A Synchronization Condition for Coupled Nonlinear Systems with Time-delay
-A Circle Criterion Approach- *

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Abstract: This paper considers the synchronization problem for coupled chaotic systems with
time-delay. We assume that the error dynamics can be rewritten by a feedback connection
of a linear delay system with multiple inputs and outputs and nonlinear elements which are
decentralized and satisfy a sector condition. Then, we derive a synchronization condition for
coupled systems with delay by applying a multivariable circle criterion. Unlike the conventional
synchronization criteria, the derived criterion is based on a frequency-domain condition and
can avoid the use of the Lyapunov-Krasovskii approach. As a result, the condition does not
contain the conservativeness caused by the Lyapunov-Krasovskii approach. The effectiveness of
the proposed criterion is shown by examples of coupled Chua systems with delay coupling.

Keywords: chaos synchronization, time-delay, absolute stability, the circle criterion

1. INTRODUCTION

Synchronization in networks of chaotic systems has been widely investigated in applied physics, mathematical bi-
ology, social sciences and control science and interdisciplinary fields since the early works by Fujisaka and
Yamada (Fujisaka and Yamada (1983)) and by Pecora and Carroll (Pecora and Carroll (1990)), which directly
provoked considerable interest in the phenomenon from the practical perspective of secure communications.
The study on synchronization of coupled systems (Pikovsky et al. (2001); Wu (2002)) has extensively dealt with
coupled systems with delay-free coupling. In addition, from a viewpoint of control science and engineering, the synchro-
nization problem has been also investigated via control theory (Nijmeijer and Mareels (1997); Cruz and Nijmeijer
(2000); Nijmeijer and Rodriguez-Angeles (2003)). More recently the interest is spreading to synchronization phe-
nomena of chaotic systems with time-delay and the effect of time-delay in synchronization (Oguchi and Nijmeijer
(2005); Huijberts et al. (2007); Oguchi et al. (2008) and others). In practical situations, time-delays are caused by
signal transmission between coupled systems and affect the behavior of coupled systems. It is therefore important to
study the effect of time-delay in existing synchronization schemes. The effect of time-delay in synchronization of
coupled systems has been investigated both numerically and theoretically by a number of researchers (Li and Chen
(2004); Gao et al. (2006); Amano et al. (2006); Wu and Jiao (2008)); these works concentrate on synchronization
of systems with a coupling term typically described by
\[ u_i(t) = \sum_{j=1, j \neq i}^{N} K_{ij}(x_i(t) - x_j(t)) \]
and there are few results for the case in which the coupling term is described by
\[ u_i(t) = \sum_{j=1, j \neq i}^{N} K_{ij}(x_i(t) - x_j(t - \tau)) \]. In either case, the synchronization error dynamics can be described by
a difference-differential equation, and the synchronization problem can be reduced to the stabilization problem for
the origin of the error dynamics with suitable conditions on the coupling gain and the time-delay. However most
existing conditions for synchronization are based on the Lyapunov-Krasovskii functional approach and are given by
linear matrix inequalities (LMIs), and these criteria tend to have conservative results due to the -almost inherently-
conservativeness of the Lyapunov-Krasovskii approach.

In this paper, we propose a less conservative criterion for synchronization of coupled systems with time-delay.
Throughout, we assume that the error dynamics can be rewritten by a feedback connection of a linear delay system
with multiple inputs and outputs and nonlinear elements which are decentralized and satisfy the sector condition.
Then, we derive a synchronization condition for coupled systems with delay by applying the multivariable circle
criterion (Bliman (2000)) which is an extension of the result by Popov and Halanay (Popov and Halanay (1962)).
Unlike the conventional synchronization criteria, the derived criterion is based on a frequency-domain condition
and avoids the Lyapunov-Krasovskii approach. As a result, although the class of applicable systems may narrow, the
condition does not contain the conservativeness caused by the Lyapunov-Krasovskii approach. The effectiveness of

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the proposed criterion is shown by examples of coupled Chua systems with delay coupling.

2. PRELIMINARIES

In this section, we briefly introduce some notions related with the stability of nonlinear systems, which will be used later.

