Biomechanics

Concepts and Computation

Extra exercises and Answers

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Extra exercises of chapter 1

1.1 Exercises

1.1 Consider a Cartesian $xyz$-coordinate system, specified with the orthonormal basis vectors \{\vec{e}_x, \vec{e}_y, \vec{e}_z\}.

(a) Within this system a plane is spanned by the vectors $\vec{a}$ and $\vec{b}$, specified by:

\[
\vec{a} = 16\vec{e}_y - 15\vec{e}_x
\]
\[
\vec{b} = 20\vec{e}_z - 15\vec{e}_x
\]

Determine the unit normal to this plane (vector with length 1 perpendicular to the plane). Why is the solution not unique?

(b) In addition the following vectors $\vec{b}, \vec{c}$ en $\vec{d}$ are defined:

\[
\vec{b} = 3\vec{e}_x + 2\vec{e}_y
\]
\[
\vec{c} = 5\vec{e}_x - \vec{e}_z
\]
\[
\vec{d} = \vec{e}_x + \vec{e}_z
\]

Express $(\vec{b}\vec{c}) \cdot \vec{d}$ in the basis vectors \{\vec{e}_x, \vec{e}_y, \vec{e}_z\}, where $(\vec{b}\vec{c})$ is the dyadic product of the vectors $\vec{b}$ and $\vec{c}$. 

1.1 (a)

\[ \vec{n} = \pm (0.64\hat{e}_x + 0.6\hat{e}_y + 0.48\hat{e}_z) \]

The solution is not unique; the normal can be oriented in two opposite directions.

(b)

\[ (\vec{b} \vec{c}) \cdot \vec{d} = 12\hat{e}_x + 8\hat{e}_y \]
Extra exercises of chapter 2
3
Extra exercises of chapter 3

3.1 Exercises

3.1 A board with length $6a$ is fixed to the wall in point $A$ with a hinge. The board is able to rotate freely around the joint. In a point $B$ at a distance $3a$ of $A$, the board is kept in horizontal position by means of a cable, tied to the board in $B$ and fixed to the rigid wall in point $C$. The mass of the board is $M_P$ (see figure). Cable and board can be assumed to be rigid (undeformable) structures. A box, with length $2a$ and mass $M_K$ is placed on the board. The front of the box is exactly placed at the front edge of the board. The gravitation acceleration is $g$.

![Diagram](image)

Fig. 3.1.

(a) Draw a free body diagram of the construction to enable the calculation of the reaction forces in points $A$ and $C$.

(b) Calculate the reaction forces in point $A$ and $C$.

3.2 An athlete hangs, arms wide and legs horizontal, in the flying-rings.
The athlete is motionless (see figure). The total body mass of the athlete is $M_L$. The mass of the arm is $M_A$. The total length of the arm, from the shoulder joint to his hand is $a$. The gravitational acceleration is $g$. To make a rough estimate of the forces and moments in the shoulder muscles a simple model is proposed for the arm in the form of a bar that is rigidly clamped near the shoulder (point S) and loaded with a vertical force $F$ at the other end.

(a) Determine the magnitude of the force $F$.

(b) Determine the reaction forces and moments at point S. Assume that the centre of gravity of the arm lies in the middle, at a distance $\frac{1}{2}a$ from the point S.

To stay in balance, with the legs horizontal, it is necessary for the athlete to incline forward a little. In that case his trunk is at a small angle $\alpha$ with the vertical axis. To determine this angle we use the model that is depicted in the figure below of two rigidly connected bars connected to a fixed point R with a hinge (the construction can rotate freely around the point R. The length of the legs is $d$ the length of the trunk $c$. The mass of the trunk is $M_R$ with the centre of gravity in the middle. The mass of the legs is $M_B$, with the centre of gravity in the middle as well.

(c) Assume that in a state of equilibrium the legs are horizontal. Determine the angle $\alpha$ for which this is the case a as a function of $M_B$, $M_R$, $c$ en $d$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\end{figure}
3.1

(a) See figure.

(b)

\[ H_A = -g \left( \frac{3}{4}M_P + \frac{5}{4}M_K \right) \]
\[ V_A = -\frac{2}{3}gM_K \]
\[ H_C = g \left( \frac{3}{4}M_P + \frac{5}{4}M_K \right) \]
\[ V_C = g \left( M_P + \frac{5}{3}M_H \right) \]
3.2 answers

(a) \[ F = \frac{1}{2} g M_L \]

(b) See figure.

\[ H_S = 0 \]
\[ V_S = g(M_A - \frac{1}{2} M_L) \]
\[ M_S = \frac{1}{2} g a (M_A - M_L) \]

(c) \[ \sin(\alpha) = \frac{dM_B}{c M_R + 2 c M_B} \]
Extra exercises of chapter 4

not yet available
5

Extra exercises of chapter 5

5.1 Exercises

5.1 In the figure a schematic of a Kelvin-Voigt model is given. The model comprises a spring with constant $c$ and a dashpot with constant $c_\eta$.

(a) What is the strain after a very long time (meaning when $t \to \infty$), if the system is loaded with a step force $F_0$ at $t = 0$?
(b) Draw a schematic of the strain history $\varepsilon(t)$ after applying the load history as depicted in the figure below. Also show the limit in the figure that was calculated in item (a).

(c) Draw a schematic of the force $F(t)$ after applying a strain rate $\dot{\varepsilon}$ as given in the figure below. Give an expression for the force $F(t)$ for $t > t^*$.
5.2 Consider the Maxwell model in the figure below consisting of a spring with constant $c$ and a dashpot with constant $c_n$.

(a) Make a sketch of the strain $\varepsilon_s$ of the spring and a sketch of the strain $\varepsilon_d$ of the dashpot as a function of time, if a force history is prescribed as indicated in the figure below. Give the values of the strains immediately after the application of the force.

(b) Make a sketch of the force $F_s$ in the spring and a sketch of the force $F_d$ in the dashpot as a function of time if a strain history is prescribed as sketched in the figure below. Give the values of the forces immediately after the application of the strain.
5.3 In the figure below four different, linear spring-dashpot models have been drawn that could be used to describe the mechanical behaviour of a biological material. The spring and dashpot constants are given in the figure. Each of the models represents some kind of linear viscoelastic behaviour.

![Diagram of linear spring-dashpot models](image)

(a) (b) (c) (d)

Assume that the length of a tissue specimen is changed by means of a step \( \varepsilon_0 \) as given in the figure below.

![Diagram of step change in length](image)

Draw the applied load \( F(t) \) as a function of time \( t \) for each of the models. Also give relevant values for the initial values of \( F \), asymptotes etc. in the figures.
5.4 For a linear visco-elastic material it is possible to calculate the strain \( \varepsilon(t) \) at a given time for a prescribed uniaxial load \( F(t) \) provided the creep function \( J(t) \) is known. Assume, that:
\[
J(t) = C(t + \alpha),
\]
with \( C \) and \( \alpha \) positive material constants. It is assumed that for \( t < t_1 \) the material is not stretched. At \( t = t_1 \) the following load is applied to the material:
\[
F(t) = \bar{F}[H(t - t_1) - H(t - t_2)],
\]
with \( \bar{F} \) constant and \( t_2 > t_1 \), and whereby \( H(\cdot) \) represents the Heaviside step function.
(a) What is the physical interpretation of the creep function \( J(t) \)?
(b) Sketch the force as a function of time.
(c) Determine the strain \( \varepsilon(t) \) as a function of time and represent the result in a drawing for the case that \( \bar{F} \) is positive. Note that for \( t > t_2 \) the strain is constant.

5.5 The strain \( \varepsilon(t) \) of a linear visco-elastic material can be calculated by means of the creep function \( J(t) \) provided the load history \( F(t) \) is known. Assume, that:
\[
J(t) = J_0 + C \left[ 1 - e^{(-t/\tau)} \right],
\]
with: \( C = \frac{\bar{F}}{\tau} \) and \( J_0 \) and \( \tau \) material constants. No deformation or force is applied to the material for \( t < 0 \). From the time \( t = 0 \) the material is loaded with a closed-loop loading process specified by:
\[
F(t) = \bar{F}[H(t) - 2H(t - \beta \tau) + H(t - 2\beta \tau)],
\]
with \( H(\cdot) \) the Heaviside step function and \( \bar{F} \) a positive constant, which is a measure for the magnitude of the force. The constant \( \beta \) is a measure for the duration of the loading process.
(a) Give the load as a function of time without using the Heaviside function.
(b) Determine the strain \( \varepsilon(t) \) as a function of time and draw a sketch of the result. Note that for \( t \gg 2\beta \tau \) the strain approaches to zero again.
(c) Prove that, independent of the numerical values of the relevant parameters, for \( t \geq 2\beta \tau \) always holds: \( \varepsilon(t) < 0 \).
5.6 For a linear visco-elastic material the following relaxation function $G(t)$ is given:

$$G(t) = Ge^{-\alpha t},$$

with $G$ and $\alpha$ positive material constants. No deformation or force is applied to the material for $t < 0$. The material is uniaxially stretched. The history of stretching the material is shown in the figure below. The magnitude of the strain is $\varepsilon_b$.

