Biomechanics

Concepts and Computation

Answers to the Exercises

Version 3.0

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Contents

1
Answers to the exercises of chapter 1

Exercises

1.1 (a) $|\vec{e}_i| = 1$
(b) $\vec{e}_i \cdot \vec{e}_j = 0$ if $i \neq j$
$\vec{e}_i \cdot \vec{e}_j = 1$ if $i = j$
(c) $\vec{e}_x \cdot (\vec{e}_y \times \vec{e}_z) = 1$
(d) Definition of a right-handed orthonormal basis.

1.2 $\vec{F}_z = \vec{e}_x - 3\vec{e}_y - 5\vec{e}_z$

1.3 (a)

$$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \qquad \underset{\sim}{b} = \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix} \qquad \underset{\sim}{c} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

(b)

$$\begin{aligned} \vec{a} + \vec{b} &= -3\vec{e}_y + 8\vec{e}_z \\ 3(\vec{a} + \vec{b} + \vec{c}) &= 3\vec{e}_x - 9\vec{e}_y + 30\vec{e}_z \end{aligned}$$

(c)

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \vec{b} \cdot \vec{a} = 16 \\ \vec{a} \times \vec{b} &= 12\vec{e}_x \\ \vec{b} \times \vec{a} &= -12\vec{e}_x \end{aligned}$$

(d)

$$\begin{aligned} |\vec{a}| &= 4 \\ |\vec{b}| &= 5 \end{aligned}$$

$$\begin{aligned} |\vec{a} \times \vec{b}| &= 12 \\ |\vec{b} \times \vec{a}| &= 12 \end{aligned}$$

(e) $\phi = \arccos(\frac{4}{5})$
(f) $\vec{e}_x$ or $-\vec{e}_x$
(g)

$$\begin{aligned} \vec{a} \times \vec{b} \cdot \vec{c} &= 12 \\ \vec{a} \times \vec{c} \cdot \vec{b} &= -12 \end{aligned}$$

(h)

$$\begin{aligned} \vec{a}\vec{b} \cdot \vec{c} &= 32\vec{e}_z \\ (\vec{a}\vec{b})^T \cdot \vec{c} &= -24\vec{e}_y + 32\vec{e}_z \\ \vec{b}\vec{a} \cdot \vec{c} &= -24\vec{e}_y + 32\vec{e}_z \end{aligned}$$

(i) The vectors are independent, but not perpendicular. So the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a suitable but non-orthogonal basis.

1.4

$$\begin{aligned} \vec{d} + \vec{e} &= 3\vec{a} + 2\vec{b} - 3\vec{c} \\ \vec{d} \cdot \vec{e} &= 24 \end{aligned}$$

1.5 (a) $\vec{a}_z = -25\vec{e}_z$
(b) $|\vec{a}_x| = |\vec{a}_y| = 5$; $|\vec{a}_z| = 25$
(c) $\vec{a}_x \times \vec{a}_y \cdot \vec{a}_z = 625$
(d) $\phi = \frac{\pi}{2}$
(e)

$$\begin{aligned} \vec{\alpha}_x &= 4/5\ \vec{e}_x + 3/5\ \vec{e}_y \\ \vec{\alpha}_y &= 3/5\ \vec{e}_x - 4/5\ \vec{e}_y \\ \vec{\alpha}_z &= -\vec{e}_z \end{aligned}$$

So the basis $\{\vec{\alpha}_x, \vec{\alpha}_y, \vec{\alpha}_z\}$ is right-handed and orthogonal.

(f)

$$\vec{b} = 2\vec{e}_x + 3\vec{e}_y + \vec{e}_z$$

so with respect to $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$, $\underset{\sim}{b} = [\ 2\ 3\ 1\ ]^T$

$$\vec{b} = \frac{17}{25}\,\vec{a}_x - \frac{6}{25}\,\vec{a}_y - \frac{1}{25}\,\vec{a}_z$$

so with respect to $\{\vec{a}_x, \vec{a}_y, \vec{a}_z\}$, $\underset{\sim}{b} = \frac{1}{25}[\ 17\ \ -6\ \ -1\ ]^T$

$$\vec{b} = \frac{17}{5}\,\vec{\alpha}_x - \frac{6}{5}\,\vec{\alpha}_y - \vec{\alpha}_z$$

so with respect to $\{\vec{\alpha}_x, \vec{\alpha}_y, \vec{\alpha}_z\}$, $\underset{\sim}{b} = \frac{1}{5}[\ 17\ \ -6\ \ -1\ ]^T$

1.6 The triple product is zero. This means that the vectors are not independent. The vector $\vec{a}$ is lying in the plane, that is defined by the vectors $\vec{b}$ and $\vec{c}$. Relation: $2\vec{a} - \vec{b} - \vec{c} = \vec{0}$.

1.7 Both operators are associated with a rotation.

1.8 (a) $a_x\vec{e}_x$
(b) $a_x\vec{e}_x + a_y\vec{e}_y$
(c) no effect
(d) $a_y\vec{e}_x - a_x\vec{e}_y + a_z\vec{e}_z$
(e) $a_x\vec{e}_x - a_y\vec{e}_y + a_z\vec{e}_z$

2

Answers to the exercises of chapter 2

Exercises

2.1 (a) $\vec{F} = \frac{5}{2}\sqrt{2}\ \vec{\epsilon}_1 + \frac{1}{2}\sqrt{2}\ \vec{\epsilon}_2 - 4\ \vec{\epsilon}_3$
(b) $|\vec{F}| = \sqrt{29}$

2.2 (a) $\vec{M}_P = \vec{0}$; $\vec{M}_R = -2\ \vec{e}_z$
(b) $\underset{\sim}{M}_P = [\ 0\ 0\ 0\ ]^T$; $\underset{\sim}{M}_R = [\ 0\ 0\ \ -2\ ]^T$

2.3 (a) 0
(b) $2F\ell$ positive if counterclockwise
(c) $F\ell$ positive if counterclockwise
(d) $2F\ell$ positive if counterclockwise
(e) $2F\ell$ positive if counterclockwise

2.4 $f = \frac{R}{r}\ F$

2.5 $\vec{M}_P = 6\vec{e}_x - 9\vec{e}_y$

2.6 $\vec{M}_S = 12\vec{e}_z$; $\vec{M}_E = 3\vec{e}_z$

2.7 (a) $\vec{M}_S = 3\vec{e}_y + 3\vec{e}_z$
(b) $\vec{M}_O = -2\vec{e}_x + 22\vec{e}_y + 13\vec{e}_z$

3
Answers to the exercises of chapter 3

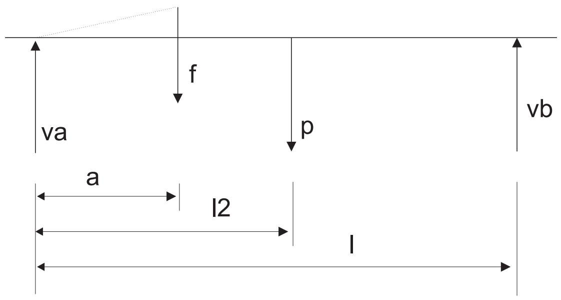
Exercises

3.1

$$\vec{M}_P = -6\vec{e}_z \qquad ; \qquad \vec{M}_R = -6\vec{e}_z$$

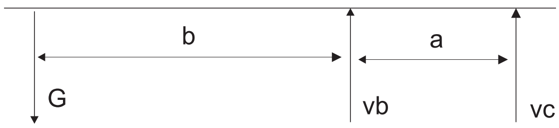
3.2 Define the reaction force vector at A as $\vec{F}_A$ and the moment vector $\vec{M}_A$. For both constructions: $F_{Ax} = -F; F_{Ay} = 0;$ $F_{Az} = 0; M_{Ax} = 0; M_{Ay} = -bF; M_{Az} = cF.$

3.3 For the direction of the forces see the figure below.



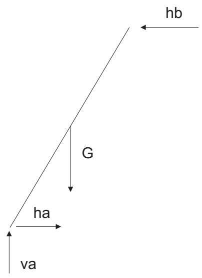
$$\begin{aligned} V_A &= (1 - \frac{\alpha}{\ell})F + \frac{P}{2} \\ V_B &= \frac{\alpha}{\ell}F + \frac{P}{2} \end{aligned}$$