We consider the following nonlinear retarded system

\[
\Sigma : \begin{cases}
\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + B_0 u(t) \\
y(t) = C_0 x(t) \\
u(t) = -\psi(t, y)
\end{cases}
\]

(1)

where \( x \in \mathbb{R}^n, u, y \in \mathbb{R}^p, \tau \in \mathbb{R}^+, \) and \( A_0, A_1, B_0 \) and \( C_0 \) are constant matrices of corresponding dimensions. Then the transfer matrix from \( u \) to \( y \) is given by

\[
G(s) = C_0 (sI - A_0 - A_1 e^{-s\tau}) B_0.
\]

(2)

Concerning the nonlinear function \( \psi \), we assume that there exists a convex open neighborhood \( \mathcal{O} \) of the origin in \( \mathbb{R}^p \) for which the following assumptions hold.

- **N-1** the function \( \psi(t, y) \) is locally Lipschitz with respect to \( y \in \mathcal{O} \).
- **N-2** the nonlinearity \( \psi \) is decentralized, that is \( \psi_i(t, y) = \psi_i(t, y_i), \forall i \in \{1, \ldots, p\} \), and there exists a diagonal matrix \( \Lambda = \text{diag}\{\lambda_i\} \geq 0 \) such that

\[
\forall(t, y) \in \mathbb{R}^{+} \times \mathcal{O}, \quad \psi(t, y)^T (\psi(t, y) - \Lambda y) \leq 0.
\]

(3)

**Definition 1.** The system \( \Sigma \) is said to be absolutely stable with a finite domain if the origin is uniformly asymptotically stable for any nonlinearity in the given sector \( \mathcal{S} \). In addition, if \( \mathcal{O} = \mathbb{R}^n \), it is said to be absolutely stable.

A sufficient condition for absolute stability of the class of linear systems with time-invariant nonlinear feedback satisfying a sector condition was given by Popov (1961) and the condition was extended for retarded systems by Popov and Halanay (1962). These results were extended to multi-variable systems and generalized for time-varying nonlinear systems by a number of researchers. In particular, for time-varying nonlinear systems, a condition for absolute stability is the circle criterion.

**Theorem 2.** Suppose that system \( \Sigma \) with a nonlinear feedback satisfies the conditions **N-1** and **N-2**. Assume the following conditions are fulfilled.

1. All roots of the equation \( \det(sI - A_0 - A_1 e^{-s\tau}) = 0 \) have negative real part.
2. \( H(s) = I + \Delta G(s) \) is strictly positive real, i.e., for all real \( w \) for which \( jw \) is not a pole of any element of \( H(s) \), the matrix \( H(jw - \varepsilon) + H^*(jw - \varepsilon) \) is positive semidefinite for some \( \varepsilon > 0 \), where \( H^* \) denotes the conjugate transpose of \( H \).

then the origin of the system is uniformly locally asymptotically stable. Moreover if \( \mathcal{O} = \mathbb{R}^p \), then the origin of system is uniformly globally asymptotically stable.

The above theorem is a simple extension of the result for multi variable rational transfer function systems into time-delay systems and is included in Theorem 1 of Bliman (2000) as a special case.

**Remark 3.** If the system has a rational transfer function matrix, the positive realness which is the second condition can be equivalently replaced with the well-known linear matrix equations arising from the Kalman-Yakubovich-Popov lemma (KYP lemma) and the circle criterion mentioned above can be rewritten by linear matrix inequalities (LMIs).

3. SYNCHRONIZATION PROBLEM

We consider \( N \)-coupled chaotic systems with time-delay in the coupling.

\[
\Sigma_i : \begin{cases}
\dot{x}_i(t) = A x_i(t) + B_0 f(y_i(t)) + B_1 u_i(t) \\
y_i(t) = C_0 x_i(t) \\
u_i(t) = \sum_{j=1, j \neq i}^{N} K_{ij} (y_j(t) - y_i(t - \tau_{ij}))
\end{cases}
\]

(4)

where \( x_i \in \mathbb{R}^n, \) output \( y_i \in \mathbb{R}^p, \) input \( u_i \in \mathbb{R}^m \) for \( i = 1, \ldots, N \), nonlinear term \( f(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^q \), and \( A, B_0, B_1 \) and \( C_0 \) are real matrices of appropriate dimensions, and coupling gain \( K_{ij} \in \mathbb{R}^{m \times p} \). \( \tau_{ij} \) denotes a time-delay caused by transferring the signal from system \( j \) to system \( i \), and for simplicity of discussion, all delays are assumed to be identical, \( \tau_{ij} = \tau \).