(a) Make a sketch of the force $F(t)$ in the material as a function of time for $0 \leq t \leq 6$.
(b) Calculate the force at time $t = 3$.
(c) Calculate the force at time $t = 5$. 
5.7 A lot of research is aimed at developing bio-degradable bone screws (see figure).

One disadvantage for the application of this type of polymer screws is their visco-elastic material behaviour. In a laboratory test a researcher fixes two stiff plates (they may be considered as rigid bodies) to each other. Initially the screw is in a stress and strain free state. The screw is manufactured from a linearly visco-elastic material. The mechanical behaviour of the material can be described by means of the relaxation function $G(t)$:

$$G(t) = 5 \, e^{(-0.07 \, t^{0.3})}$$

where the time $t$ is expressed in [s] and $G(t)$ in [$10^6$ N].

At a certain time the screw is tightened (fast) such that the transmitted extensional force equals 1 [kN].

(a) Calculate the axial strain $\varepsilon_1$ in the screw after it is tightened.
(b) Calculate the force $F_1$ in the screw 24 hours after it is tightened.

After 24 hours the screw is tightened again fast, such that the extensional force again equals 1 [kN].

(c) Calculate the associated strain $\varepsilon_2$ in the screw.
(d) Calculate the force $F_2$ in the screw again 24 hours later.
5.8 In a laboratory a tendon with a length of 50 [mm] is mechanically tested. Initially the tendon is in a stationary state (the force in the tendon equals zero and the tendon has a constant length) which is used as a reference state. The mechanical behaviour of the tendon can be described with a linear visco-elastic model, characterized by the creep function \( J(t) \) according to:

\[
J(t) = J_0(2 - e^{-t/\tau}) ,
\]

with \( \tau = 250 \) [s], \( J_0 = 0.0001 \) [N\(^{-1}\)]. In an experimental set-up the tendon is loaded during 500 [s] with a force of 100 [N] and after that fully unloaded.

(a) Calculate the strain \( \varepsilon_1 \) of the tendon immediately after the load is removed.

(b) Calculate the strain \( \varepsilon_2 \) of the tendon 100 [s] after the load has been removed.

5.9 To determine the mechanical properties of biological tissues often a rheometer is used (see figure). A cylindrical tissue specimen is clamped between two plates. Subsequently the top plate is rotated and the moment (torque) that is acting on the bottom plate is measured. Because the deformation is completely determined by the geometry and not by the mechanical behaviour, this is called a visco-metric deformation. For the rotational angle and the measured torque the material behaviour can be derived. A researcher has determined the properties of fat tissue in this way. At small strains the behaviour can be described with a linear visco-elastic model according to:

\[
F(t) = \int_{\xi=-\infty}^{t} G(t - \xi) \dot{\varepsilon}(\xi) \, d\xi ,
\]

with \( G(t) \) the relaxation function, \( F \) a force with dimension [N] and
\( \dot{\varepsilon} \) the strain rate \([s^{-1}]\). The following relaxation function could be derived from the experiments:

\[
G(t) = 10 + 5 \ e^{-t/5} \quad \text{expressed in \text{kPa}}
\]

To the fat tissue a strain is applied as depicted in the figure at the right. By using the Heavyside function \(H(t)\) this can be formulated as:

\[
\varepsilon(t) = 0.001 \ [H(t) - H(t - 10)]
\]

(a) Give an expression for the force \(F_\tau(t)\) as a consequence of the applied strain for \(t < 10\ [s]\).

(b) What is the force, immediately after the strain is removed at \(t = 10^+\ [s]\)? Is it a compressive or extensional force?

(c) What is the force at \(t = 30\ [s]\)?
5.2 Answers

5.1

(a) If \( t \to \infty \) then \( \varepsilon \to \frac{F_0}{\varepsilon} \)

(b) See figure.

(c) See figure.
Extra exercises of chapter 5

5.2

<table>
<thead>
<tr>
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</tr>
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</tr>
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<tr>
<td>$(b)$</td>
<td>$(b)$</td>
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<td>$c \varepsilon_0$</td>
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</table>

5.3

<table>
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<td>$F$</td>
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<tr>
<td>$k\varepsilon_0$</td>
<td>$2k\varepsilon_0$</td>
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<tr>
<td>$\frac{1}{2}k\varepsilon_0$</td>
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</tbody>
</table>

<table>
<thead>
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<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$F$</td>
</tr>
<tr>
<td>$k\varepsilon_0$</td>
<td>$2k\varepsilon_0$</td>
</tr>
</tbody>
</table>
5.4

(a) The creep function $J(t)$ can be interpreted as the strain $\varepsilon(t)$ resulting from a unit step in the force $F$ at time $t = 0$, so:

$$\varepsilon(t) = J(t)$$

is the response on $F(t) = H(t)$

(b) for $t < t_1$ : $F(t) = 0$
for $t_1 \leq t < t_2$ : $F(t) = F$
for $t \geq t_2$ : $F(t) = 0$

(c) for $t < t_1$ : $\varepsilon(t) = 0$
for $t_1 \leq t < t_2$ : $\varepsilon(t) = FJ(t - t_1) = C F(t - t_1 + \alpha)$
for $t \geq t_2$ : $\varepsilon(t) = F[J(t - t_1) - J(t - t_2)] = C F(t_2 - t_1)$
Extra exercises of chapter 5

5.5

(a) for \( t < 0 \) : \( F(t) = 0 \)
for \( 0 \leq t < \beta \tau \) : \( F(t) = F \)
for \( \beta \tau \leq t < 2\beta \tau \) : \( F(t) = -F \)
for \( t \geq 2\beta \tau \) : \( F(t) = 0 \)

(b) \[
\varepsilon(t) = F \cdot \frac{J_0}{2} \left[ 3 - e^{(\frac{\tau}{t})} \right] \quad \text{for} \quad 0 \leq t < \beta \tau \\
\varepsilon(t) = F \cdot \frac{J_0}{2} \left[ -3 - e^{(\frac{\tau}{t})} + 2e^{(\beta - \frac{\tau}{t})} \right] \quad \text{for} \quad \beta \tau \leq t < 2\beta \tau \\
\varepsilon(t) = F \cdot \frac{J_0}{2} \left[ -e^{(\frac{\tau}{t})} + 2e^{(\beta - \frac{\tau}{t})} - e^{(2\beta - \frac{\tau}{t})} \right] \quad \text{for} \quad t \geq 2\beta \tau 
\]

5.6

(a) \[
2 \leq t \leq 4 \quad \Rightarrow \quad F(t) = \varepsilon_b \cdot G \cdot e^{-\alpha(t-2)} \\
4 \leq t \leq \infty \quad \Rightarrow \quad F(t) = \varepsilon_b \cdot G \cdot e^{-\alpha(t-2)} - \varepsilon_b G e^{-\alpha(t-4)} 
\]

(b) \( F(t = 3) = \varepsilon_b G e^{-\alpha} \)

(c) \( F(t = 5) = \varepsilon_b G e^{-3\alpha} - \varepsilon_b G e^{-\alpha} \)

5.7

(a) \( \varepsilon_1 = 0.2 \cdot 10^{-3} \) [\( \cdot \)]

(b) \( F_1 = 120 \) [N]

(c) \( \varepsilon_2 = 0.2 \cdot 10^{-3} + 0.18 \cdot 10^{-3} \) [\( \cdot \)]

(d) \( F_2 = 179 \) [N]
5.8
(a) \( \varepsilon_1 = 100 \times 0.0001(e^{-500/250} - 1) = 0.0086 \) [-]
(b) \( \varepsilon_2 = 100 \times 0.0001(e^{-600/250} + e^{-100/250}) = 0.0058 \) [-]

\[
\begin{align*}
\text{Force} & \quad \text{100 [N]} \\
\text{Strain} & \quad 0.01(2 - e^{-500/250}) = 0.0186 \\
& \quad 0.0186 - 0.01 = 0.0086
\end{align*}
\]

5.9
(a) \( F_\tau(t) = 0.001 \left( 10 + 5e^{-t/5} \right) \) [kN]
(b) \( F_\tau(t = 10^+) = -4.32 \) [N] (compressive)
(c) \( F_\tau(t = 30) = -0.0796 \) [N]
6

Extra exercises of chapter 6

6.1 Exercises

6.1 A research assistant wants to determine the mechanical properties of heart valve tissue. For this, a long slender rectangular tissue specimen is cut from a valve (see figure) and clamped in a uniaxial-testing machine. After clamping it appears that the specimen is a little narrower in the middle (point M) than it is near the clamps (point K). The assistant decides to make a model. Because of symmetry reasons only half of the strip is analyzed, resulting in the model below. The specimen is considered to have a rectangular cross section with constant thickness $t$ and a position dependent height $h(x)$, for which:

$$h(x) = h_0(1 + \frac{\alpha x}{\ell}) ,$$
with \( h_0 \) and \( \alpha \) positive constants. The material is assumed to be homogeneous and linear elastic with Young’s modulus \( E \). The length of the specimen (in fact half of the specimen) is \( \ell \). At the point \( x = \ell \) an external force \( F \) is applied.

(a) What are the differential equation and boundary conditions, describing the displacement \( u(x) \) of every point in the specimen?

(b) Which cross section in the material has the highest stress?

(c) Calculate the displacement at \( x = \ell \)?