3.4 For the direction of the forces see the next figure.



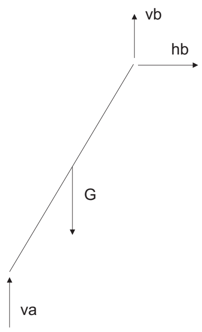
$$V_B = \frac{G(a+b)}{a} \quad ; \quad V_C = -G\frac{b}{a}$$

3.5 **Problem at the left**:
For the direction of the forces see the figure below.



$$H_A = \frac{G\cos(\alpha)}{2\sin(\alpha)} \quad ; \quad V_A = G \quad ; \quad H_B = \frac{G\cos(\alpha)}{2\sin(\alpha)}$$

Problem at the right:
For the direction of the forces see the following figure.



$$V_A = \frac{1}{2}G \quad ; \quad H_B = 0 \quad ; \quad V_B = \frac{1}{2}G$$

4
Answers to the exercises of chapter 4

Exercises

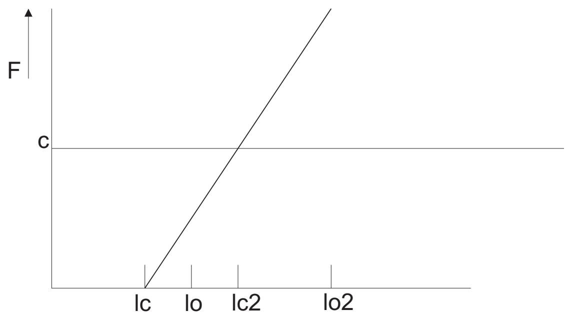
4.1 (a)

$$F = c\frac{\delta}{\ell_c} + c\left(\frac{\ell_0}{\ell_c} - 1\right)$$

(b)

$$F = c\left(\frac{\ell_0}{\ell_c} - 1\right)$$

4.2 See the graph below.



4.3 (a) The number of cross bridges that are able to make a connection is decreasing.

(b) $c \approx 220$ [N]

4.4

$$\delta = \left(\frac{\ell_c^m}{c^m} + \frac{\ell_0^t}{c^t} \right) F + \ell_c^m - \ell_0^m$$

4.5

$$\vec{F}_B = -\vec{F}_A = 160\vec{e}_x - 120\vec{e}_y$$

4.6

$$\vec{F} = c \left(\frac{\sqrt{L^2 + R^2 + 2RL\sin(\phi)}}{\sqrt{L^2 + R^2}} - 1 \right) \frac{R\cos(\phi)\vec{e}_x - (R\sin(\phi) + L)\vec{e}_y}{\sqrt{L^2 + R^2 + 2RL\sin(\phi)}}$$

5
Answers to the exercises of chapter 5

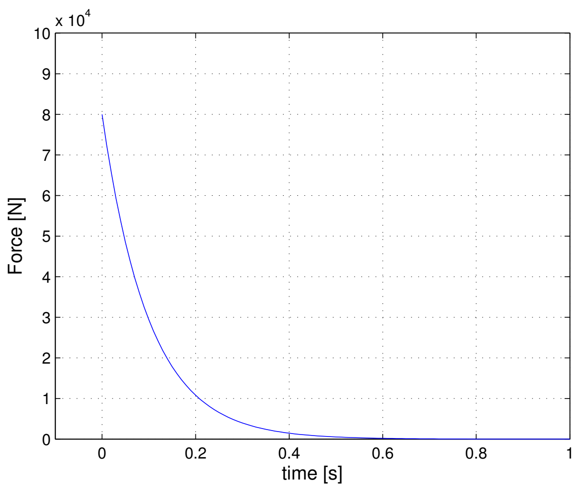
Exercises

5.1 (a)

$$\tau \dot{F} + F = c\dot{\varepsilon} \quad \text{with} \quad \tau = \frac{c_\eta}{c}$$

(b)

$$F = c\, \mathrm{e}^{-t/\tau}$$



(c)

$$E^*(\omega) = \frac{j\omega c}{1 + j\omega\tau}$$

$$\phi = \arctan(\frac{1}{\omega\tau})$$

(d)

$$|F(\omega)| = |E^*(\omega)|\varepsilon_0$$

$$F(10\pi) = 7.6 * 10^3 \quad [\mathrm{N}]$$

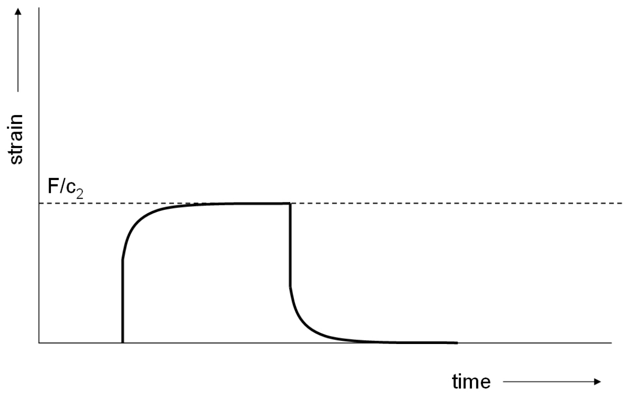
5.2 (a)

$$J(t) = a(1 - b\ \mathrm{e}^{-t/\tau})$$

with:

$$a = \frac{1}{c_2}\ ;\ b = \frac{c_1}{c_1 + c_2}\ ;\ \tau_K = \frac{c_1 + c_2}{c_1 c_2} c_\eta$$

(b) Use superposition



(c)

$$\left.\frac{d\varepsilon}{dt}\right|_{t=0} = \frac{ab}{\tau}$$

5.3 (a)

$$E^*(\omega) = \frac{j\omega c}{1 + j\omega\tau}$$

$$\tan(\delta) = \frac{1}{\omega\tau}$$

$$\tau = \frac{10}{\tan(10)} = 15.4[\text{s}]$$

$$\frac{F(0.1)}{\varepsilon(0.1)} = |E^*(0.1)| = \frac{0.1\ c}{\sqrt{1+\frac{15.4^2}{100}}} = 20 \quad \rightarrow \quad c = 36.7[\text{N}]$$

The phase shift at an angular frequency of 0.2 [s^{-1}] is:

$$\delta = \arctan\left(\frac{1}{0.2 * 15.4}\right) = 0.3139$$

(b)

$$F(0.2) = \frac{0.2 * 36.7}{\sqrt{1 + 0.04 * 15.4^2}} = 2.27[\text{N}]$$

(c)

$$F(t) = c\ \mathrm{e}^{-t/\tau}\ 0.02$$

$$\begin{aligned} F(t=0) &= 0.734 \quad [\text{N}] \\ F(t=10) &= 0.383 \quad [\text{N}] \\ F(t=100) &= 0.0008 \quad [\text{N}] \end{aligned}$$

5.4

$$\varepsilon(t) = \frac{F}{c}\left(1 - \mathrm{e}^{-t/\tau}\right)$$

From the graph it follows that: $c = 500$[N] and $\tau = 10$ [s].

(a)

$$\varepsilon(60) = 0.04\left(1 - \mathrm{e}^{-60/10}\right) - 0.04\left(1 - \mathrm{e}^{-30/10}\right) = 0.019$$

(b)

$$\varepsilon_0 = \frac{F}{c}\left(\frac{1}{\sqrt{1+\omega^2\tau^2}}\right)$$

Because the parameters are not chosen very well, the amplitude of the strain decreases very fast for higher frequencies. At $f = 0.01$ [Hz] we find $\varepsilon = 0.0017$. At $f = 0.1$ [Hz] the strain already decreased to $\varepsilon = 3.1 * 10^{-4}$. This is not strange because the dashpot becomes infinitely stiff when the strain rate increases.

