control
problem as defined in Definition 3.1, if and only if
\[
\rho \left[ \begin{bmatrix} \lambda I - A & B \\ -C & D \end{bmatrix} \right] = n_x + n_y, \quad \forall \lambda \in \sigma(Aw),
\]  
(19)

**Proof:** This result can be directly derived from the results available for the general servo compensator by noting that a repetitive controller is a servo compensator for periodic signals, which is a special class of persistent signals. A full proof for the general case can be found in [18], see also [17].

Due to the established duality, a dual version holds for the learning controller:

**Theorem 4.2** Consider the learning controller (17) and assume that \( K \) is chosen such that \( A + BK \) is asymptotically stable, and assume that \( \{ L_k, L \} \) is chosen such that the system (18) is asymptotically stable. Then the controller (17) solves the robust periodic control problem as defined in Definition 3.1, if and only if
\[
\rho \left[ \begin{bmatrix} \lambda I - A & B \\ -C & D \end{bmatrix} \right] = n_x + n_y, \quad \forall \lambda \in \sigma(Aw),
\]  
and
\[
\rho(B) + n_y = \rho \left[ \begin{bmatrix} B \\ D \end{bmatrix} \right].
\]  
(20)

**Proof:** Follows by similar — yet dual — reasoning from the proof of Theorem 4.1, noting that a learning controller is a special class of disturbance observers for persistent periodic signals.

Theorem 4.1 requires that \( P(z) \) does not have transmission zeros located at the spectrum of \( Aw \) and that \( P(z) \) has at least as many inputs as outputs. Theorem 4.2 requires that \( P(z) \) does not have transmission zeros located at the spectrum of \( Aw \) and that \( P(z) \) has at least as many outputs as inputs. These conditions follow by requiring the controllability of the series connection of plant \( P(z) \) and memory dynamics, and by requiring the observability of the series connection of memory dynamics and plant \( P(z) \), respectively. However, the second condition in Theorem 4.2 — which results from the requirement of asymptotic tracking of \( r(z) \) in the output space, as opposed to canceling an estimated disturbance in the input space — is true only if \( P(z) \) has at least as many inputs as outputs. Thus the learning controller (17) will show asymptotic tracking of \( r(z) \) only if \( P(z) \) is square and invertible.

A further question is whether it is feasible to design the repetitive controller as shown in Figure 9 in two separate steps: First, determine \( C(z) \) as an estimated-state feedback compensator by choosing \( K \) and \( L \), given \( \{ A, B, C, D \} \); second, determine \( K_r \) which specifies the key properties of the ”add-on” periodic disturbance accommodating part, which is the
Figure 17: Sensitivity function of separate design.

series connection of $\bar{G}_R(z)$ and $K_r(z I_{N_y} - A_r)^{-1} B_r$. Stability of the composite system resulting from the interconnection of both parts with the system can be investigated by a sufficient condition provided by the small-gain theorem. This requires that the transfer function between \textit{in} and \textit{out} in Figure 10 is smaller than unity in terms of an induced norm (see relation (14)): 
\[
\| (I + (I + P(z)C(z))^{-1} P(z) \bar{G}_R(z) R'(z) )^{-1} \|_i \leq 1,
\]
which is equivalent to:
\[
\| (I + \bar{G}_K(z) R'(z) )^{-1} \|_i \leq 1,
\]
where $\bar{G}_K(z)$ is given by (15). This condition implies that the sensitivity function of the loop in Figure 17 — the transfer function between \textit{in} and \textit{out} — has induced norm smaller than 1. Here, $\bar{G}_K(z)$, which is fixed by $K, L$ and $R'(z)$, is to be designed and will determine $K_r$. As it is impossible for linear systems to have a sensitivity function which has induced norm smaller than one — whereas the strict inequality is required as $z^{-N}$ attains the absolute value 1 for all harmonics of the basic frequency — a design of $K_r$ must be precisely tuned in conjunction with the properties of $K$ and $L$. This requires a joint, simultaneous, design of $K$ and $K_r$. The separation principle allows \{ $K_r, K$ \} and $L$ to be designed separately.

One possibility is to use an LQG approach. Consider the system (16), written as:
\[
\eta_{k+1} = \begin{bmatrix} A_r & -B_r C \\ 0 & A \end{bmatrix} \eta_k + \begin{bmatrix} -B_r D \\ B \end{bmatrix} \mu_k \quad \eta_0 \text{ given}
\mu_k = \begin{bmatrix} K_r & K \end{bmatrix} \eta_k.
\]
Determine $[K_r \; K]$ such that
\[
\sum_{k=0}^{\infty} \eta_k^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \eta_k + \mu_k^T R \mu_k
\]
is minimal. Then $[K_r \; K]$ follows from the appropriate discrete Riccati equation. By selecting $Q_1$ relatively small, we can express the desire that the memory variables should
contribute to the control performance as specified for \( \{A, B, C, D\} \) in terms of a choice of \( Q_2 \) and \( R \). In addition, \( L \) can be determined on the basis of pole assignment or as a Kalman gain, based on assumed noise statistics for system and measurement noise for \( \{A, B, C, D\} \).

The approach for learning control can follow a completely similar — yet dual — approach.

## 5 Example: MIMO Aircraft Model

Consider the aircraft model described in the Appendix of [11]. This has three inputs, three outputs and five states. The specification is to achieve a high bandwidth of approximately 10 rad/sec for each loop. In [11] this example is used to compare different MIMO feedback design techniques. For this example we use LQG with integral control as the MIMO feedback design method. By selecting the appropriate weighting functions, we can duplicate the high-bandwidth feedback design. Figure 18 shows the maximum and minimum singular values of the sensitivity and complementary sensitivity function. Figure 19 shows the corresponding step responses to unit step demands for each loop. The responses are fast (settling within 2 seconds), well damped (overshoot < 25%), and interaction is greatly reduced, compared to the open-loop plant.