(d) Where is the highest strain found in the specimen and how high is this strain?
Extra exercises of chapter 6

6.2 answers

(a)

\[
\frac{d}{dx} \left( EA \frac{du}{dx} \right) = 0 ,
\]

with \( A = th_0(1 + \alpha x/\ell) \).

The boundary conditions are: \( u(0) = 0 \) and \( EA \frac{du}{dx} \big|_{x=\ell} = F \).

(b) At \( x = 0 \), because the cross section is smallest at that point.

(c)

\[
u(x) = \frac{F \ell}{\alpha t h_0 E} \ln(1 + \frac{\alpha x}{\ell})
\]

(d)

\[
\varepsilon_{max} = \varepsilon(x = 0) = \frac{F}{t h_0 E}
\]
7

Extra exercises of chapter 7

7.1 Exercises

7.1 Consider a cubic domain in three-dimensional space, given as:

\[-2R \leq x \leq 2R\]
\[-2R \leq y \leq 2R\]
\[-2R \leq z \leq 2R\]

with \( R \) a constant and \( x, y \) and \( z \) the Cartesian coordinates. Within this domain a temperature field is defined by:

\[ T = T(x, y, z) = T_0 \left( 1 - \frac{x + y}{6R} \right) , \]

with \( T_0 \) the temperature in the origin. In the domain a circle is defined by means of the parameter curve:

\[ x = R \cos(\phi) \]
\[ y = R \sin(\phi) \]
\[ z = 0 , \]

with \( \phi \) the angle between the positive \( x \)-axis and the position vector to points on the circle \((0 \leq \phi \leq 2\pi)\).

Determine the directional derivative of the temperature along the circle as a function of \( \phi \) and calculate the angles \( \phi \) for which the absolute value of this directional derivative is maximal.
Extra exercises of chapter 7

7.2 In a Cartesian $xyz$-coordinate system a body is considered, that rotates with constant angular velocity $\omega$ around the $y$-axis. Consider the point $P$ in this body with the current position vector:

$$\vec{x}_P = 3\ell \, \vec{e}_x - 5\ell \, \vec{e}_y + 4\ell \, \vec{e}_z,$$

with $\ell$ constant.

Determine the magnitude of the velocity of point $P$.

7.3 Within a spherical domain (with radius $2R$) in three-dimensional space, defined by $x^2 + y^2 + z^2 \leq 4R^2$, with $x$, $y$ and $z$ the Cartesian coordinates, a temperature field is given by:

$$T = T(x, y, z) = T_0 \left(1 + \frac{x + y + z}{8R}\right),$$

with $T_0$ a constant.

In the domain a circle is defined in the $xy$-plane, given by: $x^2 + y^2 = R^2$ and $z = 0$. On this circle a point $P$ with $x_p = 3R/5$ and $y_p = 4R/5$ is given.

Determine the derivative of the temperature field in the direction of the circle in point $P$. Why is the answer to this question not unique?

7.4 A line $\ell$ connects two fixed point $A$ and $B$ in space. The position vectors of those points in a Cartesian $xyz$-coordinate system are given as:

$$\vec{x}_A = 6 \, \vec{e}_x,$$
$$\vec{x}_B = 9 \, \vec{e}_x - 4 \, \vec{e}_y + 12 \, \vec{e}_z.$$

A body rotates around this line $\ell$ with constant angular velocity $\omega$.

Consider a point $P$ of this body that is currently in the origin of the coordinate system.

Determine the magnitude of the velocity of point $P$. 

7.5 In a Cartesian $xyz$-coordinate system a (two-dimensional) stationary fluid flow is considered. The velocity field is described by:

$$\vec{v}(x, y, z) = v_x(x, y) \vec{e}_x + v_y(x, y) \vec{e}_y$$

At three points in space (A, B and C) the velocity is measured (see table). The coordinates are in [cm], the velocities in [cm/s].

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
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<td>9</td>
</tr>
</tbody>
</table>

Determine the velocity gradient tensor $\mathbf{L}$ assuming that this tensor is constant (independent of $x$, $y$ and $z$).

7.6 In a Cartesian $xyz$-coordinate system, with basis vectors $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$, the particle track of a material point $P$ is given by means of a parameter description:

$$\vec{x} = \vec{x}(\xi) = x(\xi)\vec{e}_x + y(\xi)\vec{e}_y + z(\xi)\vec{e}_z,$$

with $0 \leq \xi \leq 4\pi$ and:

$$x(\xi) = R \cos(\xi)$$
$$y(\xi) = R \sin(\xi)$$
$$z(\xi) = 2R \xi,$$

with $R$ a constant. The point $P$ moves along the track with a velocity vector $\vec{v}$ with constant magnitude $V$ (positive constant): $|\vec{v}| = V$.

Determine the velocity vector $\vec{v}$ of the point $P$ at the time that $P$ is spatial position, defined by $\xi = \pi/2$. 

7.7 Consider a rectangular area ABCD in the $xy$-plane. For the geometry see the figure (Coordinates in $[\text{cm}]$). Within the area ABCD a two-dimensional temperature field is found with constant gradient, yielding:

$$\frac{\partial T}{\partial x} = 3 \ [^\circ \text{C}/\text{cm}] ; \quad \frac{\partial T}{\partial y} = -2 \ [^\circ \text{C}/\text{cm}]$$

The temperature in point A is 40 $[^\circ \text{C}]$.

Calculate the temperature in point C.

7.8 In the three-dimensional space a temperature field is described with the equation:

$$T(x, y, z) = T_0 + T_1 \frac{r}{L}$$

with $x$, $y$, and $z$ the Cartesian coordinates, $T_0$ and $T_1$ constants with the dimension temperature and $L$ a positive constant length. Consider a straight line $\ell$ from a point A to point B. The position vectors of these points are given:

$$\vec{x}_A = -L\vec{e}_x + L\vec{e}_y$$
$$\vec{x}_B = L\vec{e}_x + 2L\vec{e}_y + 2L\vec{e}_z$$

Determine in point B the derivative of the temperature field in the direction of the line $\ell$. 

7.9 For a certain material point the position in space is given as a function of time \( t \) by means of the position vector:

\[
\mathbf{x} = c \begin{bmatrix} t(t+2) \\ 3t \\ t(t-6) \end{bmatrix},
\]

with \( c \) a constant and \( t \) expressed in [s].

Calculate, the unit vector tangent to the track of the particle at \( t = 5 \) [s].

7.10 On a domain in three-dimensional space, given by \(-2\ell \leq x \leq 2\ell, -2\ell \leq y \leq 2\ell, -2\ell \leq z \leq 2\ell\) a temperature field is given by:

\[
T = 15 - \frac{(x^2 + 2y^2)}{\ell^2}
\]

Determine the directional derivative of the temperature \( T \) in the direction of a unit vector \( \mathbf{e} \) that is given as a function of the coordinates:

\[
\mathbf{e} = \frac{3}{5} e_x + \frac{4}{5} e_y
\]

7.11 On a two-dimensional domain, given by: \( 0 \leq x < 3, 0 \leq y \leq 3 \) the following temperature field is found:

\[
T = 4 \left( 1 - \frac{x + y}{6} \right)
\]

In addition a parameter curve in space is given as:

\[
\mathbf{x}(\xi) = (\xi^2 - 1)e_x + (\xi + 1)e_y
\]

Determine the directional derivative of the temperature \( T \) along the curve as a function of \( \xi \).
Extra exercises of chapter 7

7.2 Answers

\[ \frac{dT}{de} = \frac{T_0}{6R} [\sin(\phi) - \cos(\phi)] \]

This directional derivative is maximal/minimal for \( \phi = 3\pi/4 \) and \( \phi = 7\pi/4 \).

7.2 The magnitude of the velocity in point P is: \( 5 \omega \ell \).

7.3

\[ \frac{dT}{de} = \pm \frac{T_0}{40R} \]

7.4 The magnitude of the velocity of point P is: \( \frac{24}{13} \omega \sqrt{10} \)

7.5

\[ \mathbf{L} = 7 \mathbf{\hat{e}}_x \mathbf{\hat{e}}_y + 3 \mathbf{\hat{e}}_y \mathbf{\hat{e}}_x + 5 \mathbf{\hat{e}}_y \mathbf{\hat{e}}_y \]

7.6

\[ \mathbf{v} = \frac{1}{5} \sqrt{5}(-\mathbf{\hat{e}}_x + 2\mathbf{\hat{e}}_z) \]

7.7 The temperature in point C is 48 \( ^\circ C \)

7.8 The derivative of \( T \) in the direction of line \( \ell \) in point B is:

\[ \frac{dT}{de} = \frac{8 T_1}{5 \ell} \]

7.9

\[ \xi = \frac{1}{13} \begin{bmatrix} 12 \\ 3 \\ 4 \end{bmatrix} \]

7.10

\[ \frac{dT}{de} = \mathbf{\hat{e}} \cdot \nabla T = -\frac{1}{5\ell^2} (6x + 16y) \]
7.11

\[
\frac{dT}{d\xi} = \frac{dx}{d\xi} \cdot \nabla T = -\frac{4}{3} \xi - \frac{2}{3}
\]

\[
\frac{dT}{d\varepsilon} = -\frac{\frac{2}{3} \xi + \frac{2}{3}}{\sqrt{4\xi^2 + 1}}
\]
8.1 Consider a material element as given in the figure below. The stress state in the element is homogeneous, defined by means of the stress matrix $\sigma$ according to:

$$
\sigma = \begin{bmatrix}
150 & 0 & 0 \\
0 & 200 & 100 \\
0 & 100 & 300 \\
\end{bmatrix}
$$

expressed in [MPa]

Calculate the magnitude $s_l$ of the shear stress $\vec{s}_l$ acting upon the plane ABCD.
8.1 Exercises

33

8.2 In a material point the stress state is specified by means of the stress matrix $\sigma$ according to:

$$
\sigma = \begin{bmatrix}
36 & 0 & 48 \\
0 & 100 & 0 \\
48 & 0 & 64 \\
\end{bmatrix} \text{ [MPa]}
$$

Calculate the maximum shear stress $(\sigma_t)_{\text{max}}$.