Now we assume that the systems satisfy the following conditions:

**C-1** the coupling gains \( K_{ij} \) satisfy

\[
\sum_{j=2}^{N} K_{ij} = \sum_{j=1, j \neq 2}^{N} K_{ij} = \cdots = \sum_{j=2}^{N} K_{Nj} := K.
\]

**C-2** \( f(y_i) \) is locally Lipschitz in \( y_i \).

Here we formulate synchronization of coupled systems as follows (Oguchi et al. 2008).

**Definition 4.** If there exists a positive real number \( r \) such that the trajectories \( x_i(t) \) of the systems \( \Sigma_i \) with initial conditions \( \phi_i, \phi_j \) such that \( \|\phi_i - \phi_j\| \leq \varepsilon \) satisfy \( \|x_i(t) - x_j(t)\| \rightarrow 0 \) as \( t \rightarrow \infty \) for all \( i, j \) then the coupled systems \( \Sigma_i \) and \( \Sigma_j \) are asymptotically synchronized.

Therefore \( ||\phi||_{c} := \max_{t \geq 0} \|\phi(t)\| \) stands for the norm of a vector function \( \phi \), where \( ||\cdot|| \) refers to the Euclidean vector norm.

Substituting equation (5) into system (4), we obtain

\[
\dot{x}_i = (A + B_1 K_0 C_0) x_i(t) - \sum_{j=1, j \neq i}^{N} B_1 K_{ij} C_0 x_j(t - \tau_{ij}) + B_0 f(y_i(t))
\]

and in turn

\[
\dot{x}_i = (I_N \otimes (A + B_1 K_0 C_0)) x_i(t) - \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} G_{ij}(K_{ij}) x_j(t - \tau_{ij}) + (I_N \otimes B_0) F(y_i(t))
\]

(6)

where \( x = (x_1^T, \ldots, x_N^T)^T \in \mathbb{R}^{nN} \), the notation \( \otimes \) denotes the Kronecker product, \( I_N \) is the identity matrix of dimension \( N \), \( G_{ij}(K_{ij}) = E_{ij} \otimes B_1 K_{ij} C_0 \), \( E_{ij} \in \mathbb{R}^{N \times N} \).
is the matrix whose \((i,j)\)-entry is 1 with all other entries 0, and
\[
F(y(t)) = \begin{bmatrix}
    f(y^1(t)) \\
    \vdots \\
    f(y^N(t))
\end{bmatrix}.
\]

Defining the synchronization error as
\[
e = \begin{bmatrix}
    x^1 - x^2 \\
    \vdots \\
    x^1 - x^{N-1}
\end{bmatrix} = (M_0 \otimes I_n)x
\]
where
\[
M_0 = \begin{bmatrix}
    1 & -1 \\
    1 & 0 & -1 & 0 \\
    \vdots & \vdots & \ddots & \ddots \\
    1 & 0 & \cdots & 0 & -1
\end{bmatrix},
\]
under assumption C-1, the error dynamics can be rewritten by
\[
\dot{e}(t) = (I_{N-1} \otimes A_0)e(t) + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} MG_{ij}(K_{ij})e(t - \tau) + (I_{N-1} \otimes B_0)v(t)
\]
where \(A_0 = A + B_1KC_0\) and
\[
\psi(x^1, e_{1j}) := f(y^1) - f(y^j) = -f(C_0x^1) + f(C_0(x^1 - e_{1j}))
\]
for \(j = 2, \ldots, N\). Therefore applying the coordinate transformation \(\begin{bmatrix} x^1 \\ e \end{bmatrix} := \begin{bmatrix} I_n & 0 \\ 0 & M_0 \otimes I_n \end{bmatrix} x\), the total system (6) can be rewritten by
\[
\begin{align*}
\dot{\bar{x}}^1 &= (I_N \otimes A_0) \begin{bmatrix} x^1(t) \\ e(t) \end{bmatrix} \\
+ &\begin{bmatrix}
    -B_1KC_0 & B_1K_{12}C_0 & \cdots & B_1K_{1N}C_0 \\
    0 & \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} G_{ij} \\
    \end{bmatrix} \begin{bmatrix}
    x^1(t - \tau) \\
    e(t - \tau)
\end{bmatrix} + (I_N \otimes B_0) \begin{bmatrix}
    -\psi(x^1, x^1) \\
    -\psi(x^1, e_{12}) \\
    \vdots \\
    -\psi(x^1, e_{1N})
\end{bmatrix}.
\end{align*}
\]