5.5 (a)

$$\hat{G}(s) = \frac{1}{s} + \frac{1}{1+s}$$

$$\hat{J}(s) = \frac{1}{s} - \frac{1/2}{s+1/2}$$

$$\hat{G}\hat{J} = \left(\frac{1}{s} + \frac{1}{1+s}\right)\left(\frac{1}{s} - \frac{1/2}{s+1/2}\right) = \frac{1}{s^2}$$

(b) Use superposition

$$\begin{aligned}
\varepsilon(3) &= J(3) - J(3-1.5) \\
&= 1 - \frac{1}{2}\,\mathrm{e}^{-3/2} - 1 + \frac{1}{2}\,\mathrm{e}^{-(3-1.5)/2} \\
&= \frac{1}{2}\,(-0.2231 + 0.4724) = 0.1246
\end{aligned}$$

(c) Use superposition

$$\begin{aligned}
F(3) &= 0.01\ G(3) - 0.01\ G(3-1.5) \\
&= 0.01\ \left(1 + \mathrm{e}^{-3} - 1 - \mathrm{e}^{-3+1.5}\right) \\
&= -0.0017\ [\mathrm{N}]
\end{aligned}$$

5.6 (a)

$$G(t) = c_2 + c_1 \mathrm{e}^{-t/\tau_R}$$

With Eq. (5.105):

$$J(t) = \frac{1}{c_2}\left(1 - \frac{c_1}{c_1+c_2}\mathrm{e}^{-t/\tau_K}\right)$$

$$\varepsilon(\infty) = \frac{5000}{c_2} = 0.01 \quad \rightarrow \quad c_2 = 5 * 10^5 [\mathrm{N}]$$

$$\varepsilon(0) = \frac{5000}{c_1+c_2} = 0.005 \quad \rightarrow \quad c_1 = 5 * 10^5 [\mathrm{N}]$$

$\tau_K = 5$ [s] $\quad \rightarrow \quad \tau_R = 2.5$ [s]

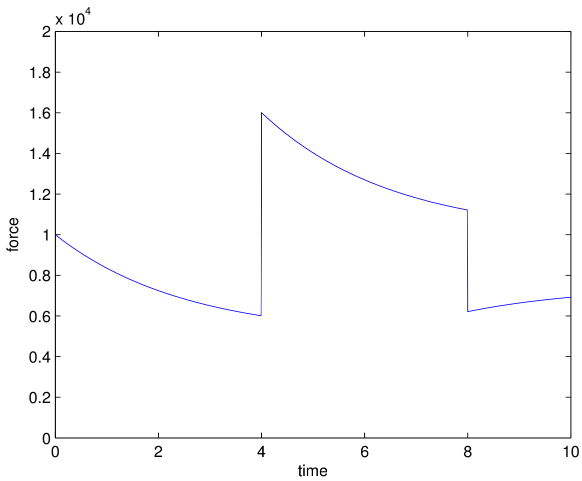
(b) Use superposition. This can be written out in a closed form solution. To make the graph below a MATLAB script was used that also shows nicely how superposition was used. The MATLAB script is:

```
clear all, close all
t=0.01:0.01:10;
Ntim=length(t)

c1  = 5e5;
c2  = 5e5;
tau = 2.5;

for i=1:Ntim
    if t(i)<4
        eps(i)=0.01*(5+5*exp(-t(i)/2.5))*1e5;
    else
        if t(i)<8
        eps1(i)=0.01*(5+5*exp(-t(i)/2.5))*1e5;
        eps2(i)=0.01*(5+5*exp(-(t(i)-4)/2.5))*1e5;
        eps(i)=eps1(i)+eps2(i);
        else
        eps1(i)=0.01*(5+5*exp(-t(i)/2.5))*1e5;
        eps2(i)=0.01*(5+5*exp(-(t(i)-4)/2.5))*1e5;
        eps3(i)=-0.005*(5+5*exp(-(t(i)-8)/2.5))*1e5;
        eps(i)=eps1(i)+eps2(i)+eps3(i);
        end
    end
end

plot(t,eps)
axis([0,10,0,2e4])
ylabel('force')
xlabel('time')
```


x 10^4
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
force
0
2
4
6
8
10
time

6

Answers to the exercises of chapter 6

Exercises

6.1

$$\varepsilon(x) = 2ax + b$$

6.2

$$u(x) = \frac{1}{3}\,a\,x^3 + \frac{1}{2}\,b\,x^2 + c\,x$$

6.3 (a)

$$\sigma(x) = \frac{F}{A_0}\mathrm{e}^{\beta x}$$

(b)

$$\frac{du}{dx} = \frac{F}{EA_0}\mathrm{e}^{\beta x}$$

$$u(x) = \frac{F}{\beta EA_0}\mathrm{e}^{\beta x} + c_1$$

$$u(0) = 0 \rightarrow c_1 = -\frac{F}{\beta EA_0}$$

$$u(x) = \frac{F}{\beta EA_0}\left(\mathrm{e}^{\beta x} - 1\right)$$

6.4 (a)

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = -q = -\alpha \mathrm{e}^{\beta x}$$

$$\sigma(x) = E\frac{du}{dx} = -\frac{\alpha}{\beta A}\mathrm{e}^{\beta x} + \frac{c_1}{A}$$

$$\sigma(\ell) = 0$$

$$\sigma(x) = \frac{\alpha}{\beta A}\left(\mathrm{e}^{\beta\ell} - \mathrm{e}^{\beta x}\right)$$

(b)

$$\frac{du}{dx} = -\frac{\alpha}{\beta EA}\mathrm{e}^{\beta x} + \frac{\alpha}{\beta EA}\mathrm{e}^{\beta\ell}$$

$$u(x) = -\frac{\alpha}{\beta^2 EA}\mathrm{e}^{\beta x} + \frac{\alpha}{\beta EA}\mathrm{e}^{\beta\ell}x + c_2$$

$$u(0) = 0$$

$$u(x) = \frac{\alpha}{\beta^2 EA}\left(1 - \mathrm{e}^{\beta x} + \beta x\mathrm{e}^{\beta\ell}\right)$$

6.5

$$q = \rho g A$$

$$u(x) = -\frac{1}{2}\frac{\rho g}{E}x^2 + \frac{c_1 x}{EA} + c_2$$

$$u(0) = 0 \qquad ; \qquad EA\frac{du}{dx}\bigg|_{x=\ell} = 0$$

$$u(x) = \frac{\rho g}{2E}\left(2\ell x - x^2\right)$$

$$u(l) = \frac{\rho g\ell^2}{2E}$$

6.6 The distributed load q is neglected and $A = \frac{1}{4}\pi(D^2 - d^2)$.

(a) Integrating once leads to:

$$EA\frac{du}{dx} = c_1 = -P$$

Note the negative sign in front of P (compressive force)!

$$\sigma = -\frac{P}{A} = \frac{-P}{\frac{1}{4}\pi(D^2 - d^2)}$$

(b)

$$u(x) = -\frac{P}{EA}x + c_2$$

$$u(\ell) = 0 \qquad \rightarrow \qquad c_2 = \frac{P\ell}{EA}$$

$$u(x) = \frac{P(\ell - x)}{\frac{1}{4}\pi E(D^2 - d^2)}$$

Supplementary, the strain can be determined:

$$\varepsilon = \frac{du}{dx} = \frac{-P}{\frac{1}{4}\pi E(D^2 - d^2)}$$

Note that the strain is negative!

6.7 (a) Everywhere in the muscle tendon complex the internal force is F.

(b) Between A and B: $\sigma_1 = F/A_1$. Between B and C: $\sigma_2 = F/A_2$.

(c) Forces and stresses remain unchanged.

(d)

$$\begin{aligned} u_B &= \frac{F\ell_1}{E_1A_1} \\ u_C &= \frac{F\ell_1}{E_1A_1} + \frac{F\ell_2}{E_2A_2} \end{aligned}$$

7
Answers to the exercises of chapter 7

Exercises

7.1

$$\underset{\sim}{\nabla} c^T \approx (-5, 5, 0)$$

7.2

$$\underset{\sim}{v}_B = \frac{1}{3}\sqrt{3}\omega a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

7.3 If $\vec{\nabla}\rho$ were constant $\rho_B - \rho_A$ would be equal to $\rho_C - \rho_D$ and $\rho_D - \rho_A$ would be equal to $\rho_C - \rho_B$. This is clearly not the case. Apparently $\vec{\nabla}\rho$ is a function of the coordinates.