8.3 For a material point in a Cartesian $xyz$-coordinate system the stress tensor is specified as:

$$
\sigma = 2\vec{e}_x\vec{e}_x + 2\vec{e}_y\vec{e}_y + 4\vec{e}_z\vec{e}_z \quad \text{in [MPa]}
$$

Prove, that upon all arbitrary planes parallel to the $z$-axis only a normal stress $\sigma_n = 2$ [MPa] is acting and no shear stress $(\sigma_s = 0)$.

8.4 For a material cube in a Cartesian $xyz$-coordinate system the stress tensor is given by:

$$
\sigma = \sigma_{xx} \vec{e}_x\vec{e}_x + \sigma_{yy} \vec{e}_y\vec{e}_y + \sigma_{zz} \vec{e}_z\vec{e}_z + 2(\vec{e}_x\vec{e}_y + \vec{e}_y\vec{e}_x + \vec{e}_x\vec{e}_z + \vec{e}_z\vec{e}_x + \vec{e}_y\vec{e}_z),
$$

so, the shears stresses have a quantitative value (in [MPa]), but the normal stresses are not specified yet.

Is it possible to give the normal stresses a value such that the three principal stresses are equal to each other ($\sigma_1 = \sigma_2 = \sigma_3$)? If the answer is yes, the question is how? If the answer is no, the question is, why not?

8.5 For a material particle in a Cartesian $xyz$-coordinate system the following stress tensor (expressed in [MPa]) is given:

$$
\sigma = 5 \vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z + 4\vec{e}_y\vec{e}_z + \vec{e}_z\vec{e}_x + 4\vec{e}_z\vec{e}_y
$$

In addition, it is given that the middle principal stress $\sigma_2 = 3$ [MPa].

Determine the, to this principal stress associated, principal stress direction vector $\vec{n}_2$, normalized to a unit length.
Extra exercises of chapter 8

8.6 In a material point the following stress state is found:

\[ \sigma = -pI + \sigma^d, \]

with:

\[ \sigma^d = q(4\varepsilon_x\varepsilon_x - \varepsilon_y\varepsilon_y - 3\varepsilon_z\varepsilon_z), \]

with \( I \) the unit tensor and with: \( p = 3 \) [MPa] and \( q = 2 \) [MPa].

Determine the equivalent stress \( \sigma_T \) according to Tresca.

8.7 Consider a wedge shaped piece of material, positioned in a Cartesian \( xyz \)-coordinate system as depicted in the figure.

Perpendicular to the plane ABCD an external load is acting in the form of a known pressure \( p \). Attention is focussed on the stress matrix \( \sigma \) (with components \( \sigma_{xx}, \sigma_{xy}, \ldots, \sigma_{zz} \)) for the material bounded to the plane ABCD.

The components of this stress matrix have to satisfy three scalar equations. Derive these relations!
8.8 Consider a material volume in the form of a tetrahedron ABCD. The coordinates of the corner point, with respect to a Cartesian $xyz$-coordinate system are given in the following table (lengths expressed in [mm]).

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>20</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y$</td>
<td>0</td>
<td>16</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$z$</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td>0</td>
</tr>
</tbody>
</table>

In the element the stress state is homogenous, defined by means of the stress matrix $\sigma$ according to:

$$
\sigma = \begin{bmatrix}
50 & 0 & 0 \\
0 & 50 & -25 \\
0 & -25 & 0 \\
\end{bmatrix}
$$

expressed in [MPa]

Calculate the stress vector $\sigma$ working on the plane ABC.

8.9 In a material point the stress state is specified by means of the stress matrix $\sigma$ according to:

$$
\sigma = \begin{bmatrix}
4 & 2 & 1 \\
2 & 7 & -2 \\
1 & -2 & 8 \\
\end{bmatrix}
$$

expressed in [MPa]

In the same point a plane is considered with unit outward normal $n$:

$$
n = \frac{1}{\sqrt{3}} \begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}
$$

Show that no shear stresses are acting on this plane.
8.10 For a material element the stress state is defined by means of the stress matrix $\sigma$:

$$
\sigma = \begin{bmatrix}
6 & 1 & -2 \\
1 & 2 & 2 \\
-2 & 2 & 5 \\
\end{bmatrix}
$$

expressed in [MPa]

It can be derived that the associated principal stresses are:

$$
\sigma_1 = (4 - \sqrt{14}) \quad \sigma_2 = 5 \quad \sigma_3 = (4 + \sqrt{13})
$$

Determine the unit outward normal $\mathbf{n}_2$ to the plane upon which the principal stress $\sigma_2 = 5$ [Mpa] is acting.

8.11 Consider a cube of material, with the edges oriented in the direction of the axes of a $xyz$-coordinate system (see figure). In the figure also the normal and shear stresses are given (expressed in [MPa]) that act on the side faces of the cube.

Determine the equivalent stress $\sigma_M$ according to von Mises.
8.2 Answers

8.1

\[ s_t = 20 \text{ [MPa]} \]

8.2

\[ (s_t)_{max} = 50 \text{ [MPa]} \]

8.3 All arbitrary planes parallel to the z-axis are characterized by the normal vector \( \vec{n} \):

\[ \vec{n} = \cos(\phi)\vec{e}_x + \sin(\phi)\vec{e}_y \quad \text{with: } 0 \leq \phi < \pi/2 \]

From: \( \vec{s} = \sigma \cdot \vec{n} \) it follows that the stress vector is in the direction of the normal vector \( \vec{n} \). The expression \( s_n = \vec{s} \cdot \vec{n} \) leads to a normal stress \( s_n = 2 \text{ [MPa]} \).

8.4 Impossible; in the case of three equal principal stresses Mohr’s circles degenerate to a single point, meaning that no plane exists upon which shear stresses are acting.

8.5

\[ \bar{n}_2 = \frac{1}{3} \sqrt{3}(\bar{e}_x - \bar{e}_y - \bar{e}_z) \]

8.6

\[ \sigma_T = 14 \text{ [MPa]} \]

8.7

\[ 3 \sigma_{xx} + 4 \sigma_{xz} = -3p \]
\[ 3 \sigma_{xy} + 4 \sigma_{yz} = 0 \]
\[ 3 \sigma_{xz} + 4 \sigma_{zz} = -4p \]
38 Extra exercises of chapter 8

8.8
\[ \vec{s} = \begin{bmatrix} 24 \\ 14 \\ -15 \end{bmatrix} \text{ expressed in [MPa]} \]

8.9 The stress vector \( \vec{s} \) acting upon this surface is:
\[ \vec{s} = \frac{7}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ expressed in [MPa]} \]
which is in the direction of the normal \( \vec{n} \).

8.10
\[ \vec{n}_2 = \pm \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

8.10
\[ \sigma_M = 5\sqrt{6} \text{ [MPa]} \]
9

Extra exercises of chapter 9

9.1 Exercises

9.1 Consider a fluid flowing through the three-dimensional space (with an
xyz-coordinate system). In a number of fixed points in the space the
temperature $T$ and the fluid velocity $v$ are measured as a function of
time $t$. Based on these measurements the temperature and velocity
can be approximated as follows:

$$T = (ax + by)e^{(1-ct)}$$

with $a$, $b$ and $c$ constant

$$v = \begin{bmatrix}
0 \\
\alpha x \\
\beta y
\end{bmatrix}$$

with $\alpha$ and $\beta$ constant

Determine with these relations the material time derivative $\dot{T}$ of the
temperature $T$ as a function of the spatial coordinates and time.

9.2 Consider a cylinder with a circular cross section, positioned in an $xyz$-
coordinate system as shown in the figure below. The bottom surface
is fixed in space. The top surface is rotating around the $z$-axis. The deformation can be described as **torsion**. The position vectors $\mathbf{x}$ of the material points in the deformed configuration can be expressed in the position vectors $\mathbf{x}_0$ in the undeformed (reference) configuration by means of:

$$
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} =
\begin{bmatrix}
  \cos(\phi) & -\sin(\phi) & 0 \\
  \sin(\phi) & \cos(\phi) & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x_0 \\
  y_0 \\
  z_0
\end{bmatrix},
$$

with $\phi = \omega z_0$ and $\omega$ the angular change around the $z$-axis per unit length in $z$-direction.

Determine the deformation matrix $F$ as a function of the reference coordinates $\mathbf{x}_0$.