Recognizing \(x^1\) as a time-varying perturbation, the error dynamics are represented as a feedback connection of a linear retarded dynamical system and a nonlinear element as shown in Figure 1.

\[
\dot{\bar{x}}^1 = (I_N \otimes A_0) \bar{x}^1(t) + (I_N \otimes B_0)v(t)
\]

The function \(\psi(y^1, z_j)\) can be considered as a time-varying nonlinear function in \(z_j, \psi(t, z_j)\).

\[
\psi(y^1, z_j) = \begin{bmatrix}
    ψ(y^1, z_1) \\
    \vdots \\
    ψ(y^1, z_{N-1})
\end{bmatrix}
\]

Fig. 1. Configuration of the error dynamics

Now we assume that the nonlinear function \(ψ\) satisfies the following sector condition for all \(y^1\) and \(z_j \in E_j \subset \mathbb{R}^p\)
\[
\psi(t, z_j)^T(\psi(t, z_j) - \Lambda z_j) \leq 0 \forall t
\]
where \(\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)\) with \(\lambda_i > 0\). Then from the construction, \(Ψ\) satisfies
\[
Ψ^T(Ψ - (I_{N-1} \otimes \Lambda)z) \leq 0
\]
for any \(y^1\) and \(z \in \prod_{j=1}^{N} E_j \subset \mathbb{R}^{p(N-1)}\).

From Definition 4, we know that the synchronization problem can be reduced to the stability of the origin \(e = 0\) of the error dynamics (7) with the nonlinear feedback (8) satisfying the sector condition (10). Applying the circle criterion mentioned in the foregoing section, we obtain the following synchronization criterion.

Theorem 5. Given the error dynamics (7) with the nonlinear feedback (8) satisfying the sector condition (10). Then, the synchronization of coupled systems can be accomplished if the coupling gain matrices \(K_{ij}\) are chosen such that

(1) the roots of the equation
\[
det\left(sI - (I_{N-1} \otimes A_0) - \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} MG_{ij}(K_{ij})e^{-\tau s} \right) = 0
\]
have negative real part, and

(2) \(I + NG(s)\) is strictly positive real, where
\[
G(s) := (I_{N-1} \otimes C_0)(sI - (I_{N-1} \otimes A_0)) - \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} MG_{ij}(K_{ij})e^{-\tau s} \}
\]
and \(\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)\).

The first condition for synchronization in Theorem 5 means that the linear part of the error dynamics is internal stable. When \(N=2\), the error dynamics is described by a single input and single output system and the second condition in Theorem 5 reduces to
\[
\Re[G(j\omega - e)] \geq \frac{1}{\lambda} \forall \omega \in \mathbb{R}, \exists \varepsilon > 0,
\]
which holds when the Nyquist plot of \( G(j\omega) \) lies in the closed right-half complex plane.

Remark 6. Theorem 5 only guarantees that the coupled systems asymptotically synchronize. Therefore even if the synchronization can be accomplished, the behavior of the resulting dynamics can be quite different from the original uncoupled case.

4. ILLUSTRATIVE EXAMPLES

Consider the Chua circuit with an input port described by the following equation

\[
\begin{align*}
S_i : & \quad \begin{cases}
  \dot{x}_1(t) = a\{-m_1 x_1(t) + x_2(t) - \eta(x_1(t))\} \\
  \dot{x}_2(t) = x_1(t) - x_2(t) + x_3(t) + u_i(t) \\
  \dot{x}_3(t) = -bx_3(t)
\end{cases}
\end{align*}
\]

in which the nonlinear function

\[
\eta(x_1) = \frac{1}{2}(m_0 - m_1)(|x_1| + c - |x_1| - c)
\]

and parameters \( a = 9, \ m_0 = -3/7, \ m_1 = 4/7, \ b = 14.28 \) and \( c = 1 \). With these parameters, this system behaves chaotically with a double scroll attractor.