7.4 The parameter equations for the curve - a circle with radius ℓ and the centroid in the point $(x, y) = (\ell, 0)$ - are:

$$\begin{aligned} x &= \ell \cos(\xi) + \ell \\ y &= \ell \sin(\xi) \end{aligned}$$

Thus:

$$\frac{dT}{d\xi} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \xi} = +\theta\ell\cos(\xi) = 0$$

$$\xi = ...\pi/2, 3\pi/2, ...$$

This renders the loacations:

$$x = \ell \quad ; \quad y = \pm\ell; \quad z = \ell$$

7.5

$$\boldsymbol{L} = -\frac{\alpha}{2\ell^2}\ (\vec{e}_x\vec{e}_y + \vec{e}_y\vec{e}_x)$$

8

Answers to the exercises of chapter 8

Exercises

8.1

$$\sigma_{xy} = -2axy + 2a$$

8.2 (a)

$$\boldsymbol{\sigma} = 10\vec{e}_x\vec{e}_x + 20\vec{e}_y\vec{e}_y + 5(\vec{e}_x\vec{e}_y + \vec{e}_y\vec{e}_x)$$

(b)

$$\vec{s} = \frac{1}{2}\sqrt{2}(15\vec{e}_x + 25\vec{e}_y)$$

(c)

$$\begin{aligned}\vec{s}_n &= 10\sqrt{2}\ (\vec{e}_x + \vec{e}_y)\\ \vec{s}_t &= \frac{5}{2}\sqrt{2}(-\vec{e}_x + \vec{e}_y)\end{aligned}$$

8.3 (a)

$$\underline{\sigma}_a = \begin{bmatrix} 10 & 0 \\ 0 & -20 \end{bmatrix} \quad ; \quad \underline{\sigma}_b = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}$$

(b)

$$\begin{aligned}\vec{s}_a &= 10\cos(\alpha)\vec{e}_x - 20\sin(\alpha)\vec{e}_y\\ \vec{s}_b &= 5\sin(\alpha)\vec{e}_x + 5\cos(\alpha)\vec{e}_y\end{aligned}$$

(c)

$$\begin{aligned}|\vec{s}_a| &= 10\sqrt{\cos^2(\alpha) + 4\ \sin^2(\alpha)}\\ |\vec{s}_b| &= 5\end{aligned}$$

8.4

$$\boldsymbol{\sigma} = 9\vec{e}_x\vec{e}_x + 12(\vec{e}_x\vec{e}_y + \vec{e}_y\vec{e}_x) + 16\vec{e}_y\vec{e}_y \qquad \text{expressed in [MPa]}$$

8.5

$$\overline{\sigma}_M = 3\sqrt{7}$$

8.6

$$\boldsymbol{\sigma} = 5\vec{e}_x\vec{e}_x + 5\vec{e}_y\vec{e}_y + 10(\vec{e}_x\vec{e}_y + \vec{e}_y\vec{e}_x + \vec{e}_x\vec{e}_z + \vec{e}_z\vec{e}_x)$$

8.7

$$|\vec{s}_t| = 2\sqrt{6} \qquad \text{expressed in [MPa]}$$

9
Answers to the exercises of chapter 9

Exercises

9.1

$$\underset{\sim}{v}(\underset{\sim}{x}) = \begin{bmatrix} a + b(y - at) \\ a \\ 0 \end{bmatrix}$$

9.2

$$\underset{\sim}{a}(\underset{\sim}{x}, t) = \frac{1}{(1+\alpha t)^2} \begin{bmatrix} bcx - \alpha ay - \alpha bz \\ 0 \\ -\alpha cx + acy + bcz \end{bmatrix}$$

9.3

$$t = 3/2 \text{ [s]}$$

9.4

$$\vec{a} = -\omega^2 x \vec{e}_x - \omega^2 y \vec{e}_y$$

9.5

$$\vec{a} = c^2 x \vec{e}_x + c^2 y \vec{e}_y$$

9.6

$$\underset{\sim}{x}_P = \begin{bmatrix} \frac{1}{2}\ell + \frac{1}{6}\ell \sin\left(2\pi \frac{t}{T}\right) \\ \frac{1}{3}\ell \\ 0 \end{bmatrix}$$

10

Answers to the exercises of chapter 10

Exercises

10.1

$$\underset{\sim}{x}_{0_Q} = \begin{bmatrix} -1 \\ 9 \\ 6 \end{bmatrix}$$

10.2

$$\vec{a}_0 \cdot \vec{a} = |\vec{a}_0||\vec{a}| \cos(\phi)$$

$$\vec{a}_0 \cdot \boldsymbol{F} \cdot \vec{a}_0 = \sqrt{\vec{a}_0 \cdot \vec{a}_0} \sqrt{(\vec{a}_0 \cdot \boldsymbol{F}^T) \cdot (\boldsymbol{F} \cdot \vec{a}_0)} \cos(\phi)$$

$$\cos(\phi) = \frac{\vec{a}_0 \cdot \boldsymbol{F} \cdot \vec{a}_0}{\sqrt{(\vec{a}_0 \cdot \vec{a}_0)(\vec{a}_0 \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \vec{a}_0)}}$$

10.3

$$\vec{x}_Q = 11\vec{e}_x + 6\vec{e}_y + 3\vec{e}_z$$

10.4

$$J = 2$$

10.5

$$\ell = 3 \text{ e}^{0.02t} \qquad [\text{cm}]$$

10.6

$$
\begin{aligned}
\dot{\boldsymbol{\epsilon}}_F &= \frac{1}{2}\left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^T + \boldsymbol{F} \cdot \dot{\boldsymbol{F}}^T\right) \\
&= \frac{1}{2}\left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T + \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{F}}^T\right) \\
&= \frac{1}{2}\left(\boldsymbol{L} \cdot \boldsymbol{B} + \boldsymbol{B} \cdot \boldsymbol{L}^T\right)
\end{aligned}
$$

11

Answers to the exercises of chapter 11

Exercises

11.1

$$\rho_A V_A = \rho_B V_B$$

11.2

$$\rho = \frac{1500}{1 + 0.02t} \qquad \text{expressed in [kg/m}^3\text{]}$$

12

Answers to the exercises of chapter 12

Exercises

12.1

$$\varepsilon^v = 1/8$$

12.2

$$F_v = -\frac{65}{24} G \ell^2$$

12.3

$$E = \frac{5\ h\ K_v}{6\pi\ \delta\ R^2}$$

12.4

$$\nu = 0.2$$

12.5 (a) $\varepsilon_{yy} = 0.1$

(b) $\varepsilon_{xx} = -0.067$

(c) $\sigma_{yy} = 0.8$ [MPa]

12.6

$$\boldsymbol{\varepsilon} = \frac{1}{2} \frac{\alpha R}{L} \left(\vec{e}_x \vec{e}_z + \vec{e}_z \vec{e}_x \right)$$

$$\boldsymbol{\sigma} = -\frac{E}{2(1+\nu)} \frac{\alpha R}{L} \left(\vec{e}_x \vec{e}_z + \vec{e}_z \vec{e}_x \right)$$

$$\overline{\sigma}_M = \frac{E}{2(1+\nu)} \frac{\alpha R}{L} \sqrt{3}$$

13

Answers to the exercises of chapter 13

Exercises

13.1 The surfaces with a normal in y-direction are stress free. So at these surfaces $\sigma_{yz} = \sigma_{zy} = 0$. This contradicts the assumptions on the stress field that were given.

13.2 Rigid body motions of the plate are not suppressed.

13.3 The following requirements are not satisfied:

- Local force equilibrium in x-direction.
- The "oblique" sides of the trapezium shaped plate are not stress free.

13.4

$$c_2 = -\frac{6P}{hb^3} \quad ; \quad c_1 = \frac{12P}{hb^3}$$

13.5 The given velocity field leads to a flow where all fluid particles rotate around the z-axis with the same angular velocity α. This means that the position with respect to each other of the particles does not change and so $\boldsymbol{D} = \boldsymbol{0}$. Every fluid particle moves uniformly along a circular track around the z-axis in a plane parallel to the xy-plane. This means that every particle has an acceleration with magnitude $\alpha^2 r$ with $r = \sqrt{x^2 + y^2}$ in the direction of the z-axis. To realize this acceleration a force per unit mass $\vec{q}$ in the direction of the z-axis is necessary.

13.6 For the points $x = \pm a, y = b$ the fluid velocity is not uniquely defined. From a view point based on the rigid wall: $\vec{v} = \vec{0}$. When looking at the moving wall $\vec{v} = V\vec{e}_x$.