### 9.3

A cubic continuum deforms homogeneously. The reference (time $t = 0$) and current configuration (time $t > 0$) are shown in the figure, in a Cartesian $xyz$-coordinate system.

The extension ratios in $x$-, $y$- and $z$-direction are:

$$
\lambda_x = 1 \quad ; \quad \lambda_y = \frac{t + \tau}{\tau} \quad ; \quad \lambda_z = \frac{\tau}{t + \tau}
$$

with $\tau$ a positive constant. The components of the velocity of the material depend on the position in space, specified by $x$, $y$ and $z$ (Eulerian formulation) and the time $t$.

Determine the velocity in the $y$-direction as a function of position and time: $v_y(x, y, z, t)$. 
9.4 In a Cartesian $xyz$ coordinate system a cylindrical blood vessel (length $\ell$) is considered. The blood is flowing with constant velocity: $\vec{v} = V \vec{e}_y$, see figure. The temperature $T$ in the blood vessel is a linear function of the spatial coordinate $y$:

$$T = T_0 + \frac{y}{\ell} (T_\ell - T_0) \quad \text{with } T_0 \text{ and } T_\ell \text{ constant}$$

Consider a fluid particle with position $\vec{x} = \ell/2 \vec{e}_y$ at time $t = \tau$.

What is the (infinitesimal) change $dT$ of the temperature of that particle after an (infinitesimal) increase $dt$ of the time?

9.5 Consider a deforming body in the reference and current configuration. In the reference configuration the position vector of a material point is given as $\vec{x}_0$ and in the deformed configuration as $\vec{x} = \vec{x}_0 + \vec{u}$ with $\vec{u}$ the displacement vector. Using the Lagrangian description the displacement field is given by:

$$\vec{u} = \vec{u}(\vec{x}_0) = \alpha [(\vec{x}_0 \cdot \vec{e}_y) \vec{e}_x - (\vec{x}_0 \cdot \vec{e}_x) \vec{e}_y] ,$$

with $\vec{e}_x$ and $\vec{e}_y$ the unit vectors along the $x$- and $y$-axes of a $xyz$-coordinate system and with $\alpha$ a constant.

Determine the associate Eulerian description of the displacement field:

$$\vec{u} = \vec{u}(\vec{x}).$$
9.6 From plate ABCD in an $xy$-coordinate system the reference and current configuration are given, see figure below. The position of a material point $P$ in the reference configuration is specified by the vector $\vec{x}_0 = x_0 \vec{e}_x + y_0 \vec{e}_y$ and in the current configuration with the vector $\vec{x} = x \vec{e}_x + y \vec{e}_y$. The deformation is homogeneous. The temperature field $T(x, y)$ is given in the current configuration (Eulerian description) as: $T(x, y) = \theta + \alpha x + \beta y$ with $\theta$, $\alpha$ and $\beta$ constant.

Determine the temperature field $T(x_0, y_0)$ with respect to the reference configuration (Lagrangian description).

9.7 Consider a flow between two plates in a fixed $x, y, z$-coordinate system (see figure). The top plate is brought at a temperature $T_2$, the bottom plate at a temperature $T_1$. The flow can be considered to be fully developed with a velocity profile:

$$\vec{v} = v(1 - y^2) \vec{e}_x,$$

with $v$ a constant. Near point $P$ a temperature profile along the $y$ axis
is measured during the heating phase of the system. For this profile the following relation is found:

\[ T = \frac{1}{2} [T_1(1 - y) + T_2(1 + y)] \left(1 - e^{-t}\right) \]

Determine the material time derivative \( \dot{T} \) of the temperature in point P.
Extra exercises of chapter 9

9.2 Answers

\[ \dot{T} = [(ab - ac)x - cby]e^{(1 - ct)} \]

\[ E = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & \omega[-x_0 \sin(\phi) - y_0 \cos(\phi)] \\ \sin(\phi) & \cos(\phi) & \omega[x_0 \cos(\phi) - y_0 \sin(\phi)] \\ 0 & 0 & 1 \end{bmatrix} \]

\[ v_y = \frac{y}{t + \tau} \]

\[ dT = \frac{V(T_t - T_0)}{\ell} \, dt \]

\[ \bar{u}(x) = \frac{1}{(1 + \alpha^2)} [\alpha(\alpha x + y)x - \alpha(x - \alpha y)y] \]

\[ T(x, y) = \theta + \left( 2\alpha + \frac{\beta}{2} \right)x_0 + \frac{\beta}{2}y_0 \]

\[ \dot{T} = \frac{1}{2}(T_1 + T_2)e^{-t} \]
Extra exercises of chapter 10

10.1 Exercises

10.1 Consider a cube of bone material, see figure. The bone cube is loaded with a shear force. The current position $\mathbf{x}$ of material points in the bone depends on the time $t$.

![Image of a cube with shear force](image)

The relation between the current position $\mathbf{x}$ and the position $\mathbf{x}_0$ in the reference configuration (at time $t = 0$) is given by:

$$
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
1 & \beta t & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_0 \\
y_0 \\
z_0
\end{pmatrix}
$$

with $\beta$ constant.

Determine the associate rate of deformation matrix $D$ as a function of time $t$.

10.2 Consider a deformable body in three-dimensional space (with $xyz$-coordinate system). Around a material point in the reference configuration a parallelepiped is defined (small with respect to the size of
the whole body), by means of the three vectors:

\[
\begin{align*}
\Delta \xi_{ja} &= \Delta \ell \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \\
\Delta \xi_{jb} &= \Delta \ell \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \\
\Delta \xi_{jc} &= \Delta \ell \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
\end{align*}
\]

with \(\Delta \ell\) constant. After a locally homogeneous deformation, these vectors are transformed to:

\[
\begin{align*}
\Delta \xi_{ja} &= \Delta \ell \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \\
\Delta \xi_{jb} &= \Delta \ell \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \\
\Delta \xi_{jc} &= \Delta \ell \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}.
\end{align*}
\]

Determine the local deformation matrix \(\mathbf{E}\).

10.3 Consider the deformation of a body in the current state with respect to the reference state. In a part of the volume of the body the deformation is homogeneous, specified by the deformation matrix \(\mathbf{F}\) according to:

\[
\mathbf{F} = \begin{bmatrix} 2 & 1 & 0 \\
3 & 2 & 0 \\
0 & 0 & 1 \end{bmatrix}
\]

In the current state two points P and Q, within the considered volume, are depicted with a distance \(\delta\) and connected by:

\[
\xi_Q - \xi_P = \delta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

Determine the (original) distance of the points P and Q in the reference state.

10.4 For a material particle the deformation in the current state with respect to the reference state is described by means of the deformation matrix \(\mathbf{F}\) according to:

\[
\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \end{bmatrix}
\]

The question is: can this deformation matrix exist in reality? If the
answer is no, why not? If the answer is yes, what is the physical meaning?

10.5 For a material particle the deformation process with respect to a defined reference state is described by means of the deformation matrix \( F(t) \) as a function of time \( t \). Based on this specification it is possible in this case to determine a spin matrix \( \Omega(t) \).

Consider an extra rotating process that is defined by the deformation matrix \( F^*(t) \) for which:

\[
F^*(t) = \mathcal{P}(t) F(t),
\]

where \( \mathcal{P}(t) \) is a time-dependent rotation matrix. The spin matrix associated with the extra rotating process is \( \Omega^*(t) \).

Proof the following relation between \( \Omega \) and \( \Omega^* \):

\[
\Omega^* = \mathcal{P} \Omega \mathcal{P}^T + \dot{\mathcal{P}} \mathcal{P}^T
\]

10.6 An incompressible, homogeneously deforming bar is placed in an \( xyz \)-coordinate system in such a way that the axis of the bar coincides with the \( x \)-axis. The length \( \ell \) of the bar is prescribed as a function of \( t \) according to:

\[
\ell = \ell_0 (1 + \alpha t) \quad \text{with } \alpha \text{ constant},
\]

where \( \ell_0 \) represents the length in the reference state (at \( t = 0 \)).

Determine the deformation rate matrix \( \dot{D} \) at time \( t = 1/\alpha \).

10.7 For a material particle, the deformation of the current state with respect to the reference configuration is given by means of the deformation matrix \( F \) according to:

\[
F = \begin{bmatrix}
3 & -4 & 0 \\
-2 & 3 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The components are referred to a Cartesian \( xyz \)-coordinate system.
Consider a material line segment that is parallel to the y-axis in the current state.

Calculate the Green-Lagrange strain $\varepsilon_{GL}$ for this line segment when deforming from the reference state to the current state.

10.8 The midplane of a thin rectangular membrane coincides with the xy-plane of a Cartesian xyz-coordinate system. The membrane is stretched equally in x- and y-direction. The extension ratio $\lambda$ as a function of time $t$ for both directions is given:

$$\lambda(t) = 1 + \alpha t ,$$

where $\alpha$ is a constant measure for the deformation velocity. The thickness of the membrane can adapt freely. The material behaves incompressible, leading to a deformation matrix $F$ of the following form:

$$F = F(t) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}$$

Determine the rate of deformation matrix $D = D(t)$, expressed in $\alpha$ (and $t$).