This system can be rewritten as follows.

\[
\begin{align*}
\dot{x}_1(t) &= A x_1(t) + B_0 f(y^i(t)) + B_1 u_i(t) \\
\dot{y}^i(t) &= C_0 x_1(t)
\end{align*}
\]

where

\[
A = \begin{bmatrix}
-a m_0 & a & 0 \\
1 & -1 & 1 \\
0 & -b & 0
\end{bmatrix}, \quad B_0 = \begin{bmatrix}
-a(m_0 - m_1)/2 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
B_1^T C_0 = [0 \ 1 \ 0],
\]

\[
f(y^i(t)) = |y^i(t) + c| - |y^i(t) - c|.
\]

4.1 Case \( N = 2 \)

Firstly, we consider the synchronization problem for two coupled Chua systems. From the condition C-I, the only network structure in which the synchronization error dynamics has a trivial solution at the origin, is in case we use a bidirectional coupling with the same gain \( K_{12} = K_{21} = k \in \mathbb{R}_+ \). The coupling term is given by

\[
u_i(t) = -k(y^i(t) - y^j(t - \tau))
\]

for \( (i,j) = \{(1,2),(2,1)\} \).

Substituting (12) into system (11), we obtain for \( (i,j) = \{(1,2),(2,1)\} \)

\[
\begin{align*}
\dot{x}_1(t) &= (A + A_1)x_1(t) + B_0 f(y^i(t)) - A_1 x_1(t - \tau) \\
\dot{y}^i(t) &= C_0 x_1(t)
\end{align*}
\]

where

\[
A_1 = B_1 K C_0 = \begin{bmatrix}
0 & 0 & 0 \\
0 & -k & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Define the synchronization error \( e_{12}(t) = x_1(t) - x_2(t) \), the error dynamics is given by

\[
\begin{align*}
\dot{e}_{12}(t) &= (A + A_1)e_{12}(t) + A_1 e_{12}(t - \tau) + B_0 v(t) \\
\Sigma_L : & = A_0 e_{12}(t) + A_1 e_{12}(t - \tau) + B_0 v(t) \\
z(t) &= [1 \ 0 \ 0] e_{12}(t) = C e_{12}(t)
\end{align*}
\]

(13)

where \( z \in \mathbb{R} \) and

\[
\psi(y^i, z) = f(y^i) - f(y^j - z).
\]

Since \( \psi \) satisfies that \( \psi(y^i, 0) = 0 \) and

\[
\psi(y^i, z) (\psi(y^i, z) - 2z) \leq 0,
\]

the nonlinear function \( \psi \) belongs to the sector \([0, 2] \).

Therefore the error dynamics (13) with the nonlinear part (14) can be identified with the system (7) with (8).

Figure 2 shows \((k, \tau)\) regions in which each condition in Theorem 5 holds. The dashed-line represents the maximum allowable time-delay satisfying the condition (1) in Theorem 5 and the line is for the condition (2). This figure means from Theorem 5 that if the coupling gain \( k \) and the time-delay \( \tau \) are chosen in the region of SPR, the synchronization can be accomplished. On the other hand, the gray-colored region represents pairs \((k, \tau)\) for which the linear part of the total system, that is

\[
\begin{bmatrix}
x^1 \\
\dot{x}_{12}^i
\end{bmatrix} = (I_2 \otimes A) \begin{bmatrix}
x^1 \\
\dot{x}_{12}
\end{bmatrix} + \begin{bmatrix}
-A_1 & A_1 \\
0 & A_1
\end{bmatrix} \begin{bmatrix}
x^1(t - \tau) \\
\dot{x}_{12}(t - \tau)
\end{bmatrix},
\]

is asymptotically stable. Since the characteristic equation of this dynamics is decomposed into the characteristic equations of \( \dot{x}^1 = A_0 x^1(t) - A_1 x^1(t - \tau) \) and the linear part of the error dynamics (13), this region means that the condition (1) in Theorem 5 holds and the origin of the coupled system may be stable.

Fig. 2. Estimated synchronization region (dashed-line: maximum time-delay for condition (1), solid line: maximum time-delay for condition (2)). The gray-colored region represents a region in which the coupled systems may be stabilized due to the coupling terms.