13.7 The requirement that no fluid can pass the wall of the vessel is not satisfied.

13.8

$$\vec{q} = -\alpha^2 \left(x\vec{e}_x + y\vec{e}_y \right)$$

14
Answers to the exercises of chapter 14

Exercises

14.1 (a) Since:

$$\underset{\sim}{p} = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}$$

for $n = 1$ the matrix $\underline{K}$ can be elaborated as:

$$\underline{K} = \int_0^1 \underset{\sim}{p}\underset{\sim}{p}^T \, dx = \int_0^1 \begin{bmatrix} 1 & x \\ x & x^2 \end{bmatrix} dx = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

Likewise:

$$\underset{\sim}{f} = \int_0^1 \underset{\sim}{p} f \, dx = \int_0^1 \begin{bmatrix} 1 \\ x \end{bmatrix} 3 \, dx = \begin{bmatrix} 3 \\ \frac{3}{2} \end{bmatrix}$$

From $\underline{K}\underset{\sim}{a} = \underset{\sim}{f}$ it can be derived:

$$\underset{\sim}{a} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

(c)
```
% Exercise 14.1
clear all ; close all
x=0:0.01:1 ;
xmax=length(x) ;

% Exact function
for i=1:xmax
    if x(i)<=0.5
        f(i)=1;
```

```
        else
            f(i)=0;
        end
    end
    plot(x,f,'r-');
    grid on ; xlabel('x') ; ylabel('h(x)') ;
    title('Answer to exercise 1')
    hold on
    % loop for different order of the polynomial from n=2 to n=10
    for n=2:10
        f=zeros(n+1,1); K=zeros(n+1,n+1);
        for i=1:n+1
            f(i)=(1/i)*(0.5)^i;
            for j=1:n+1
            K(i,j)=1/(i+j-1);
            end
        end
        a=inv(K)*f;  condnum(n)=cond(K); plot(x,polyval(a(n+1:-1:1),x),'b-')
    end
    % graph of condition number against order of the polynomial
    figure ; plot(condnum) ; xlabel('polynomial order') ;
    ylabel('condition number')
```

For a number of values of n the interpolation polynomial $h(x)$ is given in figure 14.1.

14.2 The weighted residuals form is given by:

$$\int_a^b w\left[u + \frac{du}{dx} + \frac{d}{dx}(c\frac{du}{dx}) + f\right]\, dx = 0$$

After partial integration we find the so-called weak form

$$\int_a^b w\left[u + \frac{du}{dx}\right] dx - \int_a^b \frac{dw}{dx}c\frac{du}{dx}\, dx = -\int_a^b wf\, dx - wc\frac{du}{dx}\bigg|_a^b$$

14.3

```
% Exercise 14.3
clear all ; close all ; clf

n=10; delt=1/(n-1); x=0:delt:1;
for i=1:n
     if x(i)<=0.5
```

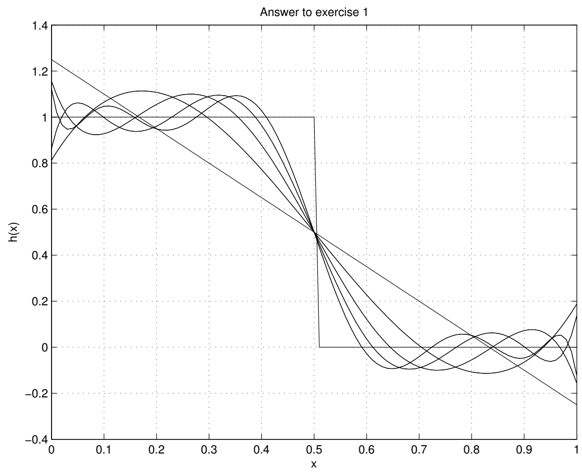


Fig. 14.1. $h(x)$ for a number of different polynomial orders

```
        g(i)=1;
    else
        g(i)=0;
    end
    for j=1:n
    K(i,j)=x(i)^(j-1);
    end
end
a=inv(K)*g';

% for plotting purposes functions are defined at finer intervals
xx=0:0.01:1
for i=1:length(xx)
    if xx(i)<=0.5
        gg(i)=1;
    else
        gg(i)=0;
    end
end
plot(xx,polyval(a(n:-1:1),xx),'b-')
title(['Approximation for polynomial of order ',num2str(n)])
xlabel('x'); ylabel('h(x)'); hold on ; grid on
```

```
plot(xx,gg,'r-')
```

The matrix is given in the lecture notes. The polynomial fit for a polynomial of order 10 is given in figure 14.2. There is a substantial difference between this fit and the one given in figure 14.1. The fit of figure 14.2 captures precisely the points $f(x_i)$, which is not the case for the fit based on the weighted residuals form. Moreover, although the fit of figure 14.2 is exact at these points, the average deviation from $f(x)$ is much larger than in figure 14.1.

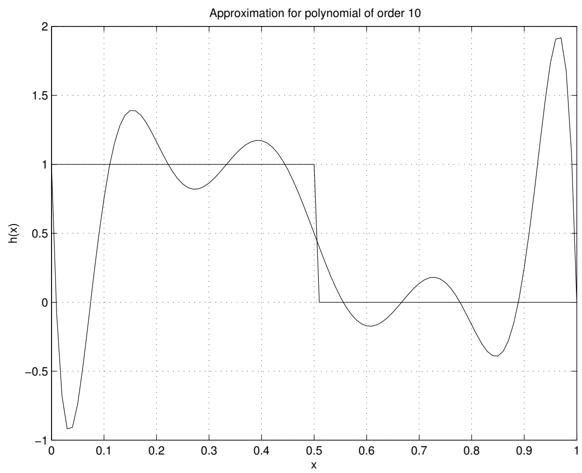


Fig. 14.2. polynomial fit of order 10

14.4 (a)

$$
\begin{aligned}
a_0 &= u_2 \\
a_1 &= \frac{1}{2}(u_3 - u_1) \\
a_2 &= \frac{1}{2}(u_1 + u_3 - 2u_2)
\end{aligned}
$$

(b) The shape functions are:

$$
\begin{aligned}
N_1 &= -\frac{1}{2}x(1-x) \\
N_2 &= (1+x)(1-x)
\end{aligned}
$$

$$N_3 = \frac{1}{2}x(1+x)$$

(d) No, as $u_h(x_i)$ should be equal to u_i.

(e) No, same reason

14.5 (a)

$$\int_{x=0}^{h_1+h_2} \frac{dw}{dx} c \frac{du}{dx} dx = -\int_{x=0}^{h_1+h_2} w dx + wc\frac{du}{dx}\Big|_0^{h_1+h_2}$$

for all $w(x)$

(b)

$$\begin{aligned} N_1^1 &= 1 - \frac{x}{h_1} \\ N_2^1 &= \frac{x}{h_1} \\ N_1^2 &= -\frac{1}{h_2}x + 1 + \frac{h_1}{h_2} \\ N_2^2 &= \frac{1}{h_2}x - \frac{h_1}{h_2} \end{aligned}$$

(f)

$$c \begin{pmatrix} \frac{1}{h_1} & 0 & -\frac{1}{h_1} \\ 0 & \frac{1}{h_2} & -\frac{1}{h_2} \\ -\frac{1}{h_1} & -\frac{1}{h_2} & \frac{1}{h_1} + \frac{1}{h_2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} h_1+? \\ h_2+? \\ h_1 + h_2 \end{pmatrix}$$

$$c\left(\frac{1}{h_1} + \frac{1}{h_2}\right) u_3 = -\frac{1}{2}(h_1 + h_2)$$

$$u_3 = -\frac{h_1 h_2}{2c}$$

14.6 (b)

$$\frac{dN_i}{dx} = \frac{dN_i}{d\xi}\frac{d\xi}{dx}$$

(c)

$$\frac{d\xi}{dx} = \frac{1}{\frac{3}{2} + \xi}$$

(d)