Next, consider the following disturbance acting on input 1:

\[
w(t_k) = 10\sin(5t_k) + 5\sin(10t_k + \phi_1) + 2\sin(20t_k + \phi_2) + 2n(t_k), \quad t_k = 0, \Delta T, 2\Delta T, \ldots
\]
with $\Delta T = 5e^{-3}$-sec sampling time, $\phi_1, \phi_2$ phase shifts of 45 and 30 degrees, respectively, and $n(t_k)$ a Gaussian distributed random noise with mean value 0 and $3\sigma$ value 1. Basically, this disturbance constitutes a 5Hz periodic signal with two higher harmonics. Figure 20 shows the closed-loop step responses to step demands on all three outputs with the disturbance acting on input 1. Clearly, because of closed-loop interaction at 5, 10 and 20 Hz, significant oscillations occur in all three channels.

To accommodate this disturbance, either a repetitive or learning controller can be designed. As the plant is square (3 inputs, 3 outputs) and does not have transmission zeros at 5, 10 or 20 Hz, both conditions in Theorem 4.1 and Theorem 4.2 are satisfied, i.e., either a repetitive or learning controller exists that can accommodate this disturbance. As the disturbance is a continuous periodicity, as opposed to a repeated operation, we will design a repetitive controller.

For this, the plant with integral states is augmented with an internal model for 5 Hz periodic signals according to (11). With $\Delta T = 5e^{-3}$-sec, this results in a 40 state internal model in each loop. Note that this internal model is capable to compensate for all 20 harmonics of the 5 Hz base frequency; in case there are only a few harmonics to be suppressed a reduced internal model can be used consisting of a few oscillators. By solving the LQG problem for the augmented system, a 3-input, 3-output, 120-state repetitive controller is designed in conjunction with the original observer-state feedback controller. Figure 21 shows the maximum and minimum singular values of the sensitivity function and complementary sensitivity function of the closed-loop system with repetitive controller. This
Figure 20: Closed-loop step responses to step demand on output 1 (black curves), output 2 (dark grey curves) and output 3 (light grey curves) with periodic disturbance on input 1.

The figure shows how the repetitive controller introduces high gain at frequency 5 Hz, and all its higher harmonics. Figure 22 shows the closed-loop step responses to step demands on all outputs. Note that the oscillation is greatly reduced, especially in the interaction. It takes the repetitive controller a few seconds to accommodate this disturbance (or to learn it), but the disturbance will eventually vanish completely with time. The speed of accommodation (convergence) is directly related to the magnitude of the weighting function in the LQG control problem. The trade-off is the increase in overshoot (30 to 35%), which can be explained from comparison of Figure 18 with Figure 21: the gain reduction at 5 Hz and its higher harmonics causes a gain increase at other, surrounding, frequencies. The peak in the sensitivity function in Figure 21 is larger than in Figure 18, which causes a less damped step response.

This trade-off shows exactly the reason why it is so important to design the repetitive controller in conjunction with the observer-state feedback controller: damping, periodic disturbance rejection, speed of convergence, and other control objectives can be traded-off in one general powerful framework. For example, if the overshoot is not acceptable, the convergence speed of the repetitive controller can be decreased, or the integral gain at low frequencies can be reduced. Note that the resulting MIMO 120-state repetitive controller can still be implemented as an add-on device according to Figure 9.
6 Conclusions and Preview on Further Research

This paper gives a general framework for the analysis and design of repetitive and learning controllers explicitly derived from results available for the internal model principle. The internal model framework gives necessary and sufficient conditions for existence of a solution to the problem of robust asymptotic tracking and rejection of periodic signals. The existence conditions allow for a proper choice between a repetitive or learning controller, dependent on location of zeros and number of inputs and outputs of the plant. Once existence of a repetitive or learning controller has been verified, the design of such a controller boils down to the design of a stabilizing compensator for the series connection of the plant and an internal model of the periodic signal, using any model-based control design technique, such as LQG, $H_{\infty}$ and/or $\mu$-synthesis. Also, model-based predictive control (MBPC) ILC schemes, as used and exploited by e.g., [2], can be straightforwardly brought into this framework because of its model-based nature.

Consequently, the analysis and design of these model-based repetitive and ILC approaches can be generalized to the powerful analysis and design procedure of the internal model framework, allowing to trade-off the convergence speed for periodic-disturbance cancellation versus other control objectives, such as stochastic disturbance suppression by using appropriate weighting functions and design parameters. An example for a MIMO aircraft model showed the importance of these trade-offs.

Further research will focus on the actual use in control design, for which it will be necessary to address the computational complexity. This is because the internal model of the
Figure 22: Closed-loop step responses with repetitive controller to step demand on output 1 (black curves), output 2 (dark grey curves) and output 3 (light grey curves) with periodic disturbance on input 1.

A periodic signal can have large dimensions: the state dimension equals the number of samples \( N \) of one period. In real life applications this number can easily exceed 1000. For this reason it would be interesting to investigate the use of basis functions to arrive at reduced order internal models. Note that for the aircraft example, the order could have been reduced by only including an internal model for the oscillators at 5, 10 and 20 Hz, which boils down to including a 6-state compensator in each loop instead of a 40-state compensator. If only one or two harmonics are contributing to the output response, there is an advantage in not using the full repetitive controller: the gain increase in the unimportant harmonics can be used to balance the trade-off in favor of other performance requirements. From implementation point of view, the repetitive controller might still be favourable: its internal model can be implemented as a First-In, First-Out (FIFO) buffer or memory loop, as opposed to an internal model for oscillators. Further investigation in this area is necessary to balance all aspects in a systematic way.
References