10.9 Consider a deforming body in a xyz-coordinate system. The position of the material points in the reference configuration is specified with the vector $\vec{x}_0$ and in the current configuration at time $t$ by $\vec{x}_c$. In the Lagrangian description the deformation process can be described by $\vec{x}_c = \tilde{x}(\vec{x}_0, t)$. For a special category of deformation processes we can write:

$$\vec{x}_c = \tilde{x}(\vec{x}_0, t) = A(t) \vec{x}_0 + \vec{s}(t) ,$$

where the components of $A$ and $\vec{s}$ are only functions of time.

Prove for the velocity gradient $L$, defined as $L = (\nabla v)^T$, that:

$$L = \frac{dA}{dt} A^{-1}$$

If the matrix $A$ equals the unit matrix $I$ substitution leads to the
10.1 Exercises

following equations:

\[ \mathbf{x} = \mathbf{x}_0(t_0, t) = \mathbf{x}_0 + \mathbf{s}(t) \quad \text{and} \quad \mathbf{L} = \mathbf{0} \]

What is the physical interpretation of these two relations?

10.10 For a material particle the deformation of the current state with respect to the reference state is specified by the deformation matrix \( \mathbf{F} \) according to:

\[ \mathbf{F} = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}, \]

where the components are related to a Cartesian \( xyz \)-coordinate system. Consider two material line segments. In the current state the line segments are parallel to the \( x \)-axis and the \( y \)-axis, respectively (the angle between the line segments is \( \alpha = \pi/2 \)).

Determine the angle \( \alpha_0 \) between the line segments in the reference state.

10.11 Around the origin of a Cartesian \( xyz \)-coordinate system the velocity field of fluid in a stationary flow is given by:

\[ \mathbf{v} = \begin{bmatrix}
v_x \\
v_y \\
v_z
\end{bmatrix} \quad \text{with:} \quad v_x = \frac{\alpha x}{(y - \beta)^2} , \quad v_y = \frac{\alpha}{y - \beta} , \quad v_z = 0 , \]

with \( \alpha \) and \( \beta \) positive constants. Consider an infinitesimally small material line segment, currently in the origin of the coordinate system and directed along the \( y \)-axis.

Determine the logarithmic strain rate \( \dot{\ln}(\lambda) = \dot{\lambda}/\lambda \), with \( \lambda \) the extension ratio.
A test that is used frequently for porous materials like cartilage, skin or spongious bone is the confined compression test. In a rigid cylindrical confining ring (radius of the cross section $R$) a porous material is placed (see figure). The porous material fits exactly in the ring. At time $t = 0$ [s] the height of the porous specimen is: $h = 1$ [cm]. From time $t = 0$ on the material is compressed by moving an indentor downwards with constant velocity $v_{st} = 0.1$ [cm/s] in negative $z$-direction. Assuming that the material in the cylinder deforms homogeneously, the only relevant component of the velocity gradient matrix $L$ will be $\partial v_z / \partial z$.

Determine the value of $\partial v_z / \partial z$ for $t = 2$ [s].

In a small volume of a material continuum the deformation tensor in the current state with respect to the reference state is constant. The following deformation tensor is given:

$$F = I - 4\tilde{e}_x \tilde{e}_x + 2\tilde{e}_x \tilde{e}_y + \tilde{e}_z \tilde{e}_z$$

with $I$ the unit tensor. Consider two material points $P$ and $Q$ in the current state:

$$\vec{x}_P = \tilde{e}_x + \tilde{e}_y + \tilde{e}_z ; \quad \vec{x}_Q = 2\tilde{e}_x + 3\tilde{e}_y + 2\tilde{e}_z$$
10.1 Exercises

The position of point P in the reference state is given by:

$$\vec{x}_{0P} = \vec{e}_x$$

Determine the position $\vec{x}_{0Q}$ in the undeformed reference configuration.
Extra exercises of chapter 10

10.2 Answers

10.1

\[ D = \frac{\beta}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

10.2

\[ F = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \]

10.3

\[ |d\vec{x}_0| = \delta \sqrt{13} \]

10.4 Yes, matrix \( F \) represents a rotation (over an angle \( \pi \)) around the \( x \)-axis

10.6

\[ D = \frac{\alpha}{4} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]

10.7

\[ \epsilon_{GL} = -\frac{12}{25} \]

10.8

\[ \widetilde{D} = \frac{\alpha}{1 + \alpha t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \]
10.2 Answers

10.9 If $A = I$ we have a rigid body translation

10.10

$$\cos(\phi) = \frac{3}{10}\sqrt{10}$$

10.11

$$\frac{\dot{\lambda}}{\lambda} = -\frac{\alpha}{\beta^2}$$

10.12

$$\frac{\partial v_z}{\partial z} = -0.125 \quad [s^{-1}]$$

10.13

$$\bar{x}_{0Q} = 2\bar{e}_x + 2\bar{e}_y + \frac{1}{2}\bar{e}_z$$
11
Extra exercises of chapter 11

11.1 Exercises

11.1 A porous material is subjected to a confined compression test. For this a cylindrical specimen is placed into a tight fitting ring (radius of the cross section R). At time $t = 0$ the material specimen is compressed by moving an indentor downwards in negative $z$-direction. In the reference state the material has a density $\rho_0$ [kg/m$^3$]. The rate of deformation matrix $\mathcal{D}$ of the material specimen is given by:

$$\mathcal{D} = \alpha \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right],$$

with $\alpha$ [m/s] a constant and with $\ell_0$ the initial height of the specimen.

Give an expression for the density $\rho = \rho(t)$ as a function of time $t$. 
11.2 An experimental set-up to test the behaviour of fluids is a so-called cross slot flow device. In the fluid reservoir four cylinders are located, all rotating with the same angular velocity. See figure.

In the set-up an incompressible fluid is examined. In the neighborhood of the origin of the $xyz$-coordinate system the flow field is described by:

$$\vec{v} = c(-x\vec{e}_x + y\vec{e}_y),$$

where $\vec{e}_x$ and $\vec{e}_y$ are unit vectors along the $x$- and $y$-axis.

The stress $\sigma$ in the fluid is related to the rate of deformation matrix $\mathbf{D}$ by means of: $\sigma = 2\eta \mathbf{D}$ (Newtonian fluid), where $\eta$ is a constant.

Determine the internal mechanical power per unit volume that is absorbed by the fluid in this point.
Extra exercises of chapter 11

11.2 Answers

11.1

\[ \rho = \frac{\rho_0 \ell_0}{\ell_0 - ct} \]

11.2

\[ P_{\text{int}} = 4\pi c^2 \]
12

Extra exercises of chapter 12

12.1 Exercises

12.1 The midplane of a very thin rectangular membrane coincides with the $xy$-plane of a Cartesian $xyz$-coordinate system. The membrane is stretched in $x$-direction (extension ratio $\lambda$. The displacement in $y$-direction is prevented. The membrane is able to stretch freely in $z$-direction. The material is incompressible, leading to a deformation matrix of the form:

$$F = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix}$$

For the stress matrix $\sigma$ a neo-Hookean relation holds:

$$\sigma = -pI + GBd$$

with $B = FF^T$, where $p$ is the hydrostatic pressure and $G$ the shear modulus.

Give an expression for $\sigma_{xx}$ in $x$-direction as a function of $\lambda$ and $G$.

12.2 A part of a blood vessel is modelled as a cylindrical tube with length $2\ell$. The central axis of the tube coincides with the $z$-axis of a Cartesian $xyz$-coordinate system, while $-\ell \leq z \leq \ell$. The cross section of the tube for $z = 0$ is depicted in the figure below. The cross section is symmetrical with respect to the $x$- and $y$-axis. The tube is loaded mechanically at the ends $z = \pm\ell$ in an unspecified way. At the same time an internal pressure is applied. The outer surface of the cylinder is unloaded. At point A at the outer surface (on the $y$-axis, see figure)
the following strain components are measured:

\[ \varepsilon_{xx} = 0.003 \quad \varepsilon_{zz} = 0.001 \quad \varepsilon_{xz} = \varepsilon_{zx} = 0.002 \]

Assume that the material behaviour of the vessel wall can be described with Hooke’s law, for which: \( E = 8 \) [GPa] and \( \nu = \frac{1}{3} \).

Determine the stress component \( \sigma_{zz} \) in point A.

**12.3** The midplane of a very thin rectangular membrane coincides with the \( xy \)-plane of a Cartesian \( xyz \)-coordinate system. The membrane is stretched in \( x \)- and \( y \)-direction in the same way (extension ratio \( \lambda \)). The membrane is able to stretch freely in \( z \)-direction. The material is incompressible, leading to a deformation matrix of the form:

\[
F = \begin{bmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \frac{1}{\lambda}
\end{bmatrix}
\]

For the stress matrix \( \sigma \) a neo-Hookean relation holds:

\[
\sigma = -p I + GB \theta^d \quad \text{with} \quad B = FF^T,
\]

where \( p \) is the hydrostatic pressure and \( G \) the shear modulus.

Give an expression for \( \sigma_{xx} \) in \( x \)-direction as a function of \( \lambda \) and \( G \).

**12.4** In a confined compression test a specimen of a porous biological material is placed into a tight fitting ring (circular cross section with radius \( R \), see figure). The specimen is compressed by means of a rigid indenter in such a way that the volume of the specimen is reduced.
with a factor 2. The force required to do this is \( P \). Assume that the porous material deforms in a homogeneous way.