$$h_1 = -\frac{1}{6}$$

14.7 (a)

```
% Adjusted demo_fem1d for exercise 7%

clear, close all

%------ Define the mesh, material properties and boundary conditions ------
% The domain spans xmin <= x <= xmax
xmin=  0;
xmax=  1;

% number of elements
nelem= 5;

% norder: order of the interpolation polynomial
% norder=1, linear element
% norder=2, quadratic element
norder=1;

% generate the mesh
if norder==1,
   % simple 1d uniform mesh, coordinates
   dx=(xmax-xmin)/nelem;
   coord=(xmin:dx:xmax)';
   % element topology
   top=[(1:nelem)' (2:nelem+1)' ones(nelem,2)];
elseif norder==2,
   % again a simple 1d uniform mesh,
   % coordinates:
   dx=(xmax-xmin)/(2*nelem);
   coord=(xmin:dx:xmax)';
   % element topology
top=[(1:2:nelem*2)' (2:2:(nelem*2+1))' (3:2:(nelem*2+2))' ones(nelem,2)];
end

% materials properties and some other useful information
% if c<0: a non-constant c is generated by elm1d_c,
```

```
% else elm1d_c returns c
c=1;
f=0;
mat.mat(1)=c;          % c
mat.mat(3)=f;          % f
mat.mat(4)=norder;     % element order
mat.types=['elm1d'];

% boundary conditions: u=0 at left end
bndcon=[1 1 0];  % [node number ; degree of freedom ; value ]

% nodal forces f=1 at right end
nodfrc=[6 1 1];  % [node number ; degree of freedom ; value ]


%------ Solve the problem using the 1D fem program FEM1D ------
fem1d


%------ postprocess the data ------
%* plot the solution (sol) as well as the exact solution (u_exact)

subplot(1,2,1),
plot(coord,sol,'-');
xlabel('x'), ylabel('u');
nel=num2str(nelem);
sorde=num2str(norder);
tekst=['aantal elementen = ',nel,' orde van elementen = ',sorde]
title(tekst)

subplot(1,2,2)
if norder==1,
   plot([coord(1:nelem) coord(2:nelem+1)]',sigma')
elseif norder==2,
plot([coord(1:2:2*nelem) coord(2:2:2*nelem+1)
coord(3:2:2*nelem+2)]',sigma')
end
xlabel('x'), ylabel('\sigma');
```

(b) `ii=pos(2,:) ; u = sol(nonzeros(ii))  ; u = [0.2 ; 0.4]`

(c) `ii=dest(3,:) ; u=sol(ii) ; u=0.4`

14.8 (a)

$$\begin{aligned} \frac{d}{dx}\left(c\frac{du}{dx}\right) - f &= 0 \\ u(0) &= 5 \\ u(l) &= 5 \end{aligned}$$

(b)

$$u(x) = \frac{f}{2c}x^2 - \frac{fl}{2c}x + 5$$

(c) When the thickness $t = 5$mm the glucose level in the middle of the construct will become less than zero (in reality of course zero) meaning that eventually cells will die in the middle of the construct.

14.9 (a) Add:

```
xx=n(int,:)*nodcoord;
   a1=1.6;    % in [cm]
   a2=0.15;  % in [cm^-1]
   a3=0.8;    % dimensioneless
   E=10;       % in [Ncm^-2]
   r=a1*(sin(a2*xx+a3))^3;
   A=pi*r^2 %calculate surface of cross section
   c=E*A;   % product of cross section and Youngs modulus
```

in the element routine.

(b) See figure 14.3.

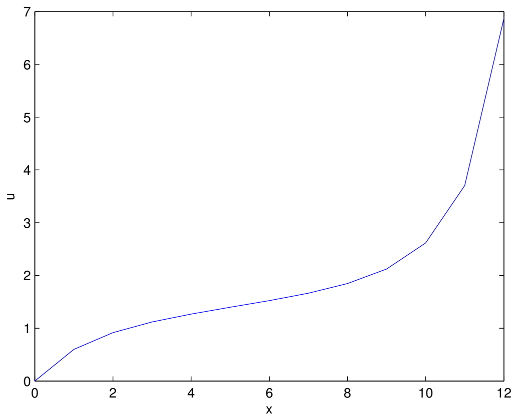


Fig. 14.3. Exercise 14.9: Displacement as a function of x

15

Answers to the exercises of chapter 15

Exercises

15.1 (b) See Fig. 15.1. For small a the solution is approximately the same as the solution of the diffusion equation. For larger a particles drag along with the fluid and thus the higher concentrations shift to the right. For values higher than 10 the solution becomes unstable.

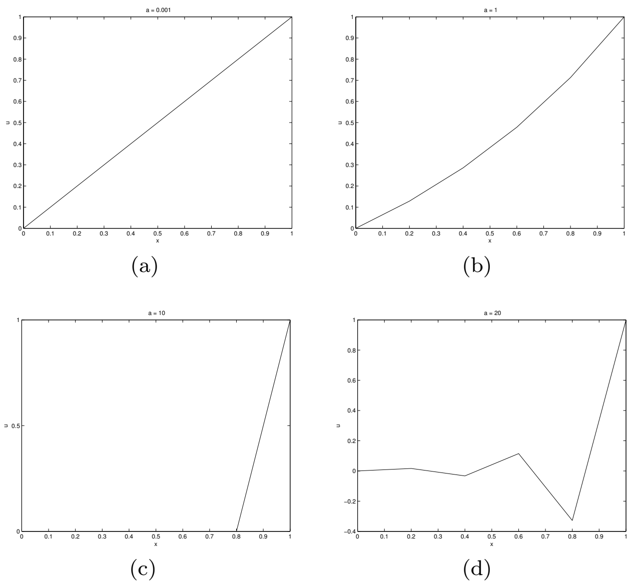


Fig. 15.1. Solution of the steady convection diffusion equation for different values of a using 5 linear elements

(c) With $h = 0.2$ and $c = 1$ it follows that $Pe_h > 1$ if $a > 10$.

15.2 (a) It can be seen that using $\theta = 0.5$ the solutions remain stable, even for large time steps. When $\theta < 0.5$ the solution starts to oscillate when the time step becomes to large.

(b) When $a > 20$ solutions will become unstable.

(c) When $\Delta t = 0.001$ the discretization of u no longer allows an "exact" description of the u as a function of x.

(d) Reducing the convective velocity does not influence the stability much. Increasing the number of elements (i.e. reducing the size of an element) again leads to stable solutions.

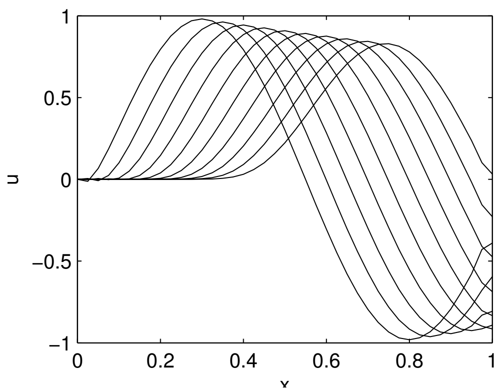


Fig. 15.2. Solution for different time steps

15.3 (a) Initial conditions. Use sol(:,1) (second index is the time step) For initial conditions the following script can be used:

```
for inode = 1: nnodes
    idof=dest(inode,1)
    x = coord(inode,1)
    sol(idof,1)=sin(2*pi*x)
end
```

Essential boundary condition $u = 0$ at $x = 0$;

```
bndcon=[1 1 0]
```

Natural boundary condition: $c\,\partial u/\partial x = 0$ at $x = 1$. It is not necessary to prescribe this, because default this condition is set when no essential boundary conditions are prescribed.

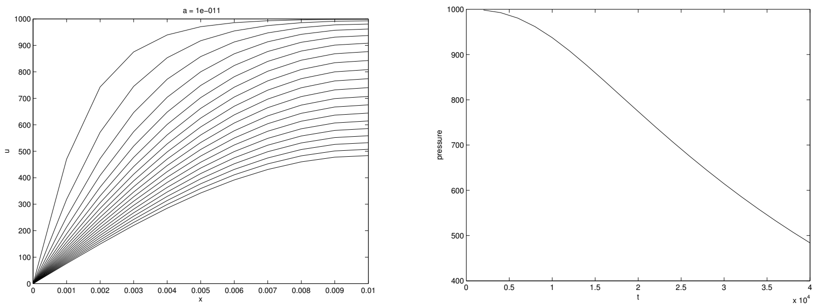
(b) The same as in item (a) but with different initial conditions.