As shown in Figure 3, the norm of the synchronization error \( \|e(t)\| \) for \( k = 0.3 \) and \( \tau = 0.24 \) converges to zero and the synchronization of coupled systems is accomplished. In case of this coupling, even if the synchronization is
accomplished, the coupling term does not vanish and each system receives the effect of the coupling term. Therefore the attractor of the coupled systems may be different from one of the uncoupled system. Figures 4 and 5 present the difference of the attractors between the uncoupled system and the coupled system under synchronization. Due to the coupling, the attractor of the system slightly shrinks in size but the system still behaves chaotically.

4.2 Case \( N = 3 \)

Now we consider the synchronization problem of a unidirectional ring network of three Chua circuits coupled by the coupling

\[
u_i(t) = -k(x_3^i(t) - x_2^i(t - \tau))
\]

for \((i, j) \in \{(1, 3), (2, 1), (3, 2)\}\).

Then, defining the synchronization error as

\[
e := \begin{bmatrix} e_{12} \\ e_{13} \end{bmatrix} := \begin{bmatrix} x_1^1 - x_2^1 \\ x_1^1 - x_3^1 \end{bmatrix},
\]

the error dynamics is described by

\[
\dot{e} = (I_2 \otimes (A + A_1))e(t) + \left( \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \otimes A_1 \right)e(t - \tau) \\
+ (I_2 \otimes B_0) \begin{bmatrix} -\psi(x^1, e_{12}) \\ -\psi(x^1, e_{13}) \end{bmatrix}
\]

The function \(\psi(x^1(t), e_{ij})\) can be considered as a time-varying nonlinear function with respect to \(e_{1i}\) and satisfies the following sector condition.

\[
\psi(t) \left( \psi(t) - 2e_{1i} \right) \leq 0
\]

Then the error dynamics can be rewritten as follows.

\[
\dot{e} = (I_2 \otimes (A + A_1))e(t) + \left( \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \otimes A_1 \right)e(t - \tau) \\
+ (I_2 \otimes B_0)v \\
:= A^0e(t) + A^1e(t - \tau) + Bv
\]

\[
z = (I_2 \otimes C)e(t)
\]

\[
v = -\Psi := \begin{bmatrix} \psi(x_1^1, z_1) \\ \psi(x_1^2, z_2) \end{bmatrix}
\]

where \(\Psi\) is decentralized and satisfies the sector condition described by

\[
\Psi^T \left( \Psi - \Lambda z \right) \leq 0, \quad \Lambda = \text{diag}(2)
\]

for any \(z\). Applying the circle criterion, the origin of the error dynamics is uniformly globally asymptotically stable if for the transfer function matrix \(G(s)\) that consists of \((18)\) and \((19)\), \(1 + \Lambda G(s)\) is strictly positive real. In particular, since the nonlinear function \(\Psi\) is decentralized and satisfies the sector condition for any \(e_i\) in \(\mathbb{R}^3\), we can conclude that synchronization can be accomplished uniformly and globally if the above condition is fulfilled.

In the same way as the results in \(N = 2\), Figure 6 shows \((k, \tau)\) regions in which both conditions of Theorem 5 holds for the 3 coupled system. From the obtained criterion, we can conclude that if we choose a pair \((k, \tau)\) in the SPR region, synchronization can be accomplished. When we choose \(k = 0.5\) and \(\tau = 0.1351\) satisfying the synchronization criterion, we can observe in Figure 7 that the synchronization error converges to zero in a finite time. In addition, Figure 8 shows that the coupled systems chaotically behave under synchronization.

5. CONCLUSIONS

In this paper, we considered the synchronization problem in networks of chaotic systems with time-delay. Unlike the conventional synchronization criteria based on the
Fig. 6. Estimated Synchronization region for the unidirectional ring network of \( N = 3 \) (dashed-line: maximum time-delay for condition (1), solid line: maximum time-delay for condition (2)). The gray region represents a region in which the coupled systems may be stabilized due to the coupling terms.

Fig. 7. Synchronization error \((k = 0.5, \tau = 0.1351)\) Lyapunov-Krasovskii approach, the proposed criterion is based on the circle criterion. Due to the avoidance of the use of the Lyapunov-Krasovskii theorem, the condition obtained by the criterion is less conservative than the conventional LMI condition, and the boundary condition for synchronization is relatively close to the result by numerical simulations.

REFERENCES