The mechanical behaviour of the material is described with a compressible neo-Hookean model:

\[
\sigma = K(J - 1) + \frac{G}{J} \tilde{B}^d,
\]

with \( \sigma \) the stress matrix, \( J \) the volume change and \( \tilde{B}^d \) the isochoric left Cauchy Green matrix. The compression modulus \( K \) and the shear modulus \( G \) for this material can be assumed to be equal to each other: \( K = G \).

Express the force \( P \) in \( K \) and \( R \).
Extra exercises of chapter 12

12.2 Answers

12.1

$$\sigma_{xx} = G \left( \lambda^2 - \frac{1}{\lambda^2} \right)$$

12.2

$$\sigma_{zz} = 18 \quad \text{[MPa]}$$

12.3

$$\sigma_{xx} = G \left( \lambda^2 - \frac{1}{\lambda^4} \right)$$

12.4

$$P = \frac{\pi R^2 K}{2} \left( 1 + 2\hat{\lambda} \right)$$
13
Extra exercises of chapter 13

13.1 Exercises

•
Extra exercises of chapter 13

13.2 Answers
14

Extra exercises of chapter 14

14.1 Consider the following partial differential equation on the domain $-1 \leq x \leq 1$:

$$\frac{\partial u}{\partial t} - Au + B \frac{\partial^2 u}{\partial x^2} - C \frac{\partial u}{\partial x} = \sin \phi t,$$

with $A, B, C$ and $\phi$ constants and $t$ the time. The weighted residual form of this equation yields:

$$\int_{-1}^{1} w \left( \frac{\partial u}{\partial t} - Au + B \frac{\partial^2 u}{\partial x^2} - C \frac{\partial u}{\partial x} - \sin \phi t \right) dx = 0$$

with $w$ an arbitrary weighting function.

Which term(s) of the weighted residual equations have to be subjected to partial integration and why? Why is the result called the "weak" formulation?

14.2 A researcher is trying to make "tissue engineered" cartilage. The cartilage is placed in a circular wells plate and surround with medium (see figure).

Because the thickness of the cartilage $t$ is much smaller than the diameter $D$ the transport problem of oxygen and nutrients can be considered one-dimensional. The thickness $t = 0.2$[cm]. The researcher wants to know how high the internal glucose concentration is for a given thickness of the cartilage and a given glucose concentration in the medium. This can be done by solving the one-dimensional sta-
schematic of bioreactor

FEM model

\[ \frac{du}{dx} = 0 \quad u = u_0 \]

x=0 \quad x=h

t

Stationary diffusion equation:

\[ \frac{d}{dx} \left( c \frac{du}{dx} \right) + f = 0, \]

with \( u \) the glucose concentration [Mol cm\(^{-3}\)], \( c = 9 \times 10^{-6} \text{ [cm}^2\text{ s}^{-1}] \) the diffusion coefficient and \( f = -0.156 \times 10^{-8} \text{ [Mol s}^{-1}\text{ cm}^{-3}] \) (watch the minus sign!) the glucose consumption. The glucose concentration in the medium is: \( 5 \times 10^{-6} \text{ [Mol cm}^{-3}] \). The interface between the cartilage and the bottom of the well can be considered impenetrable.

Determine in the stationary equilibrium state the concentration glucose in the point \( x = 0 \). Adjust the file demo_femid that is found in directory oned of mlfem_nac.
14.2 Answers

14.1 Only the term with $\frac{\partial^2 u}{\partial x^2}$ because we want to reduce the order of differentiation. This means that the requirements on the set of functions chosen to construct approximate solutions is less strict. That is the reason why it is called "weak" formulation.

14.2

\[ u(0) = 1.53 \times 10^{-6} [\text{mol cm}^{-3}] \]
Extra exercises of chapter 15

15.1 Exercises

15.1 We would like to solve the following ordinary differential equation for $u = u(t)$:

$$\frac{du}{dt} + Au = \sin(\omega t),$$

with $\omega$ and $A$ constants and $t$ is the time. We divide the time axis in $N$ discrete time intervals. We write $u_{n+1} = u(t_{n+1})$ and $u_n = u(t_n)$.

Derive a recursive scheme that enables us to determine $u_{n+1}$ when $u_n$ is known. Use a $\theta$-scheme with $\theta = 0.5$ (Crank-Nicholson).

15.2 In a bioreactor a tissue engineered construct is placed with medium on both sides. On the left side (see figure) a high concentration of a certain growth factor is found. On the other side the concentration of the growth factor is low. A fluid flow is forced through the porous tissue from the side with the high concentration to the side with the low concentration. (See figure)

A researcher wants to know the concentration as a function of position of the growth factor for a given velocity of the flow. He solves the one-dimensional convection diffusion equation for this purpose and uses a mesh with 10 linear elements. The length of the domain is 1 [mm]. The diffusion constant for the domain is: $c = 8 \cdot 10^{-5}$ [mm$^2$ s$^{-1}$].
At which velocity $v$ of the main flow do you expect stability problems with the numerical solution?

15.2 We would like to solve the following ordinary differential equation:

$$\frac{du}{dt} - 2u = \sin(\pi \cdot t/4),$$

with $u = u(t)$ and $u(0) = 0$ at time $t = 0$.

Write a MATLAB script to solve the equation using an Euler explicit time integration scheme for $0 \leq t \leq 8$. Calculate the solution $u(t)$ for $t = 2$ and $t = 6$.
15.1

\[ u_{n+1} = \frac{(1 - \frac{1}{2}A\Delta t)u_n + \frac{\Delta t}{2} (\sin(\omega t_{n+1}) + \sin(\omega t_n))}{(1 + \frac{1}{2}A\Delta t)} \]

15.2 Stability problem will arise when \( Pe = \frac{vh}{c} \), based on the element size \( h \) becomes higher than one. This point is reached when: \( v = 16 \cdot 10^{-4} \) [mm$^2$ s$^{-1}$]
16.1 A finite element mesh is given in the figure below.

The associated top array (in MATLAB format) is given by:

```
top = [1 2 5 4
      2 3 5 0
      4 5 6 0
      5 7 6 0
      3 7 5 0]
```

The dest array (in MATLAB format) is given by:

```
dest = [4 ; 1 ; 5 ; 3 ; 2 ; 7 ; 6]
```

Determine the associated array pos.
16.2 The following array \texttt{usercurves} is associated with the mesh in exercise 16.1 (MATLAB format):

\begin{verbatim}
usercurves = [1 2 3
   3 7 0
   7 6 0
   6 4 1]
\end{verbatim}

The solution array \texttt{sol} is given by (MATLAB format):

\begin{verbatim}
sol = [ 0.3 ; 0.4 ; -0.2 ; 0.5 ; -0.7 ; 0.8 ; 0.0 ]
\end{verbatim}

Determine the sum of the solutions in the nodes of \texttt{usercurve 4}. Also use the information given in exercise 16.1.

16.3 We want to solve the two-dimensional diffusion equation by means of the finite element method.

\[ \vec{\nabla} \cdot (c \vec{\nabla} u) + f = 0 \, . \]

The weak form leads to the following equation:

\[ \sum_{e=1}^{N_{el}} \mathbf{v}_e^T \mathbf{K}_e \mathbf{u}_e = \sum_{e=1}^{N_{el}} \mathbf{v}_e^T \mathbf{f}_e \, , \]

for which the element matrix \( \mathbf{K}_e \) is given by:

\[ \mathbf{K}_e = \int_{\Omega_e} c \left( \frac{\partial N}{\partial x} \frac{\partial N^T}{\partial x} + \frac{\partial N}{\partial y} \frac{\partial N^T}{\partial y} \right) \, d\Omega \, . \]

We chose \( c = 1 \). In addition, an element with 4 corner nodes is chosen, with a global coordinate system as given in the figure.

![Diagram of a square element with global coordinate system]
A program to determine the matrix by means of a 4-point Gauss integration might have the following structure:

c = 1;
% coordinaten van de integratiepunten
coordint=[ ; ; ; ];
% startwaarde voor de te berekenen matrix
K=zeros(4);
% loop over integratiepunten
for i=1:4
    x = coordint(i,1);
    y = coordint(i,2);
    dndx=[ ; ; ; ];
    dndy=[ ; ; ; ];
    K = ;
end
K

Finish the program and determine the matrix $K$. 
Extra exercises of chapter 16

16.2 Answers

16.1 \( \text{pos} = \begin{bmatrix} 4 & 1 & 2 & 3 \\ 1 & 5 & 2 & 0 \\ 3 & 2 & 7 & 0 \\ 2 & 6 & 7 & 0 \\ 5 & 6 & 2 & 0 \end{bmatrix} \)

16.2 The sum of the solution in the nodes of \text{usercurve} 4 is: 0.3.

16.2 \% program to determine stiffness matrix

```matlab
c=1;
a=1/sqrt(3)
coord=[-a -a ; a -a ; a a ; -a a]
nint=4

Ke=zeros(4)
for i=1:nint
    x=coord(i,1)
y=coord(i,2)
dndx=[-0.25*(1-y) +0.25*(1-y) +0.25*(1+y) -0.25*(1+y)];
    sum(dndx)
dndy=[-0.25*(1-x) -0.25*(1+x) +0.25*(1+x) +0.25*(1-x)];
    sum(dndy)
    Ke = Ke + c*(dndx*dndx'+dndy*dndy')
end

Ke
```

16.2 Answers

\[ K_e = \]

\[
\begin{pmatrix}
0.6667 & -0.1667 & -0.3333 & -0.1667 \\
-0.1667 & 0.6667 & -0.1667 & -0.3333 \\
-0.3333 & -0.1667 & 0.6667 & -0.1667 \\
-0.1667 & -0.3333 & -0.1667 & 0.6667
\end{pmatrix}
\]
17

Extra exercises of chapter 17

17.1 Exercises

17.1 A one-dimensional element has 4 nodes at the positions given in the figure below.

![Element diagram]

Give the 4 shape functions \( N_i (i = 1, 2, 3, 4) \) belonging to this element.