15.4 (a) For the initial condition add the lines:

```
ii=find(coord>0.0001) % excludes the point at x=0
sol(dest(ii,1))=1000;
```

This change can be made in the file: `fem1dcd` or in `demo_fem1dcd`. Boundary condition:

```
bndcon=[1 1 0];
```



(a) (b)

Fig. 15.3. Solution of the confined compression problem a) pressure as a function of space and time, b) pressure near indenter as a function of time

16

Answers to the exercises of chapter 16

Exercises

16.1 (a) The answers to items (a) to (c) can be determined manually or by means of MATLAB. The enclosed answer is by the use of MATLAB. The following `.m`-file was used:

```
% Answer to exercise 16.1 a) to c) by means of matlab
%location of integration points in local coordinates
%ichoice = 1 ; Gauss integration ; ichoice = 2 Newton Cotes

clear all
ichoice=1;
switch ichoice
    case 1
        a=1/sqrt(3);
        xicol= [-a ;  a ; a ; -a];
        etacol=[-a ; -a ; a ;  a];
        w=[1 1 1 1];
    case 2
        xicol= [-1 ;  1 ; 1 ; -1];
        etacol=[-1 ; -1 ; 1 ;  1];
        w=[1 1 1 1];
end

coord=[-2 -2 ;
        2 -2 ;
        2  2 ;
       -2  2 ];
M=zeros(4); C=zeros(4);
```

```
%loop over integration points
for i=1:4
xi=xicol(i) ; eta=etacol(i);
% Equation (16.42)
N=[0.25*(1-xi)*(1-eta) ;
   0.25*(1+xi)*(1-eta) ;
   0.25*(1+xi)*(1+eta) ;
   0.25*(1-xi)*(1+eta) ];

dNdxi = [-0.25*(1-eta) ;
          0.25*(1-eta) ;
          0.25*(1+eta) ;
         -0.25*(1+eta) ];

dNdeta= [-0.25*(1-xi)  ;
         -0.25*(1+xi)  ;
          0.25*(1+xi)  ;
          0.25*(1-xi)  ];

% Equation (16.48)

dxdxi  = dNdxi' * coord(:,1);
dxdeta = dNdeta'* coord(:,1);
dydxi  = dNdxi' * coord(:,2);
dydeta = dNdeta'* coord(:,2);

% Equation (16.47)

matxxi=[dxdxi  dxdeta ;
               dydxi dydeta];

% Equation (16.50)

j=det(matxxi);

% Equation (16.49)

matdxidx=inv(matxxi);
dxidx=matdxidx(1,1);
```

```
detadx=matdxidx(1,2);

%Equations (16.37) and (16.51)

M=M+N*N'*j*w(i);

C=C+N*(dNdxi'*dxidx+dNdeta'*detadx)*j*w(i);
end
```

(d) In this case it is easiest to work with a global coordinate system. In the global system the shape function can be written in the following form:

$$\begin{aligned} N_1 &= \frac{1}{16}(2-x)(2-y) \\ N_2 &= \frac{1}{16}(2+x)(2-y) \\ N_3 &= \frac{1}{16}(2+x)(2+y) \\ N_4 &= \frac{1}{16}(2-x)(2+y) \end{aligned}$$

N_1 and N_4 are 0 on the right-hand side. This leads to:

$$\int_{-2}^{2} \begin{pmatrix} 0 \\ \frac{1}{4}(2-y) \\ \frac{1}{4}(2+y) \\ 0 \end{pmatrix} q \, dx = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} q$$

16.2 (a) No, topology is not unique

(b)
```
pos= [ 5  7  12 10 ]
```

(c)
```
dest= [1 2 3 .... 16]
```

(d)
```
ii = nonzeros(pos(8,:))
ulem=sol(ii)
```

(e)
```
ii=dest(nodes,:)
ulem=sol(ii)
```

(f)

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1=\eta)$$

$$u(\frac{1}{4}, \frac{3}{4}) = 4.3$$

(g)

$$\underset{\sim}{u}_e^T = [3\ 7\ 1\ 4]$$

16.3 (a) No heat flow through these boundaries.

(b)
```
% Solution of exercise 3.3

%mesh generation
lx=1;   % length of the domain
h=1;    % height of the domain
points=[
   0 0;
   1 0;
   1 1;
   0 1];
n=5;    % number of elements in the x-direction
m=5;    % number of elements in the y-direction

curves=[
   1 2 n 1 1
   2 3 m 1 1
   3 4 n 1 1
   4 1 m 1 1];

subarea=[1 2 3 4 1];

norder=1;   % 1: bi-linear elements, 2: bi-quadratic elements

% create the mesh
[top,coord,usercurves]=crmesh(curves,subarea,points,norder);
```

```
% define the mat file
mat.mat(1)=1;
mat.mat(2)=1; %Adjust elcd_a so ax is given the value of mat.mat(2)
mat.mat(3)=0;
mat.mat(11)=norder+2;

mat.types='elcd';   % convection diffusion element

% inflow boundary
bndcon=[];
crv=4;
dof=1;
iplot=0;
string='0';
bndcon=addbndc(bndcon,coord,top,mat,usercurves,crv,dof,string,iplot);

% outflow boundary
crv=2;
dof=1;
iplot=0;
string='1';
bndcon=addbndc(bndcon,coord,top,mat,usercurves,crv,dof,string,iplot);

istat=1;    % 1: steady state solution, 2: unsteady solution

femlin_cd

figure(1)
if istat==1,
   conoddat(sol(dest),coord,top,mat);
else
   for i=1:4
      subplot(2,2,i)
      conoddat(sol(dest,2*i),coord,top,mat);
      title(['t=' num2str(2*i*dt)]);
   end
end
```

(c)

```
ii=nonzeros(usercurves(1,:))
u=sol(dest(ii))
```

```
u=[0 0.1286 0.2859 0.4780 0.7129 1.000]
```

(d) The same result.

16.4

```
% Exercise 16.4
 points=[0 0; 10 0; 10 2; 0 2 ; 0 -1; 10 -1];
 n=10;   % number of elements in the x-direction
 m=3;    % number of elements on gam2
 mm=5 ;  % number of elements on gam5
 curves=[5 6 n 1 1
         6 2 m 1 1
         2 1 n 1 1
         1 5 m 1 1
         2 3 mm 1 1
         3 4 n 1 1
         4 1 mm 1 1];
 subarea=[1 2 3 4 1 ;  -3 5 6 7 1 ];
 norder=1;   % 1: bi-linear elements, 2: bi-quadratic elements
 % create the mesh
 [top,coord,usercurves]=crmesh(curves,subarea,points,norder);
 % define the mat file
 mat.mat(1)=1;
 mat.mat(2)=1;
 mat.mat(3)=0;
 mat.mat(11)=norder+2;
 mat.types='elcd';   % convection diffusion element
 % inflow boundary
 bndcon=[]; crv=1; dof=1; iplot=0; string='1';
 bndcon=addbndc(bndcon,coord,top,mat,usercurves,crv,dof,string,iplot);
 % outflow boundary
 crv=7; dof=1; iplot=0; string='0';
 bndcon=addbndc(bndcon,coord,top,mat,usercurves,crv,dof,string,iplot);
 istat=1;    % 1: steady state solution, 2: unsteady solution
 femlin_cd
 conoddat(sol(dest),coord,top,mat);
```

For the condition for $\vec{a}$ the file **elcd_a.m** has to be changed:

```
function [ax,ay]=elcd_a(x,y,fx,fy);
ax=0;
ay=0;
```

```
if y>0
    ax=y;
end
```

17
Answers to the exercises of chapter 17

Exercises

17.1 fdgfghfsh

(c)

ξ	η	$j = \det(\underline{x}_{,\xi})$
0	0	5.75
0.5	0.5	5.625
1	0	6
1	-1	6.5

Table 17.1. *element 1*

ξ	η	$j = \det(\underline{x}_{,\xi})$
0	0	2
0.5	0.5	1.625
1	0	0.125
-1	1	-1

Table 17.2. *element 2*

For element 2 the point (-1,1) renders a negative Jacobian determinant, meaning that the element is locally inside-out.

17.2 (a)

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{2}(1+\eta)$$

(b)

$$j = \frac{1}{8}x_1$$

(c) When $x_1 < 0$ the element turns inside out which leads to a negative Jacobian determinant.

17.3

$$\begin{aligned} N_1 &= \frac{(\xi+\frac{1}{3})(\xi-\frac{1}{3})(\xi-1)}{(-1+\frac{1}{3})(-1-\frac{1}{3})(-1-1)} \\ &= \frac{9}{16}(\xi+\frac{1}{3})(\xi-\frac{1}{3})(\xi-1) \end{aligned}$$

Other shape functions can be found in the same way.

17.4 (a) $u = 0.4$
(b) $\partial u/\partial x = -0.0863$ and $\partial u/\partial y = 0.0164$

17.5

```
% 6 noded triangle
a=[1,7,5]; b=[1,3,7];
x=4; y=4;

labdet=(a(3)-a(2))*(b(1)-b(2))-(b(2)-b(3))*(a(2)-a(1));

lab11=(a(2)*b(3)-a(3)*b(2))
lab12=(b(2)-b(3))*x
lab13=(a(3)-a(2))*y
lab1=(lab11+lab12+lab13)/labdet;

lab21=(a(3)*b(1)-a(1)*b(3));
lab22=(b(3)-b(1))*x;
lab23=(a(1)-a(3))*y;
lab2=(lab21+lab22+lab23)/labdet;

lab31=(a(1)*b(2)-a(2)*b(1));
lab32=(b(1)-b(2))*x;
lab33=(a(2)-a(1))*y;
```

```
lab3=(lab31+lab32+lab33)/labdet;

icheck=lab1+lab2+lab3 % icheck should be 1

N=[ lab1*(2*lab1-1);
    lab2*(2*lab2-1);
    lab3*(2*lab3-1);
    4*lab1*lab2;
    4*lab2*lab3;
    4*lab1*lab3]

u=[2 ; 5 ; 3 ; 2 ; 0 ; 2]

u44=N'*u
```

leads to: $u44 = 0.8367$

18
Answers to the exercises of chapter 18

Exercises

18.1 (a)

$$\mathrm{tr}(\varepsilon) = \varepsilon_{xx} + \varepsilon_{yy}$$

(b)

$$\mathrm{tr}(\varepsilon) = \varepsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$$

ε_{zz} can be eliminated with the constitutive law

(c) Plane stress: $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}$

(d) Plane strain: $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$

(e)

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{zz} \end{pmatrix} = \begin{bmatrix} \kappa + \frac{4}{3}G & \kappa - \frac{2}{3}G & 0 & \kappa - \frac{2}{3}G \\ \kappa - \frac{2}{3}G & \kappa + \frac{4}{3}G & 0 & \kappa - \frac{2}{3}G \\ 0 & 0 & 2G & 0 \\ \kappa - \frac{2}{3}G & \kappa - \frac{2}{3}G & 0 & \kappa + \frac{4}{3}G \end{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{zz} \end{pmatrix}$$

Let $\kappa - \frac{2}{3}G = \beta$ and $\kappa + \frac{4}{3}G = \alpha$. With $\sigma_{zz} = 0$ and the constitutive law ϵ_{zz} can be written as a function of ϵ_{xx} and ϵ_{yy}. This will lead to:

$$\underline{H} = \begin{pmatrix} \alpha - \frac{\beta^2}{\alpha} & \beta - \frac{\beta^2}{\alpha} & 0 \\ \beta - \frac{\beta^2}{\alpha} & \alpha - \frac{\beta^2}{\alpha} & 0 \\ 0 & 0 & 2G \end{pmatrix}$$

(f) Use

$$\kappa = \frac{E}{3(1-2\nu)}$$

and

$$G = \frac{E}{2(1+\nu)}$$

Stress depends linearly on E.

18.2 (a)

$$\text{top} = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

(b)

$$\text{pos} = \begin{bmatrix} 1 & 2 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

(c)

$$\text{dest} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$$

(d) The matrix can be formed manually, but also by writing a short Matlab program: A program could be:

```
pos = [1 2 5 6 7 8 ; 1 2 3 4 5 6 ]
K=zeros(8);
k1e=zeros(6)+1;
k2e=zeros(6)+2;
%contribution element 1
ii=nonzeros(pos(1,:));
K(ii,ii)=K(ii,ii)+k1e;
%contribution element 2
ii=nonzeros(pos(2,:));
K(ii,ii)=K(ii,ii)+k2e;
K
```

The following assembled matrix can be constructed:

$$\underline{K} = \begin{bmatrix} 1+2 & 1+2 & 2 & 2 & 1+2 & 1+2 & 1 & 1 \\ 1+2 & 1+2 & 2 & 2 & 1+2 & 1+2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 \\ 2+1 & 2+1 & 2 & 2 & 2+1 & 2+1 & 1 & 1 \\ 2+1 & 2+1 & 2 & 2 & 2+1 & 2+1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(e)

```
iinodes=nonzeros(usercurves(1,:));
dofs = nonzeros(dest(iinodes,1);
ux=sol(dofs);
```

(f) q = stiffness matrix , sol is solution vector
reaction forces can be found by:

```
freact  = q*sol                    % Calculate the nodal forces
idof1   = nonzeros(dest(1,:))      % degrees of freedom of node 1
idof2   = nonzeros(dest(2,:))      % degrees of freedom of node 2
freac1  = freact(idof1)            % reaction forces in node 1
freac2  = freact(ifof2)            % reaction forces in node 2
fx      = freac1(1) + freac2(1)    % reaction forces in x-direction
fy      = freac1(2) + freac2(2)    % reaction forces in y-direction
```

18.3 The following matlab program gives the answers:

```
clear all
kappa=2.0; G=1.0;
x=0.5; y=0.5;

N=[0.25*(1-x)*(1-y)
   0.25*(1+x)*(1-y)
   0.25*(1+x)*(1+y)
   0.25*(1-x)*(1+y)]

disp('test sum N'); sum(N)

dNdx=[-0.25*(1-y)
       0.25*(1-y)
```

```
       0.25*(1+y)
      -0.25*(1+y)]

 dNdy=[-0.25*(1-x)
       -0.25*(1+x)
        0.25*(1+x)
        0.25*(1-x)]

disp('test sum dNdxy');
sum(dNdx)
sum(dNdy)

 B=[dNdx(1)  0       dNdx(2) 0       dNdx(3) 0       dNdx(4) 0
    0        dNdy(1) 0       dNdy(2) 0       dNdy(3) 0       dNdy(4)
    dNdy(1)  dNdx(1) dNdy(2) dNdx(2) dNdy(3) dNdx(3) dNdy(4) dNdx(4)]

 u=[0 0 0 0 1 0 1 0]'
 eps=B*u
 H1=kappa*[1 1 0 ; 1 1 0 ; 0 0 0]
 H2=(G/3)*[4 -2 0 ; -2 4 0 ; 0 0 3]
 H=H1+H2;
 sig=H*eps
```

18.4 (a) for an element with 4 nodes:

```
inodes=top(ielem,1:4)
idof=dest(inodes,:);
u=sol(idof);
```

This leads to a matrix `u` with two columns. The first column gives the x-displacements, the second column the y-displacements

(b) and (c) This can be done when the shape functions $N(\xi, \eta)$ are known and the coordinates ξ and η of the integration point are known. First the displacements have to be stored in one column:

```
ucol=zeros(8,1)
for i=1:4
ucol(2*i-1)= u(i,1)
```

```
ucol(2*i)  = u(i,2)
end
```

Then the strain can be calculated by means of (also see exercise: 5.3):

```
eps=B*ucol
sig=H*eps
```

18.5 In a pure bending test the stress in x-direction has to be a linear function of y. In the current example the stress is zero at $y = 0.5$, positive (extension) for $y > 0.5$ and negative (compression) for $y < 0.5$. Both quadratic elements show this behavior, but the linear triangle can only give a constant stress in an element. This means that it is too stiff.

Several ways of generating the meshes are possible. The following results, leading to an error of less than 1 % could be obtained, using different meshes:

Type of element	element divison ($L \times H$)	CPU time [s]
Quadrilateral	5×1	0.047
Quadratic Triangle	8×1	0.172
Bi-linear	100×3	1.703
Linear Triangle	100×20	22.390

18.6 (a) In the shear test the curves 2 and 4 are kept straight. This requires external reaction forces acting on these curves.

(b) In a real experiment the curves 2 and 4 are free. This means that the stress at the surface is zero. Near the corners this will lead to a problem, because in the limiting case near the corner force equilibrium is not satisfied. This has to result in a stress concentration near the corner and a highly inhomogeneous stress and strain field. This edge effect becomes less important when the length L of the sample is higher than the height H.

(c) To obtain a shear modulus G with 10 % accuracy an aspect ratio of $L/H = 5$ is needed.