17.2 Consider the following integral:

\[
\int_0^2 \frac{x^2}{\sqrt{3x + 1}} \, dx
\]

Determine an approximate solution for this integral by using a two point Gauss integration.

17.3 Proof that numerical integration of the quadratic function \( f(x) = a + bx + cx^2 \) (with \( a, b \) and \( c \) arbitrary constants) on the domain \(-1 \leq x \leq 1\) with 2-point Gauss integration leads to an exact solution of the integral.
17.4 A two-dimensional element has 9 nodes that are found on the positions as shown in the figure below.

Give the shape functions $N_i$ in the global coordinates $x$ and $y$, that belong to the nodes 1, 5 and 9.

17.4 Consider the following integral:

$$\int_{-1}^{1} \frac{x^2}{\sqrt{3x + 4}} \, dx$$

Determine an approximate solution for this integral by using 3-point Gauss integration.

17.5 From a certain element the following shape functions are given:

$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)(-\xi - \eta - 1)$$
$$N_2 = \frac{1}{4}(1 + \xi)(1 - \eta)(+\xi - \eta + 1)$$
$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)(+\xi + \eta - 1)$$
$$N_4 = \frac{1}{4}(1 - \xi)(1 + \eta)(-\xi + \eta + 1)$$

Why is it impossible that these are correct?
17.2 Answers

\[ N_1 = -\frac{1}{30}(x + 1)x(x - 2) \]
\[ N_2 = \frac{1}{6}(x + 3)x(x - 2) \]
\[ N_3 = -\frac{1}{6}(x + 3)(x + 1)(x - 2) \]
\[ N_4 = \frac{1}{30}(x + 3)(x + 1)x \]

17.2

\[ \int_{0}^{2} \frac{x^2}{\sqrt{3x + 1}} \, dx \approx 1.1578 \]

17.3 Both exact integration as well as numerical integration lead to the expression:

\[ 2a + \frac{2}{3}c \]

17.4

\[ N_1 = \frac{1}{9}x(x - 2)y(y - 2) \]
\[ N_5 = \frac{1}{4}(x + 1)(x - 2)(y + 1)(y - 2) \]
\[ N_9 = \frac{1}{36}x(x + 1)y(y + 1) \]

17.4

\[ \int_{-1}^{1} \frac{x^2}{\sqrt{3x + 4}} \, dx \approx 0.390 \]

17.5 Because:

\[ \sum_{i=1}^{4} N_i(\xi, \eta) \neq 1 \]
18
Extra exercises of chapter 18

18.1 Exercises

18.1 In a test-setup the bending of a beam is used to measure forces. To adjust the dimensions of the set-up a researcher wants to do a number of simulations with the model that is given in the figure.
In the figure is shown which userpoints are defined to construct the model. The dimensions are given in [mm]. Assume a state of plane stress. Use bi-linear rectangular elements. Subarea I can be modelled with $10 \times 5$ elements. Subarea II with $3 \times 5$ elements and subarea III with $3 \times 10$ elements. The displacement of the bottom boundary ($y = 0$) is suppressed. A force of $0.01 \text{ N}$ in $x$-direction is prescribed on userpoint 7. The Young’s modulus $E$ is 100 [N mm$^{-2}$], the Poisson’s ratio $\nu$ is 0.3 [=].

Determine the horizontal displacement of the beam in userpoint 7.

18.2 A finite element mesh for a two-dimensional elastic solid problem with three rectangular 4-node elements is given in the figure. There is no distributed load. The associated dest matrix (in MATLAB format)

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6 \\
7 & 8 \\
9 & 10 \\
11 & 12 \\
13 & 14 \\
15 & 16
\end{bmatrix}
\]

The displacement of node 1 is suppressed in both directions. The displacement of node 3 is suppressed in $y$-direction. On node 8 a force $F_1$ is working. For the direction see the figure. On node 4 a force $F_2$ is working in negative $x$-direction. After all steps of the finite element method up to assemblage, the equilibrium equations lead to a set:

\[ K u \sim = c f. \]

Show which degrees of freedom in the column $f$ on the right-hand-side of the equation are unknown, have a known prescribed value unequal to zero or have a value equal to zero.
18.3 A finite element mesh is given in the figure below. The associated top array is (in MATLAB format):

\[
\text{top} = \begin{bmatrix}
1 & 2 & 5 & 4 \\
2 & 3 & 5 & 0 \\
4 & 5 & 7 & 0 \\
5 & 8 & 7 & 0 \\
3 & 6 & 5 & 0 \\
5 & 6 & 9 & 8
\end{bmatrix}
\]

The pos array (in MATLAB format) is given by:

\[
\text{pos} = \begin{bmatrix}
4 & 1 & 3 & 5 \\
1 & 7 & 3 & 0 \\
5 & 3 & 9 & 0 \\
3 & 2 & 9 & 0 \\
7 & 8 & 3 & 0 \\
3 & 8 & 6 & 2
\end{bmatrix}
\]

Determine the associated array dest.

18.4 A finite element mesh for a two-dimensional solid problem comprises 3 rectangular 4 noded elements as given in the figure below. There is no distributed load. De associates dest-matrix is given by (in MATLAB format):

\[
\text{dest} = \begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6 \\
7 & 8 \\
9 & 10 \\
11 & 12 \\
13 & 14 \\
15 & 16
\end{bmatrix}
\]

The displacements of nodes 1 and 7 are prescribed in both directions. A force \( F_1 \) is acting on node 2 in negative \( y \)-direction. A force \( F_2 \) is acting on node 3 in negative \( y \)-direction. After all steps until assem-
equations of the form $\mathbf{K}\mathbf{y} = \mathbf{f}$.

Point out which degrees of freedom in the right-hand-side column are unknown, which have a prescribed value that is not equal to zero and which have a prescribed value equal to zero.
18.2 Answers

18.1 % This script solves extra exercise 18.1
    clear, close all
    % L: Length of the model, H: Height of the model
    L1=9; L2=10; H1=5; H2=15;
    % itype=1: quadrilateral element
    % norder=1: (bi-)linear element
    itype=1;
    norder=1;
    % definition of the points of the domain
    points=[0 0 ; L1 0 ; L2 0 ; 0 H1 ; L1 H1 ; L2 H1 ; L1 H2 ; L2 H2];
    % n: number of elements in the x-direction
    n1=10;
    n2=3;
    m1=5;
    m2=10;
    % definition of the curves (see crmesh)
    curves=[
        1 2 n1 1 1
        2 3 n2 1 1
        4 5 n1 1 1
        5 6 n2 1 1
        7 8 n2 1 1
        1 4 m1 1 1
        2 5 m1 1 1
        3 6 m1 1 1
        5 7 m2 1 1
        6 8 m2 1 1];
    % subarea definition (see crmesh)
    subarea=[1 7 -3 -6 1
             2 8 -4 -7 1
             4 10 -5 -9 1];
    % E: Young’s modulus
    mat.mat=[100 0.3 2];
    if itype<10,
        mat.mat(11)=norder+2;
    else
mat.mat(11)=itype-20+norder;
end
% element type specification
mat.types='ele';
% create the mesh
[top,coord,usercurves,userpoints,usersurfaces,topbnd]= ...
crmesh(curves,subarea,points,norder,itype)
% specification of the boundary conditions
bndcon=[];
% along curves 1 and 2, the degrees
% of freedom in the 1 and 2 direction are set to 0
crv=[1 2];
dof=[1 2];
string='0';
iplot=0;
bndcon=addbndc(bndcon,coord,top,mat, ... 
usercurves,crv,dof,string,iplot);
% nodal force at userpoint number 7
nodfrc=[userpoints(7) 1 1e-2];
% do the finite element computation
femlin_e
figure(1)
pldisp(sol,coord,top,dest,mat)
figure(2)
plelmdat(sigma_elm,coord,top,mat,2);
%determine node number of userpoint 7
ii=userpoints(7);
idof=dest(ii,1);
ux=sol(idof)
This leads to: u = 0.3429.

18.2 \( f = \begin{bmatrix} ? & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F1x & F1y \end{bmatrix} \)

18.3 dest = [4 ; 1 ; 7 ; 5 ; 3 ; 8 ; 9 ; 2 ; 6 ]

18.4 \( f = \begin{bmatrix} ? & ? & 0 & -F1 & 0 & -F2 & 0 & 0 & 0 & 0 & 0 & 0 & ? & 0 & 0 \end{bmatrix} \)